Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

Positive bound states for nonlinear Schrödinger equations in exterior domains

by

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Dedicated to my parents,
my wife
and my daughters

"Of course, the most rewarding part (of Mathematical study) is the "Aha" moment,
the excitement of discovery and enjoyment of understanding something new,
the feeling of being on top of a hill and having a clear view"
Maryam Mirzakhani, the first woman to win the Fields Medal, 1977-2017.

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Abstract

In this work, we consider two problems. First we establish the existence of a positive solution for semilinear elliptic equation in an exterior domain

$$\begin{cases} -\Delta u + V(x)u = f(u), & \text{in } \Omega \subseteq \mathbb{R}^N \\ u \in H_0^1(\Omega) \end{cases} \tag{P_V}$$

where $N \geq 2$, $\mathbb{R}^N \backslash \Omega$ is regular bounded domain but there is no restriction on its size, nor any symmetry assumption. The nonlinear term f is a non homogeneous, asymptotically linear or superlinear function at infinity. Moreover, the potential V is a positive function, not necessarily symmetric. The existence of a solution is established in situations where this problem does not have a ground state.

In the second problem we consider the Null Mass nonlinear field equation

$$\begin{cases} -\Delta u = f(u) \quad \text{in } \Omega \\ \quad u > 0 \\ \quad u \mid_{\partial\Omega} = 0 \end{cases}$$

where $\mathbb{R}^N \setminus \Omega$ is regular bounded domain and like as above there is no restriction on its size, nor any symmetry assumption. The nonlinear term f is general non-homogeneous non-linearities with double-power growth condition. The existence of bound state solution is established in situations where this problem does not have a ground state.

Keywords: Asymptotically linear, superlinear, nonlinear Schrödinger equation, exterior domain, variational methods, nonlinear Null Mass equation.

Resumo

Estamos interessados na existência de uma solução positiva para duas classes de equaçãoes não lineares de Schrödinger em domínios exteriores:

$$\begin{cases} -\Delta u + V(x)u = f(u), & \text{in } \Omega \subseteq \mathbb{R}^N \\ u \in H_0^1(\Omega) \end{cases} \tag{P_V}$$

onde $N \geq 2$, $\mathbb{R}^N \setminus \Omega$ é um domínio limitado regular, mas não há restrição sobre o seu tamanho, nem qualquer hipótese de simetria e também

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ \quad u > 0, \\ \quad u \mid_{\partial\Omega} = 0, \end{cases} \tag{$\mathcal{P}$}$$

onde $N \geq 3$, $\mathbb{R}^N \setminus \Omega$ é um domínio limitado regular, e como acima não há restrição sobre o seu tamanho, nem qualquer hipótese de simetria.

Nosso objetivo no primeiro capítulo é mostrar a existência de uma solução positiva do problema (P_V) onde o nível mínimo de energia não pode ser obtido. Usando uma nova abordagem desenvolvida recentemente por Évéquoz e Weth [31], Clapp e Maia [24] e Maia e Pellacci [37] uma solução positiva é encontrada, estendendo os resultados de existência obtidos nos artigos clássicos de Benci e Cerami [9] e Bahri e Lions [6], para não-linearidades gerais não homogêneas, superlineares ou assintoticamente lineares no infinito em um domínio exterior.

O estudo de ondas solitárias de equações de Schrödinger não lineares ou equações não lineares de Klein-Gordon é modelado por (P_V) com $\Omega = \mathbb{R}^N$. Da mesma forma, problemas de fronteira de limite exterior podem estar associados a modelos de fluxos de estado estacionário na dinâmica de fluidos (ver [32]) e ao problema eletrostáatico de capacitores (veja [27], Volume 1, Capítulo II), por exemplo.

Nossa contribuição principal no primeiro capítulo foi estender o resultado de Bahri e

Lions [6] para f não homogêneas, sem hipótese de simetria em V ou Ω. Além disso, permitimos que a função não linear f seja uma função menos suave, apenas em C^1, melhorando as hipóteses em [24] e [37] onde esta foi considerada em C^3 por razões técnicas (veja o Lemma 3.3 em [24]). O método que empregamos para resolver (P_V) tem muitas ideias em comum com [24, 37]. Do mesmo modo, o trabalho de [31] forneceu algumas ferramentas úteis e informações para estimativas, mesmo que seu problema seja para f super-linear em todo $\mathbb{R}^N$ e usa a variedade de Nehari generalizada.

Segundo o nosso conhecimento, os resultados que apresentamos aqui são novos e estendem os trabalhos anteriores encontrados na literatura para uma classe de problemas em domínios exteriores. Consideramos o problema elíptico

$$-\Delta u + V(x)u = f(u) \quad , \quad u \in H_0^1(\Omega) \tag{P_V}$$

onde $N \geq 2$, $\mathbb{R}^N \setminus \Omega \subseteq B_K(0)$ a bola do raio K e centro na origem em $\mathbb{R}^N$; de fato $\mathbb{R}^N \setminus \Omega$ é limitado, $\partial\Omega$ é regular e $u \in H_0^1(\Omega)$ e V é um potencial que satisfaça as condições:

(V_1) $V \in C^0(\Omega)$, $\inf_{x\in\Omega} V(x) > 0$ e $\lim\limits_{|x|\to+\infty} V(x) = V_\infty$;

(V_2) $V(x) \leq V_\infty + Ce^{-\gamma|x|}$, onde $C > 0$ e $\gamma > 2\sqrt{V_\infty}$.

As condições que consideramos na não linearidade f são as seguintes:

(f_1) $f \in C^1\big([0,\infty)\big)$;

(f_2) Existe $C_2 > 0$ e $1 < p_1 \leq p_2$ tal que $p_1, p_2 < 2^* - 1$ e

$$|f^{(k)}(s)| \leq C_2(|s|^{p_1-k} + |s|^{p_2-k})$$

por $k \in \{0,1\}$ e $s > 0$;

(f_3) $\lim\limits_{s\to+\infty} \dfrac{f(s)}{s} \geq m > V_\infty$;

(f_4) Se $F(s) := \displaystyle\int_0^s f(t)dt$ e $Q(s) := \dfrac{1}{2}f(s)s - F(s)$, então

$$\lim_{s\to\infty} Q(s) = +\infty;$$

(f_5) A função $s \mapsto f(s)/s$ é crescente em $s \in (0,+\infty)$;

(U) A solução positiva radialmente simétrica do problema limite

$$-\Delta u + V_\infty u = f(u) \quad , \quad u \in H_0^1(\mathbb{R}^N) \qquad (P_\infty)$$

é única.

O resultado principal do primeiro capítulo é o seguinte:

Teorema A: Sob hipóteses $(V_1) - (V_2)$, $(f_1) - (f_5)$ e (U), o problema (P_V) tem uma solução positiva u em $H_0^1(\Omega)$.

No segundo capítulo, procuramos uma solução positiva para o problema $(\mathcal{P})$ onde um nível mínimo de energia não pode ser atingido. Aqui, estudamos não linearidades não homogêneas gerais, com condição de crescimento em f de potência dupla, que se comporta como uma potência subcrítica u^p no infinito e uma potência supercrítica u^q perto da origem, onde $p < 2^* < q$, em qualquer domínio exterior. Usando as ideias introduzidas em [24, 25, 37], estendemos os resultados de V. Benci e A. Micheletti [12] removendo qualquer suposição no tamanho da abertura $\mathbb{R}^N \setminus \Omega$.

Neste capitulo o método utilizado para encontrar uma solução de $(\mathcal{P})$ como um ponto crítico do funcional associado à equação, restrito à variedade de Nehari do funcional, é bastante natural por causa da geometria deste funcional devido ao crescimento superquadrático dos termos não lineares. Entretanto, a novidade em nossa aproximação é encontrada principalmente em alguns resultados técnicos delicados, como as estimativas exatas sobre o decaimento da solução de nivel minimo de energia do problema em $\mathbb{R}^N$ e suas implicações na interação de duas cópias distintas e distantes desses solitões. Por outro lado, um novo resultado de compacidade numa nova versão do Lema de Lions, que nos permite contornar as dificuldades criadas por um domínio não simétrico ilimitado e abraçar um problema muito geral.

Problemas como $(\mathcal{P})$ com $f'(0) = 0$, o chamado caso de massa zero, aparecem no estudo das equações de Yang-Mills e tem atraído o interesse dos pesquisadores, principalmente no caso $\Omega = \mathbb{R}^N$ (veja [13, 33, 46]).

O principal objetivo do segundo capítulo é resolver o problema $(\mathcal{P})$, no caso de massa zero, quando Ω é um domínio exterior que não há restrição sobre o seu tamanho. Para fazermos isso, usamos nível mínimo de energial em todo o $\mathbb{R}^N$, qual seja w, e mostramos que existe $u \in \mathcal{D}^{1,2}(\Omega)$ que é solução de $(\mathcal{P})$, mas não uma solução de nível minimo de energia. Na verdade, não existe uma solução de $(\mathcal{P})$ que minimize a função de energia na variedade de Nehari. Estendemos os resultados em V. Benci e A. Micheletti [12], em que eles trabalharam com Ω tal que $\mathbb{R}^N \setminus \Omega \subset B_\epsilon$ quando ϵ é suficientemente pequeno. Essa hipótese no tamanho de Ω é removida em nosso trabalho.

Uma característica importante quando Ω é um domínio exterior ilimitado é que $\mathcal{D}^{1,2}(\Omega)$ não está necessariamente contido em qualquer espaço de Lebesgue $L^q(\Omega)$ com $q \neq 2^*$ e, portanto, não há imersão de Sobolev padrão como as de $H_0^1(\Omega)$. Por esse motivo, estudamos o espaço de Orlicz relacionado ao termo do lado direito f e exigimos que ele satisfaça uma condição de crescimento de dupla de potência e resulte na regularidade necessária do funcional de energia. Estes espaços de Lebesgue têm várias propriedades importantes e essenciais que desempenham o mesmo papel para o espaço de Hilbert $\mathcal{D}^{1,2}(\mathbb{R}^N)$ que os espaços comuns de Lebesgue jogam para $H_0^1(\Omega)$. Em um domínio exterior, a principal dificuldade é a falta de compacidade. Aqui, usamos o Lema de Splitting que é uma chave importante para superar a falta de compacidade. Este lema é uma variante de um resultado bem conhecido de M. Struwe (veja [45]) relacionado ao espaço $\mathcal{D}^{1,2}(\Omega)$ e também V. Benci e G Cerami [9] com uma descrição precisa do que acontece quando uma sequência de Palais-Smale não converge para seu limite fraco. Observe que, uma vez que o espaço $\mathcal{D}^{1,2}(\Omega)$ não está necessariamente contido em $H_0^1(\Omega)$, não podemos usar Lema de Lions como em [35]. Então precisamos de outra versão do Lema de Lions e lema de Splitting em espaços de Orlicz que mostramos em Lemma 2.3.3 e Lemma 2.3.5.

Finalmente, de acordo com o método que aplicamos neste segundo capítulo, precisamos comparar o nível mínimo de energia associado á equação em $(\mathcal{P})$ com o nível mínimo de energia associados com a equação em $\mathbb{R}^N$. Estimativas de decaimento adequadas para w, a solução radial positiva do problema limite e ∇w serão fundamentais para comparar todos os termos nos funcionais de energia com o nível minimo de energia. Graças a J. Vetois [47], encontramos estimativas de decaimento muito finas e exatas para w e ∇w, que desempenham papéis essenciais neste trabalho.

As condições que consideramos na não linearidade $f : \mathbb{R} \to \mathbb{R}$ são: ela é uma função ímpar e de classe $C^1(\mathbb{R}, \mathbb{R})$ tal que

(f_1) Seja $F(s) := \int_0^s f(t)dt$, então $0 < \mu F(s) \leq f(s)s < f'(s)s^2$ para qualquer $s \neq 0$ e para alguns $\mu > 2$;

(f_2) $F(0) = f(0) = f'(0) = 0$. Existem $C_1 > 0$ e $2 < p < 2^* < q$ tal que

$$\begin{cases} |f^{(k)}(s)| \leq C|s|^{p-(k+1)} & \text{for } |s| \geq 1 \\ |f^{(k)}(s)| \leq C|s|^{q-(k+1)} & \text{for } |s| \leq 1 \end{cases}$$

por $k \in \{0,1\}$, $s \in \mathbb{R}$.

O resultado principal do segundo capítulo é o seguinte:

Teorema B: Suponha que a solução positiva em todo o $\mathbb{R}^N$ é única. Então, sob as hipóteses $(f_1) - (f_2)$, o problema $(\mathcal{P})$ tem uma solução clássica positiva $u \in \mathcal{D}^{1,2}(\Omega)$.

Palavras-chave: Assintóticamente linear, superlinear, equação não linear de Schrödinger, domínio exterior, métodos variacionais, equação de massa zero não linear.

Contents

Introduction

We are interested in the existence of a positive solution, not necessarily ground states, for two classes of nonlinear Schrödinger equations in exterior domains:

$$\begin{cases} -\Delta u + V(x)u = f(u), & \text{in } \Omega \subseteq \mathbb{R}^N \\ u \in H_0^1(\Omega) \end{cases} \qquad (P_V)$$

where $N \geq 2$, $\mathbb{R}^N \setminus \Omega$ is regular bounded domain but there is no restriction on its size, nor any symmetry assumption and

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u \mid_{\partial\Omega} = 0 \end{cases} \qquad (\mathcal{P})$$

where $N \geq 3$, $\mathbb{R}^N \setminus \Omega$, like as above is regular bounded domain but there is no restriction on its size, nor any symmetry assumption.

Our goal in the first chapter is to show the existence of a positive bound state solution for problem (P_V) where a ground state cannot be obtained. Using a new approach recently developed by Évéquoz and Weth [31], Clapp and Maia [24] and Maia and Pellacci [37] a positive solution is found, extending the existence results obtained in the celebrated papers of Benci and Cerami [9] and Bahri and Lions [6], for general non-homogeneous non-linearities, either superlinear or asymptotically linear at infinity in an exterior domain.

The study of solitary waves of nonlinear Schrödinger equations or of nonlinear Klein-Gordon equations is modeled by (P_V) with $\Omega = \mathbb{R}^N$. Likewise, exterior boundary-value problems may be associated with models of steady-state flows in fluid dynamics (see [32]) and electrostatic problem of capacitors (see [27],Volume 1,Chapter II), for instance.

The primary works applying variational methods to find solutions of problems like (P_V) report to the 80's and 90's with the articles of Benci and Cerami [9] and Bahri and Lions [6]. The method applied in both works was finding critical points of a functional constrained on a manifold and absorbing a Lagrange multiplier by the homogeneity of the nonlinear term $f(x) = |u|^{p-2}u$ where $p \in (2, 2^*)$ and $2^* = \frac{2N}{N-2}$ if $N \geq 3$, $2^* = \infty$ if $N = 2$ in order to obtain a positive solution of the Euler equation in (P_V).

One of the main challenges of trying to apply the usual variational method when Ω is an unbounded domain is the lack of compactness of the Sobolev embeddings. In order to circumvent this difficulty, a deeper study of the obstruction for compactness was performed by Benci and Cerami in [9] and a clever description was obtained of what happens when a Palais-Smale sequence does not converge to its weak limit (for details see [20] and references therein). Problem (P_V) with $f(u) = |u|^{p-2}u,\ p \in (2, 2^*)$ was solved in the case that the ground state does not exist first in [9] in the autonomous case $V(x) = \lambda$ a positive constant, proving the existence of a positive solution with some restriction on the size of the hole $\mathbb{R}^N \setminus \Omega$, and posteriorly that condition was eliminated in [6] and existence was proved for potentials V which decay to a constant potential V_∞ at infinity. In the same spirit, this problem has been extensively studied if Ω is an exterior domain for power non-linearity $f(u) = |u|^{p-2}u$ in recent years (see [6]). If the non-linear term f is not a pure power with respect to u, there are few contributions in the literature. In particular, the existence of solution is proved in [23], using topological methods, in the case that f is super-linear and depends on the spatial variable but the asymptotic nonlinearity f_∞, of the autonomous problem, must satisfy a convexity assumption.

In the case Ω is spherically symmetric about some point, benefiting from the strength of the symmetry property, this problem can be solved on $H^1_{rad}(\Omega)$ (subspace of radial functions in $\mathbb{R}^N$) which embeds compactly in $L^p(\Omega)$, if $p \in (2, 2^*)$. This idea was exploited by Berestycki and Lions in [13], Coffman and Marcus in [26] and Esteban and Lions in [30] when Ω is the complement of a ball. However, symmetry of Ω does not help if we don't have radial symmetry in $V(x)$. This is the case in our problem (P_V) where we do not assume any symmetry, neither in Ω nor in $V(x)$.

In the past five decades a different approach has been successfully applied in order to obtain solutions for this class of problems with no symmetry assumption. The so-called Nehari method, [39] and [40], which consists of finding solutions of (P_V) which are critical points of a functional associated with the equation in (P_V), restricted to the Nehari manifold. This method has been extensively used in the last years in order to find ground state solutions as well as sign changing solutions of nonlinear elliptic problems in $\mathbb{R}^N$ and exterior domains (see [28, 36, 41] and references therein). When finding a solution which is a minimum of the functional restricted to the Nehari manifold, the Lagrange multiplier is proved to be zero, yielding that the constrained critical point is in fact a free critical point of the functional, and that the manifold is natural. This allows to solve the problem for non-homogeneous nonlinearties because the multiplier does not have to be absorbed in the construction of a solution for the equation. Most importantly, this approach enables to avoid the use of a technical algebraic inequality $(a+b)^p \geq a^p + b^p + (p-1)(a^{p-1}b + ab^{p-1})$ largely applied in the case $f(u) = |u|^{p-2}u$ ([5, 6, 21]). We follow these ideas, closely related to the arguments found in [24] and [37], for general non-linearities f which satisfy the assumption that $f(s)/s$ is increasing. In this setting, not all functions $u \neq 0$ are projectable on the Nehari manifold , however the class of functions which are good for projections in this environment is enough to pursue the argument.

Our main contribution in the first chapter is extending the result of Bahri and Lions [6] for non-homogeneous f, with no symmetry assumption on V or Ω . Moreover, we allow the non-linear f to be a less smooth function just in C^1, improving the hypotheses in [24] and [37] where it was considered in C^3 for technical reasons (see Lemma 3.3 in [24]). The method we employ in order to solve (P_V) has many ideas in common with [24, 37]. Likewise, the work of [31] provided some useful tools and insight for estimates, even though their problem is for super-linear f in the whole $\mathbb{R}^N$ and uses the generalized Nehari manifold.

In the second chapter we look for a positive bound state solution for problem $(\mathcal{P})$ where a ground state cannot be obtained. Here we study general non-homogeneous non-linearities with double-power growth condition on f, which behaves as a subcritical power u^p at infinity and a supercritical power u^q near the origin, where $p < 2^* < q$, in any exterior domain. Using the ideas introduced in [24, 25, 37], we extend the results of V. Benci and A. Micheletti [12] by removing any assumption on the size of hole $\mathbb{R}^N \setminus \Omega$.

The method used in this work, of finding a solution of $(\mathcal{P})$ as a critical point of the functional associated with the equation, constrained to the Nehari manifold of the functional, is rather natural because of the geometry of this functional due to the super-quadratic growth of the nonlinear terms. However, the novelty in our approach is found mostly in some clever technical results such as the sharp estimates on the decay of the positive ground state solution of the problem in $\mathbb{R}^N$ and its implications in the interaction of two distinct and distant copies of these solitons, and on the other hand, a new compactness result which allows us to circumvent the difficulties created by an unbounded non-symmetric domain and embrace a very general problem.

Problems like $(\mathcal{P})$ with $f'(0) = 0$, the so-called zero mass case, appear in the study of Yang-Mills equations and have attracted the interest of researchers mostly in the case $\Omega = \mathbb{R}^N$(see [13, 33, 46]). Also, electrostatic problem of capacitors that is modeled by exterior boundary-value problems (see [27],Volume 1,Chapter II, for instance).

When $\Omega = \mathbb{R}^N$, we distinguish three different cases; $f'(0) < 0$, $f'(0) > 0$ and $f'(0) = 0$. In the first case there is a quite large literature, where the first results on this subject can be seen in [30] and [44]. Also H. Berestycki and P-L. Lions analyzed this problem in [13] and [14]. In the second case there is no finite energy solutions in general. Finally, when $f'(0) = 0$, the so-called zero mass case, has seen a growing interest in recent mathematical literature where the zero mass limit case of noncritical elliptic problems is of the form

$$-\Delta u + V(x)u = g(u),$$

for $g'(0) = 0$, and potentials satisfying $\liminf_{x\to\infty} V(x) = 0$. The existence of solutions for a null potential $V = 0$ was obtained by H. Berestycki and P. L. Lions in [13], where they used the double-power growth condition on g and shown that there is a solution u in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Further, many authors resumed the study of this kind of equation under the double-power growth condition, after it was successfully exploited in [10] and [11].

The main purpose of the second chapter is to solve problem $(\mathcal{P})$, in the null mass case, when Ω is an exterior domain that there is no restriction on its size. In order to do so, we make use of the ground state soluton in whole the $\mathbb{R}^N$, namely w, and show that there exists $u \in \mathcal{D}^{1,2}(\Omega)$ that is solution of $(\mathcal{P})$, but not a ground state solution. In fact, there is no solution of $(\mathcal{P})$ which minimizes the energy function on the Nehari manifold. We extend the results in V. Benci and A. Micheletti [12], they worked with Ω such that $\mathbb{R}^N \setminus \Omega \subset B_\epsilon$ when ϵ is sufficiently small. This assumption on the size of Ω is removed in our work.

An important feature when Ω is an unbounded exterior domain is that $\mathcal{D}^{1,2}(\Omega)$ is not necessarily contained in any Lebesgue space $L^q(\Omega)$ with $q \neq 2^*$ and thus, there are no standard Sobolev embeddings like those of $H_0^1(\Omega)$. For this reason we study the Orlicz space related to the right hand side term f and require that it satisfies a double power growth condition and obtain the regularity required in the energy functional. These Lebesgue spaces have several important and essential properties that play the same role for the Hilbert space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ that the usual Lebesgue spaces play for $H_0^1(\Omega)$. In an exterior domain, the main difficulty is the lack of compactness. Here we used a splitting lemma that is an important key to overcome the lack of compactness. This lemma is a variant of a well known result of M. Struwe (see [45]) related to the space $\mathcal{D}^{1,2}(\Omega)$ and also V. Benci and G. Cerami [9] with a clever description obtained of what happens when a Palais-Smale sequence does not converge to its weak limit. Note that since the space $\mathcal{D}^{1,2}(\Omega)$ is not necessarily contained in $H_0^1(\Omega)$, we cannot use Lions Lemma as in [35], so we need another version of the Lions Lemma and Splitting Lemma in Orlicz spaces which we show in Lemma 2.3.3 and Lemma 2.3.5.

Finally, according to the method that we apply in this chapter, we need to compare energy functionals associated with the equation in $(\mathcal{P})$ and there associated with the equation in $\mathbb{R}^N$. Suitable decay estimates for w, the positive radial solution of limit problem and ∇w will be crucial in order to compare all the terms in the energy functionals with the ground state level. Thanks to J. Vetois [47], we find very fine and exact decay estimates for w and ∇w, that play essential roles in this work.

Chapter 1

Asymptotically linear or superlinear limit problem

We establish the existence of a positive solution for semilinear elliptic equation in exterior domains

$$\begin{cases} -\Delta u + V(x)u = f(u), & \text{in } \ \Omega \subseteq \mathbb{R}^N \\ u \in H_0^1(\Omega) \end{cases} \tag{P_V}$$

where $N \geq 2$, $\mathbb{R}^N \backslash \Omega$ is regular bounded domain but there is no restriction on its size, nor any symmetry assumption. The nonlinear term f is a non homogeneous, asymptotically linear or superlinear function at infinity. Moreover, the potential V is a positive function, not necessarily symmetric. The existence of a solution is established in situations where this problem does not have a ground state.

1.1 Introduction

To our knowledge the results we present here are new and extend the previous works in the literature for a class of problems in exterior domains. We consider the elliptic problem

$$-\Delta u + V(x)u = f(u) \quad , \quad u \in H_0^1(\Omega) \tag{P_V}$$

where $N \geq 2$, $\mathbb{R}^N \setminus \Omega \subseteq B_K(0)$ the ball of radius K and center at the origin in $\mathbb{R}^N$, in fact $\mathbb{R}^N \setminus \Omega$ is bounded, $\partial\Omega$ is regular and $u \in H_0^1(\Omega)$ and V is a potential satisfying the conditions:

(V_1) $V \in C^0(\Omega)$, $\inf_{x\in\Omega} V(x) > 0$ and $\lim_{|x|\to+\infty} V(x) = V_\infty$;

(V_2) $V(x) \le V_\infty + Ce^{-\gamma|x|}$, where $C > 0$ and $\gamma > 2\sqrt{V_\infty}$.

The conditions that we consider on the nonlinearity f are the following:

(f_1) $f \in C^1\big([0,\infty)\big)$;

(f_2) There exist $C_2 > 0$ and $1 < p_1 \le p_2$ such that $p_1, p_2 < 2^* - 1$ and

$$|f^{(k)}(s)| \le C_2(|s|^{p_1-k} + |s|^{p_2-k})$$

for $k \in \{0,1\}$ and $s > 0$;

(f_3) $\lim_{s\to+\infty} \frac{f(s)}{s} \ge m > V_\infty$;

(f_4) If $F(s) := \int_0^s f(t)dt$ and $Q(s) := \frac{1}{2}f(s)s - F(s)$, then

$$\lim_{s\to\infty} Q(s) = +\infty;$$

(f_5) The function $s \mapsto f(s)/s$ is increasing in $s \in (0,+\infty)$;

(U) The positive radially symmetric solution of limit problem

$$-\Delta u + V_\infty u = f(u) \quad , \quad u \in H_0^1(\mathbb{R}^N) \qquad (P_\infty)$$

is unique.

Remark 1.1.1 *We have*

$$Q(s) := \frac{1}{2}f(s)s - F(s) > 0, \quad \forall s > 0 \qquad (1.1.1)$$

because from (f_5), $\big(\frac{f(t)}{t}\big)' = \frac{tf'(t) - f(t)}{t^2} > 0$ *and hence*

$$f(s)s - 2F(s) = \int_0^s (f(t)t)' - 2f(t)dt = \int_0^s tf'(t) - f(t)dt > 0.$$

Remark 1.1.2 *Note that* $f(s) > 0$ *for* $s > 0$, *since by* (f_2), $f(0) = f'(0) = 0$, *on the other hand* $f'(0) = \lim_{s\to 0} \frac{f(s) - f(0)}{s - 0} = \lim_{s\to 0} \frac{f(s)}{s}$ *and so by* (f_5), $\frac{f(s)}{s} > 0$, *now we can write* $f(s) = \frac{f(s)}{s}s > 0$ *for* $s > 0$.

It is straightforward to verify that the superlinear model nonlinearity $f(s) = s^p,\ s > 0$ with $p \in (1, 2^* - 1)$, and the asymptotically linear model nonlinearity $f(s) = \dfrac{s^3}{1 + bs^2}$ with $b \in (0, V_\infty^{-1})$ satisfy the hypotheses $(f_1) - (f_5)$.

Remark 1.1.3 *The assumption*
(U') $\psi(s) := \dfrac{-V_\infty s + f(s)}{sf'(s) - f(s)}$ is non decreasing in $s \in (\tau, +\infty)$ where τ is the unique positive number such that $\dfrac{f(\tau)}{\tau} = V_\infty$, guarantees that the positive solution to the problem (P_∞) is unique (see [38], Teorem 1 or [42], Teorem 1). It may be replaced by any other assumption which guarantees the uniqueness of positive ground state solution.

The main result of this chapter is the following

Theorem 1.1.4 *Under assumptions $(V_1) - (V_2)$, $(f_1) - (f_5)$ and (U), problem (P_V) has a positive solution u in $H_0^1(\Omega)$.*

This chapter is organized as follows. In section 2, we formulate the variational setting and present some preliminary results. Section 3 is dedicated to compactness condition. In section 4, applying a topological argument, which involves the barycenter map, we show that I_V has a positive critical value.

1.2 Variational setting and exponential decay estimate

Note that by Remark 1.1.2, $f(s) > 0$ for $s > 0$, and we shall consider the extended $f(s) := -f(-s)$ for $s < 0$, so without loss of generality we may suppose that f is odd and establish the existence of positive solution for it, which in particular will be a positive solution of the problem with the original f. We will use the following notation:

$$\langle u, v\rangle_\Omega = \int_\Omega (\nabla u \cdot \nabla v + V(x)uv)dx \quad , \quad \|u\|_\Omega^2 = \int_\Omega (|\nabla u|^2 + V(x)u^2)dx$$

Our assumptions on V imply that $\|.\|$ is a norm in $H_0^1(\Omega)$which is equivalent to the standard one.We write

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V_\infty uv)dx \quad , \quad \|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty u^2)dx$$

and our assumptions on V_∞ imply that $\|.\|$ is a norm in $H^1(\mathbb{R}^N)$ which is equivalent to the standard one. If $u \in H_0^1(\Omega)$ we may define $u = 0$ in $\mathbb{R}^N \setminus \Omega$, in fact $H_0^1(\Omega) \subset H^1(\mathbb{R}^N)$ (see [16], Proposition 9.18).

The solutions of problem (P_V) are critical points of the functional

$$I_V(u) = \frac{1}{2}\|u\|_\Omega^2 - \int_\Omega F(u)dx,$$

with $u \in H_0^1(\Omega)$. Set

$$J_V(u) = I_V'(u)u = \|u\|_\Omega^2 - \int_\Omega f(u)udx,$$

$$\mathcal{N}_V := \{u \in H_0^1(\Omega)\backslash\{0\} : J_V(u) = 0\},$$

and

$$c_V := \inf_{u \in \mathcal{N}_V} I_V(u).$$

Also we denote in the same way

$$I_\infty(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(u)dx,$$

$$J_\infty(u) = I_\infty'(u)u = \|u\|^2 - \int_{\mathbb{R}^N} f(u)udx,$$

$$\mathcal{N}_\infty := \{u \in H^1(\mathbb{R}^N)\backslash\{0\} : J_\infty(u) = 0\},$$

and

$$c_\infty := \inf_{u \in \mathcal{N}_\infty} I_\infty(u).$$

Let w be the unique positive radial solution of (P_∞), see [13, 15, 42]. It is well known, see [34] that there are constants C such that

$$C(1+|x|)^{-\frac{N-1}{2}} e^{-\sqrt{V_\infty}|x|} \le |D^i w(x)| \le C(1+|x|)^{-\frac{N-1}{2}} e^{-\sqrt{V_\infty}|x|}, \quad i = 0, 1. \qquad (1.2.1)$$

Hereafter C will denote a positive constant, not necessarily the same one. The following lemma gives informations about the Nehari manifold $\mathcal{N}_V$ which are, by now, standard (see [24] Lemma 2.1). We include them here for the sake of completeness.

Lemma 1.2.1 *(a)There exists $\varrho > 0$ such that $\|u\|_\Omega \ge \varrho$ for every $u \in \mathcal{N}_V$.*

(b)$\mathcal{N}_V$ is a closed C^1-submanifold of $H_0^1(\Omega)$ and natural constraint for I_V.

(c)If $u \in \mathcal{N}_V$, the function $t \mapsto I_V(tu)$ is strictly increasing in $(0,1]$ and strictly decreasing in $(1,\infty)$. In particular,

$$I_V(u) = \max_{t>0} I_V(tu) > 0$$

Proof. (a) Property (f_2) and the Sobolev embedding theorem imply that

$$J_V(u) \geq \|u\|_\Omega^2 - C\int_\Omega |u|^{p_2+1}dx \geq \|u\|_\Omega^2 - C\|u\|_\Omega^{p_2+1}, \quad u \in H_0^1(\Omega).$$

if $u \in \mathcal{N}_V$ then $J_V(u) = 0$ and soon $\frac{C\|u\|^{p_2+1}}{\|u\|^2} > 1$ and as $p_2 > 1$ we have $\|u\|^{p_2-1} > \frac{1}{C}$, This proves (a).

(b) Since $J_V(u)$ is continuous, it follows from (a) that $\mathcal{N}_V := \{u \in H_0^1(\Omega)\backslash\{0\} : J_V(u) = 0\}$ is closed in $H_0^1(\Omega)$. Moreover, property (f_5) yields

$$J_V'(u)u = 2\|u\|_\Omega^2 - \int_\Omega f'(u)u^2 - \int_\Omega f(u)u = \int_\Omega [f(u) - f'(u)u]u < 0.$$

for every $u \in \mathcal{N}_V$. This implies that 0 is a regular value of $J_V : H_0^1(\Omega)\backslash\{0\} \to \mathbb{R}$. So, as J_V is of class C^1, $\mathcal{N}_V$ is a C^1-submanifold of $H_0^1(\Omega)$. It also implies that u is not on the tangent space of $\mathcal{N}_V$ at u and, therefore, that $\mathcal{N}_V$ is a natural constraint for I_V.

(c) Let $u \in \mathcal{N}_V$. Set $\Omega^+ := \{x \in \Omega; u(x) > 0\}$, $\Omega^- := \{x \in \Omega; u(x) < 0\}$.

Then

$$\frac{d}{dt}I(tu) = \frac{1}{t}J(tu) = t\|u\|_\Omega^2 - \int_\Omega f(tu)udx = t\int_\Omega \left[f(u) - \frac{f(tu)}{t}\right]u$$

$$= t\int_\Omega \left[\frac{f(u)}{u} - \frac{f(tu)}{tu}\right]u^2 = t\left(\int_{\Omega^-}\left[\frac{f(u)}{u} - \frac{f(tu)}{tu}\right]u^2 + \int_{\Omega^+}\left[\frac{f(u)}{u} - \frac{f(tu)}{tu}\right]u^2\right).$$

By property (f_5) we have that $\frac{f(u)}{u}$ is strictly increasing for $u \in (0,\infty)$ and strictly decreasing for $u \in (-\infty,0)$. Therefore $\frac{d}{dt}I_V(tu) > 0$ if $t \in (0,1)$ and $\frac{d}{dt}I_V(tu) > 0$ if $t \in (1,\infty)$. This proves (c). □

Now we present a sequence of lemmas that will help to show that $\mathcal{N}_V \neq \emptyset$. As before, C will always denote a positive constant, not necessarily the same one.

Lemma 1.2.2 *For every $0 < \nu < p_1 - 1$ and $\rho > 0$ there exists $C_\rho \geq 0$ such that for all $0 \leq u, v \leq \rho$ we have*

$$F(u+v) - F(u) - F(v) - f(u)v - f(v)u \geq -C_\rho(uv)^{1+\frac{\nu}{2}}.$$

Proof. The inequality is obviously satisfied if $u = 0$ or $v = 0$. By (f_5), f is increasing, which yields

$$F(u+v) - F(u) = \int_u^{u+v} f(w)dw \geq f(u)v.$$

Moreover by (f_2) for every $0 < \nu < p_1 - 1$ we have

$$f(s) = o(|s|^{1+\nu}) \quad \text{as} \quad |s| \to 0,$$

and then $\tilde{C}_\rho := \sup_{0<u\leq\rho} \frac{f(u)}{u^{1+\nu}} < \infty$. Now for $0 < v \leq u \leq \rho$, we deduce

$$\begin{aligned} F(u+v) - F(u) - F(v) - f(u)v - f(v)u &\geq -F(v) - f(v)u \\ = \int_0^v -\frac{f(w)}{w^{1+\nu}} w^{1+\nu} dw - \frac{f(v)}{v^{1+\nu}} uv^{1+\nu} &\geq -\tilde{C}_\rho \frac{v^{2+\nu}}{2+\nu} - \tilde{C}_\rho uv^{1+\nu} \\ \geq -((\frac{1}{2}(\frac{v}{u})^{1+\frac{\nu}{2}} + \frac{v}{u})^{\frac{\nu}{2}})\tilde{C}_\rho (uv)^{1+\frac{\nu}{2}} &\geq -\frac{3}{2}\tilde{C}_\rho (uv)^{1+\frac{\nu}{2}}. \end{aligned}$$

By the symmetry in u and v, the same estimate holds for $0 < u \leq v$, and the proof is complete. □

Lemma 1.2.3 *If $\mu_2 > \mu_1 \geq 0$, there exists $C > 0$ such that, for all $x_1, x_2 \in \mathbb{R}^N$,*

$$\int_{\mathbb{R}^N} e^{-\mu_1|x-x_1|} e^{-\mu_2|x-x_2|} \, dx \leq Ce^{-\mu_1|x_1-x_2|}.$$

If $\mu_2 \geq \mu_1 > 0$, and $\mu_3 > \mu_1 \geq 0$, there exists $C > 0$ such that, for all $x_1, x_2, x_3 \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} e^{-\mu_1|x-x_1|} e^{-\mu_2|x-x_2|} e^{-\mu_3|x-x_3|} \, dx \leq Ce^{-\frac{\mu_1}{2}(|x_1-x_2|+|x_2-x_3|+|x_3-x_1|)}.$$

Proof. Since $\mu_1 |x_1 - x_2| + (\mu_2 - \mu_1) |x - x_2| \leq \mu_1 (|x - x_1| + |x - x_2|) + (\mu_2 - \mu_1) |x - x_2| = \mu_1 |x - x_1| + \mu_2 |x - x_2|$, we have

$$\int_{\mathbb{R}^N} e^{-\mu_1|x-x_1|} e^{-\mu_2|x-x_2|} \, dx \leq \int_{\mathbb{R}^N} e^{-\mu_1|x_1-x_2|} e^{-(\mu_2-\mu_1)|x-x_2|} \, dx = Ce^{-\mu_1|x_1-x_2|},$$

The second inequality is obtained in a similar way. □

The next four lemmas present some description of w the positive radial solution of (P_∞) and its translates. For $\lambda \in (0, 1]$ and $r \in (0, \infty)$ set $\phi : [0, 1] \times (0, \infty) \to \mathbb{R}$ as

$$\phi(\lambda, r) := \lambda^2 \left(\|w\|^2 - \int_{\mathbb{R}^N} \frac{f(r\lambda w)}{r\lambda w} w^2 \right) \quad ,$$

and for $\lambda = 0$ define $\phi(0, r) := 0$. By (f_1) and (f_2), $\phi(\lambda, r)$ is continuous.
By (f_5), $\phi(\lambda, r)$ is decreasing with respect to r. If $r\lambda > 1$ then

$$\|w\|^2 - \int_{\mathbb{R}^N} \frac{f(r\lambda w)}{r\lambda w} w^2 < \|w\|^2 - \int_{\mathbb{R}^N} \frac{f(w)}{w} w^2 = 0$$

and so $\phi(\lambda, r)$ is decreasing with respect to λ.

Lemma 1.2.4 *There are $S_0 < 0$ and $T_0 > 0$ such that*

$$\phi(\lambda, r) + \phi(1 - \lambda, r) \leq S_0 < 0 \qquad \forall \;\; r \geq T_0, \lambda \in [0, 1].$$

Proof. The function ϕ is continuous and

$$\phi(\lambda, r) \leq \|w\|^2 \lambda^2 =: A\lambda^2 \qquad \forall \;\; r \in (0, \infty), \lambda \in [0, 1].$$

As w is a solution of problem (P_∞) we have that

$$\phi(\lambda, r) = \lambda^2 \left(\int_{\mathbb{R}^N} \left[\frac{f(w)}{w} - \frac{f(r\lambda w)}{r\lambda w} \right] w^2 \right).$$

There are two cases to study:
case 1) if $\lim\limits_{u \to \infty} \dfrac{f(u)}{u} \to \infty$ (in fact $f(u)$ is superlinear) then by (f_5) and the Monotone Convergence Theorem

$$\lim_{r \to \infty} \phi(\lambda, r) = -\infty \quad , \quad \forall \lambda \in (0, 1]$$

and if $\lambda = 0$, $\lim\limits_{r \to \infty} \phi(1, r) = -\infty$ and this case is setted.
case 2) if $\lim_{u \to \infty} \frac{f(u)}{u} \to a$ (the nonlinearity $f(u)$ is asymptotically linear) then by (f_5) and Lebesgue's Monotone Convergence Theorem

$$\lim_{r \to \infty} \phi(\lambda, r) = \lambda^2 \left(\int_{\mathbb{R}^N} \left[\frac{f(w)}{w} - a \right] w^2 \right) =: -B\lambda^2 < 0 \quad , \quad \forall \lambda \in (0, 1].$$

Due to the symmetry, with respect to λ it suffices to consider $\lambda \in [0, 1/2]$. Fix $\lambda_0 \in (0, 1/2)$ such that $A\lambda_0^2 < \frac{B}{2}(1 - \lambda_0)^2$ and by the continuity of ϕ , there exists $r_0 \in (0, \infty)$ such that

$$\phi(1 - \lambda_0, r_0) = -\frac{B}{2}(1 - \lambda_0)^2$$

Then, for all $\lambda \in [0, \lambda_0]$ and all $r \geq \max\{r_0, 2\}$ we have $r(1 - \lambda) > 1$ and

$$\phi(\lambda, r) + \phi(1 - \lambda, r) \leq A\lambda_0^2 + \phi(1 - \lambda_0, r_0) = A\lambda_0^2 - \frac{B}{2}(1 - \lambda_0)^2 < 0.$$

On the other hand, if $\lambda \in [\lambda_0, 1/2]$, by fixing $r_1 > 1/\lambda_0$, we have that $r(1-\lambda) \geq r\lambda > 1$ for all $r > r_1$. Hence,

$$\phi(\lambda, r) + \phi(1-\lambda, r) \leq \phi(\lambda_0, r) + \phi(1-\lambda_0, r) < 0 \quad , \quad \forall \lambda \in [\lambda_0, 1/2] \text{ and } r > r_0.$$

Set $T_0 := \max\{r_0, r_1\}$ and

$$S_0 := \max_{\lambda \in [0,1]} \phi(\lambda, T_0) + \phi(1-\lambda, T_0) < 0 \quad ,$$

we conclude that

$$\phi(\lambda, r) + \phi(1-\lambda, r) \leq \phi(\lambda, T_0) + \phi(1-\lambda, T_0) \leq S_0$$

for all $r > T_0$ and $\lambda \in [0,1]$, as claimed. □

Now, let $y_0 \in \mathbb{R}^N$ with $|y_0| = 1$ and $B_2(y_0) := \{x \in \mathbb{R}^N : |x - y_0| \leq 2\}$, we write for each $y \in \partial B_2(y_0)$

$$w_0^R := w(\cdot - Ry_0) \quad , \quad w_y^R := w(\cdot - Ry), \quad R > 0.$$

Lemma 1.2.5 *Let $q > 0$ and $R > 0$ be large enough then we have*

$$a) \int_{B_{2K}(0)} |w_0^R|^q \leq CR^{-q\frac{N-1}{2}} e^{-q\sqrt{V_\infty}R} \quad \textit{and} \quad \int_{B_{2K}(0)} |w_y^R|^q \leq CR^{-q\frac{N-1}{2}} e^{-q\sqrt{V_\infty}R},$$

$$b) \int_{B_{2K}(0)} |\nabla w_0^R|^q \leq CR^{-q\frac{N-1}{2}} e^{-q\sqrt{V_\infty}R} \quad \textit{and} \quad \int_{B_{2K}(0)} |\nabla w_y^R|^q \leq CR^{-q\frac{N-1}{2}} e^{-q\sqrt{V_\infty}R}.$$

Proof. In order to prove the first estimate let $2K < \frac{1}{2}R$, so that

$$\frac{1}{2}R = R - \frac{1}{2}R < |Ry_0| - |x| < |x - Ry_0| < 1 + |x - Ry_0|, \quad \forall x \in B_{2K}(0). \qquad (1.2.2)$$

Now by (1.2.1) and (1.2.2) we have

$$\int_{B_{2K}(0)} |w_0^R|^q = \int_{B_{2K}(0)} |w(x - Ry_0)|^q dx \leq C \int_{B_{2K}(0)} [(1 + |x - Ry_0|)^{-\frac{N-1}{2}} e^{-\sqrt{V_\infty}|x - Ry_0|}]^q dx$$

$$\leq CR^{-q\frac{N-1}{2}} e^{-q\sqrt{V_\infty}R}.$$

The proofs of the other estimates are similar. □

Lemma 1.2.6 *Let $p \geq q \geq 1$ then*

$$\int_{\mathbb{R}^N}(w_0^R)^q(w_y^R)^p \leq CR^{-q\frac{N-1}{2}}e^{-2q\sqrt{V_\infty}R} \quad \textit{and} \quad \int_{\mathbb{R}^N}(w_y^R)^q(w_0^R)^p \leq CR^{-q\frac{N-1}{2}}e^{-2q\sqrt{V_\infty}R}.$$

Proof. Note that

$$\begin{cases} \text{if } \ |x| > R \ \text{ then } \ R < |x|+1 \ \text{ and} \\ \text{if } \ |x| < R \ \text{ then } \ 2R - R < |R(y-y_0)| - |x| < 1 + |x - R(y-y_0)|. \end{cases} \tag{1.2.3}$$

Now by (1.2.1), (1.2.3) and Lemma 1.2.3 we have

$$\begin{aligned}
\int_{\mathbb{R}^N}(w_0^R)^q(w_y^R)^p &= \int_{\mathbb{R}^N}(w(x-Ry_0))^q w(x-Ry)^p dx = \int_{\mathbb{R}^N} w(x)^q w(x-R(y-y_0)^p dx \\
&\leq \int_{\mathbb{R}^N}(1+|x|)^{-q\frac{N-1}{2}}e^{-q\sqrt{V_\infty}|x|}(1+|x-R(y-y_0)|)^{-p\frac{N-1}{2}}e^{-p\sqrt{V_\infty}|x-R(y-y_0)|} \\
&\leq \int_{B_R(0)} e^{-q\sqrt{V_\infty}|x|}(1+|x-R(y-y_0)|)^{-p\frac{N-1}{2}}e^{-p\sqrt{V_\infty}|x-R(y-y_0)|} \\
&\quad + \int_{B_R(0)^c} e^{-q\sqrt{V_\infty}|x|}(1+|x|)^{-q\frac{N-1}{2}}e^{-p\sqrt{V_\infty}|x-R(y-y_0)|} \\
&\leq CR^{-q\frac{N-1}{2}}e^{-2q\sqrt{V_\infty}R}.
\end{aligned}$$

Similarly we can prove the second estimate. □

Lemma 1.2.7 *For $R > 0$ sufficiently large we have*

$$\int_\Omega (V(x)-V_\infty)(w_0^R\psi)^2 \leq CR^{-(N-1)}e^{-2\sqrt{V_\infty}R}, \tag{1.2.4}$$

$$\int_\Omega (V(x)-V_\infty)(w_y^R\psi)^2 \leq CR^{-(N-1)}e^{-2\sqrt{V_\infty}R}, \tag{1.2.5}$$

and

$$\int_\Omega (V(x)-V_\infty)w_y^R\psi w_0^R\psi \leq CR^{-(N-1)}e^{-2\sqrt{V_\infty}R}. \tag{1.2.6}$$

Proof. In order to prove the first inequality, it follows from (V_2) and estimative (1.2.1) that

$$\int_\Omega (V(x)-V_\infty)(w_0^R\psi)^2 \leq \int_{\mathbb{R}^N}(V(x)-V_\infty)(w_0^R)^2$$

$$\leq \int_{\mathbb{R}^N} e^{-\gamma|x|}(1+|x-Ry_0|)^{-(N-1)}e^{-2\sqrt{V_\infty}|x-Ry_0|}. \tag{1.2.7}$$

Now let $\varrho = \frac{1}{2} - \frac{\sqrt{V_\infty}}{\gamma} > 0$, so we can write (1.2.7) as

$$\int_{\mathbb{R}^N \setminus B_{\varrho R}(Ry_0)} e^{-\gamma|x|}(1+|x-Ry_0|)^{-(N-1)}e^{-2\sqrt{V_\infty}|x-Ry_0|} \tag{1.2.8}$$

$$+\int_{B_{\varrho R}(Ry_0)} e^{-\gamma|x|}(1+|x-Ry_0|)^{-(N-1)}e^{-2\sqrt{V_\infty}|x-Ry_0|}. \tag{1.2.9}$$

As $|x-Ry_0| > \varrho R$ in $\mathbb{R}^N \setminus B_{\varrho R}(Ry_0)$, applying Lemma 1.2.3 with $\mu_1 = \gamma > \mu_2 = 2\sqrt{V_\infty}$, we get

$$(1.2.8) \leq CR^{-(N-1)}e^{-2\sqrt{V_\infty}R}.$$

On the other hand, $|x+Ry_0| \geq R|y_0| - \varrho R = (1-\varrho)R$ for x in $B_{\varrho R}(0)$ and by making a change of variables, we have

$$(1.2.9) \leq \int_{B_{\varrho R}(0)} e^{-\gamma|x+Ry_0|}e^{-2\sqrt{V_\infty}|x|}$$

$$\leq e^{-\gamma(1-\varrho)R}\int_{B_{\varrho R}(0)} e^{-2\sqrt{V_\infty}|x|} \leq Ce^{-\gamma(1-\varrho)R}\int_0^{\varrho R} r^{N-1}dr$$

$$\leq Ce^{-\gamma(1-\varrho)R}R^N \leq CR^{-(N-1)}e^{-2\sqrt{V_\infty}R},$$

since by definition of ϱ, $\gamma(1-\varrho)R > \gamma(\frac{1}{2}+\frac{\sqrt{V_\infty}}{\gamma})R > (\frac{\gamma}{2}+\sqrt{V_\infty})R > 2\sqrt{V_\infty}R$. The proof of first inequality is complete. Similarly we can prove the second estimate.

Finally, in order to prove (1.2.6) we may repeat the above argument for $B_{\varrho R}(Ry) \cup B_{\varrho R}(Ry_0)$ rather than $B_{\varrho R}(Ry_0)$. Note that $|x-Ry_0|$,$|x-Ry| > \varrho R$ in $\mathbb{R}^N \setminus \{B_{\varrho R}(Ry) \cup B_{\varrho R}(Ry_0)\}$, performing a change of variables and applying Lemma 1.2.3

$$\int_\Omega (V(x)-V_\infty)w_0^R\psi w_y^R\psi \leq \int_{\mathbb{R}^N}(V(x)-V_\infty)w_0^R w_y^R$$

$$\leq \int_{\mathbb{R}^N \setminus \{B_{\varrho R}(Ry)\cup B_{\varrho R}(Ry_0)\}} e^{-\gamma|x|}(1+|x-Ry_0|)^{-\frac{N-1}{2}}e^{-\sqrt{V_\infty}|x-Ry_0|}(1+|x-Ry|)^{-\frac{N-1}{2}}e^{-\sqrt{V_\infty}|x-Ry|}$$

$$+\int_{B_{\varrho R}(Ry)} e^{-\gamma|x|}(1+|x-Ry_0|)^{-\frac{N-1}{2}}e^{-\sqrt{V_\infty}|x-Ry_0|}(1+|x-Ry|)^{-\frac{N-1}{2}}e^{-\sqrt{V_\infty}|x-Ry|}$$

$$+\int_{B_{\varrho R}(Ry_0)} e^{-\gamma|x|}(1+|x-Ry_0|)^{-\frac{N-1}{2}}e^{-\sqrt{V_\infty}|x-Ry_0|}(1+|x-Ry|)^{-\frac{N-1}{2}}e^{-\sqrt{V_\infty}|x-Ry|}$$

$$\leq CR^{-N-1}e^{-2\sqrt{V_\infty}R}.$$

□

In what follows we exploit the idea of Bahri and Li in [5] of working with a convex combination of two translated copies of w, the ground state solution of (P_∞) (see also [24], [31] and [37]).

Define

$$Z^R_{\lambda,y} := \lambda w^R_0 + (1-\lambda) w^R_y \ \ , \ \lambda \in [0,1], \ R > 0,$$

and

$$U^R_{\lambda,y} := Z^R_{\lambda,y}\psi \tag{1.2.10}$$

where $\psi \in C^\infty(\mathbb{R}^N)$ is continuous radially symmetric and increasing cut-off function

$$\psi(x) = \begin{cases} 0 & |x| \leq K \\ 0 < \psi < 1 & K < |x| < 2K \\ 1 & |x| \geq 2K. \end{cases}$$

Note that here K is the radius of the sphere $B_K(0)$ which contains $\mathbb{R}^N \setminus \Omega$. We can consider $U^R_{\lambda,y} \in H^1(\mathbb{R}^N)$ by extending $U^R_{\lambda,y} \equiv 0$ outside Ω.

Lemma 1.2.8 *$U^R_{\lambda,y} - Z^R_{\lambda,y} \to 0$ in $H^1(\mathbb{R}^N)$, as $R \to \infty$.*

Proof. First of all for R sufficiently large we claim that

$$|w^R_0 - \psi w^R_0|^2_{L^2(B_{2K}(0))} \leq C R^{-(N-1)} e^{-2\sqrt{V_\infty}R}, \tag{1.2.11}$$

$$|\nabla w^R_0 - \nabla \psi w^R_0|^2_{L^2(B_{2K}(0))} \leq C R^{-(N-1)} e^{-2\sqrt{V_\infty}R}, \tag{1.2.12}$$

$$|w^R_y - \psi w^R_y|^2_{L^2(B_{2K}(0))} \leq C R^{-(N-1)} e^{-2\sqrt{V_\infty}R}, \tag{1.2.13}$$

$$|w^R_y - \psi w^R_y|^2_{L^2(B_{2K}(0))} \leq C R^{-(N-1)} e^{-2\sqrt{V_\infty}R}, \tag{1.2.14}$$

therefore

$$\|U^R_{\lambda,y} - Z^R_{\lambda,y}\| \leq \lambda \|w^R_0 - \psi w^R_0\| + (1-\lambda)\|w^R_y - \psi w^R_y\|$$

$$= \lambda \|w^R_0 - \psi w^R_0\|_{H^1(B_{2K}(0))} + (1-\lambda)\|w^R_y - \psi w^R_y\|_{H^1(B_{2K}(0))}$$

$$= \lambda [|w^R_0 - \psi w^R_0|^2_{L^2(B_{2K}(0))} + |\nabla w^R_0 - \nabla \psi w^R_0|^2_{L^2(B_{2K}(0))}]^{\frac{1}{2}}$$

$$+(1-\lambda)[|w_y^R - \psi w_y^R|^2_{L^2(B_{2K}(0))} + |\nabla w_y^R - \nabla \psi w_y^R|^2_{L^2(B_{2K}(0))}]^{\frac{1}{2}}$$

and by the claim we have

$$\|U^R_{\lambda,y} - Z^R_{\lambda,y}\|^2 \le CR^{-(N-1)}e^{-2\sqrt{V_\infty}R}$$

and this shows that $U^R_{\lambda,y} - Z^R_{\lambda,y} \to 0$ where $R \to \infty$ as the lemma states.
Now in order to complete the proof we have to show the claim. To obtain the first estimate (1.2.11) we use Lemma 1.2.5

$$|w_0^R - \psi w_0^R|^2_{L^2(B_{2K}(0))} = \int_{B_{2K}(0)} |1-\psi||w_0^R|^2 dx$$

$$\le C \int_{B_{2K}(0)} |w_0^R|^2 dx \le CR^{-(N-1)}e^{-2\sqrt{V_\infty}R}.$$

To prove the second estimate (1.2.12) we have $\psi \in C^\infty$, then there exists positive constants C_1 and C_2 such that

$$|\nabla \psi w_0^R| = |(\nabla\psi)w_0^R + (\nabla w_0^R)\psi| \le C_1|w_0^R| + C_2|\nabla w_0^R| \quad \text{in} \quad B_{2K}(0) \tag{1.2.15}$$

and so by Lemma 1.2.5

$$|\nabla w_0^R - \nabla\psi w_0^R|^2_{L^2(B_{2K}(0))} \le \int_{B_{2K}(0)} [(C_1+1)|w_0^R| + C_2|\nabla w_0^R|]^2 dx$$

$$\le CR^{-(N-1)}e^{-2\sqrt{V_\infty}R}$$

as claimed. The proof of (1.2.13) and (1.2.14) are similar. □

Lemma 1.2.9 *For any $r > 0$, $J_\infty(rU^R_{\lambda,y}) - J_\infty(rZ^R_{\lambda,y}) \to 0$ as $R \to \infty$.*

Proof. By the definition of J_∞ we have

$$\left|J_\infty(rU^R_{\lambda,y}) - J_\infty(rZ^R_{\lambda,y})\right|$$

$$= \left| \left\|rU^R_{\lambda,y}\right\|^2 - \int_{\mathbb{R}^N} f(rU^R_{\lambda,y})rU^R_{\lambda,y} - \left\|rZ^R_{\lambda,y}\right\|^2 + \int_{\mathbb{R}^N} f(rZ^R_{\lambda,y})rZ^R_{\lambda,y} \right|$$

$$\le \left\|rU^R_{\lambda,y} - rZ^R_{\lambda,y}\right\|^2 + \left| \int_{\mathbb{R}^N} f(rZ^R_{\lambda,y})rZ^R_{\lambda,y} - f(rU^R_{\lambda,y})rU^R_{\lambda,y} \right|. \tag{1.2.16}$$

By Lemma 1.2.8 the first part of (1.2.16) is equal to $o_R(1)$, then it is enough to show that

$$\int_{\mathbb{R}^N} f(rZ^R_{\lambda,y})rZ^R_{\lambda,y} - f(rU^R_{\lambda,y})rU^R_{\lambda,y} = \int_{B_{2K}(0)} f(rZ^R_{\lambda,y})rZ^R_{\lambda,y} - f(rU^R_{\lambda,y})rU^R_{\lambda,y} = o_R(1).$$

By (f_2), Lemma 1.2.5 and the inequality $(a+b)^p \leq 2^p(a^p+b^p)$ we have

$$\int_{B_{2K}(0)} f(rZ^R_{\lambda,y})rZ^R_{\lambda,y} - f(rU^R_{\lambda,y})rU^R_{\lambda,y}$$

$$\leq \int_{B_{2K}(0)} (|rZ^R_{\lambda,y}|^{p_1} + |rZ^R_{\lambda,y}|^{p_2})rZ^R_{\lambda,y} - (|rU^R_{\lambda,y}|^{p_1} + |rU^R_{\lambda,y}|^{p_2})rU^R_{\lambda,y}$$

$$\leq \int_{B_{2K}(0)} |1-\psi|(|rZ^R_{\lambda,y}|^{p_1+1} + |rZ^R_{\lambda,y}|^{p_2+1}) \leq C\int_{B_{2K}(0)} (|Z^R_{\lambda,y}|^{p_1+1} + |Z^R_{\lambda,y}|^{p_2+1})$$

$$\leq C\int_{B_{2K}(0)} (|\lambda w^R_0 + (1-\lambda)w^R_y|^{p_1+1} + |\lambda w^R_0 + (1-\lambda)w^R_y|^{p_2+1})$$

$$\leq C\int_{B_{2K}(0)} |w^R_0|^{p_1+1} + |w^R_0|^{p_2+1} + |w^R_y|^{p_1+1} + |w^R_y|^{p_2+1}$$

$$\leq CR^{-(N-1)}e^{-2\sqrt{V_\infty}R} = o_R(1).$$

□

Our assumptions do not guarantee that every $u \in H^1_0(\Omega)$ admits a projection onto $\mathcal{N}_V$. However, the following lemma says that $U^R_{\lambda,y}$ does admit a projection onto $\mathcal{N}_V$ if R is sufficiently large.

Lemma 1.2.10 *There exist $R_0 > 0$, $T_0 > 2$ such that for each $R \geq R_0$, $y \in \partial B_2(y_0)$ and $\lambda \in [0,1]$, there exists a unique $T^R_{\lambda,y}$ such that*

$$T^R_{\lambda,y}U^R_{\lambda,y} \in \mathcal{N}_V,$$

$T^R_{\lambda,y} \in [0,T_0]$ and $T^R_{\lambda,y}$ is a continuous function of the variables λ, y and R. In particular for $\lambda = 1/2$ we have $T^R_{\frac{1}{2},y} \to 2$ as $R \to \infty$ uniformly in $y \in \partial B_2(y_0)$.

Proof. First note that, for each $u \in H^1_0(\Omega)$, $u > 0$, property (f_5) implies that

$$\frac{J_V(ru)}{r^2} = \|u\|^2 - \int_\Omega \frac{f(ru)}{ru}u^2$$

is strictly decreasing in $r \in (0,\infty)$. Therefore, if there exists $r_u \in (0,\infty)$ such that $J_V(r_u u) = 0$, this number will be unique. Observe also that $J_V(ru) > 0$ for r small enough. Next, we will show that, for R large enough and some $T_0 > 0$,

$$J_V(rU^R_{\lambda,y}) < 0 \qquad \forall r \geq T_0. \tag{1.2.17}$$

This implies that there exists $T^R_{\lambda,y} \in [0, T_0)$ such that $J_V(T^R_{\lambda,y}U^R_{\lambda,y}) = 0$, i.e. $T^R_{\lambda,y}U^R_{\lambda,y} \in \mathcal{N}_V$. Let us prove (1.2.17). For $u, v \in H^1(\mathbb{R}^N), u, v > 0$, and $r \in (0, \infty)$, by using (f_5) we have

$$\begin{aligned} J_\infty(ru + rv) &= r^2(\|u\|^2 + \|v\|^2 + 2\langle u, v\rangle) - \int_{\mathbb{R}^N} \frac{f(ru + rv)}{ru + rv}(ru + rv)^2 \\ &\leq r^2(\|u\|^2 - \int_{\mathbb{R}^N} \frac{f(ru)}{ru}u^2 + \|v\|^2 - \int_{\mathbb{R}^N} \frac{f(rv)}{rv}v^2 + 2\langle u, v\rangle). \end{aligned}$$

Setting $u := \lambda w^R_0$ and $v := (1-\lambda)w^R_y$, performing a change of variable and Lemma 1.2.6 we conclude that

$$\begin{aligned} \frac{J_\infty(r\lambda w^R_0 + r(1-\lambda)w^R_y)}{r^2} &= \phi(\lambda, r) + \phi(1-\lambda, r) + 2\lambda(1-\lambda)\langle w^R_0, w^R_y\rangle \\ &\leq S_0 + \frac{1}{2}\langle w^R_0, w^R_y\rangle = S_0 + o_R(1) \qquad \forall r \geq T_0, \quad \lambda \in [0, 1], \end{aligned}$$

where $o_R(1) \to 0$ as $R \to \infty$, uniformly in $y \in \partial B_2(y_0)$ and $\lambda \in [0, 1]$ also $S_0 < 0$ as in Lemma (1.2.4). Now since $H^1_0(\Omega) \subset H^1(\mathbb{R}^N)$ we can write

$$\frac{J_V(rU^R_{\lambda,y})}{r^2} = \frac{J_\infty(rU^R_{\lambda,y})}{r^2} + \int_\Omega (V(x) - V_\infty)(U^R_{\lambda,y})^2,$$

by Lemma 1.2.7 and Lemma 1.2.9

$$\frac{J_V(rU^R_{\lambda,y})}{r^2} \leq S_0 + o_R(1) \qquad \forall r \geq T_0, \quad \lambda \in [0, 1].$$

Hence, there exists $R_0 > 0$ such that

$$\frac{J_V(rU^R_{\lambda,y})}{r^2} \leq \frac{S_0}{2} < 0 \qquad \forall r \geq T_0, \quad \lambda \in [0, 1], \quad R > R_0.$$

This proves (1.2.17), and so we have showed that $\mathcal{N}_V \neq \emptyset$.
Now let $\varphi(u, v) = f(u + v) - f(u) - f(v)$, by mean value theorem

$$-Cv^{p_1} \leq -f(v) \leq \varphi(u, v) \leq f(u + v) - f(u) \leq f'(u + tv)v \leq Cv$$

and by Lemma 1.2.6

$$-o_R(1) = -C\int_{\mathbb{R}^N} (w^R_y)^{p_1} w^R_0 \leq \int_{\mathbb{R}^N} \varphi(w^R_0, w^R_y)w^R_0 \leq C\int_{\mathbb{R}^N} (w^R_y)w^R_0 = o_R(1)$$

or

$$\int_{\mathbb{R}^N} |\varphi(w_0^R, w_y^R)w_0^R| = o_R(1),$$

also by symmetry of u and v in $\varphi(u, v)$ we get similarly

$$\int_{\mathbb{R}^N} |\varphi(w_0^R, w_y^R)w_y^R| = o_R(1),$$

and from the two above estimates we have

$$\int_{\mathbb{R}^N} |\varphi(w_0^R, w_y^R)(w_0^R + w_y^R)| = o_R(1). \tag{1.2.18}$$

Now by Lemma 1.2.6 and (1.2.18) we can write

$$J_\infty(w_0^R + w_y^R) = \|w_0^R + w_y^R\|^2 - \int_{\mathbb{R}^N} f(w_0^R + w_y^R)(w_0^R + w_y^R)$$

$$= \|w_0^R\|^2 + \|w_y^R\|^2 + 2\langle w_0^R, w_y^R\rangle - \int_{\mathbb{R}^N} f(w_0^R)(w_0^R) - \int_{\mathbb{R}^N} f(w_y^R)(w_y^R) -$$

$$\int_{\mathbb{R}^N} f(w_0^R)(w_y^R) - \int_{\mathbb{R}^N} f(w_y^R)(w_0^R) + \int_{\mathbb{R}^N} \varphi(w_0^R, w_y^R)(w_0^R + w_y^R)$$

$$= J_\infty(w_0^R) + J_\infty(w_y^R) + o_R(1) = o_R(1)$$

since w is a solution of (P_∞). So, by Lemma 1.2.9 we have

$$J_\infty((w_0^R + w_y^R)\psi) = J_\infty(w_0^R + w_y^R) + o_R(1) = o_R(1) \quad \text{as} \qquad R \to \infty. \tag{1.2.19}$$

Therefore, by (1.2.19), Lemma 1.2.7 and $H_0^1(\Omega) \subset H^1(\mathbb{R}^N)$

$$J_V(2U^R_{\frac{1}{2},y}) = J_V((w_0^R + w_y^R)\psi))$$

$$= J_\infty((w_0^R + w_y^R)\psi) + \int_\Omega (V(x) - V_\infty)(w_0^R + w_y^R)^2\psi^2 = o_R(1)$$

since by Lemma 1.2.7

$$\int_\Omega (V(x) - V_\infty)(w_0^R + w_y^R)^2\psi^2 \le \int_{\mathbb{R}^N} (V(x) - V_\infty)(w_0^R + w_y^R)^2 = o_R(1)$$

and this proves the lemma. □

1.3 Compactness results

Lemma 1.3.1 *Any sequence* (u_k) *satisfying*

$$(u_k) \in \mathcal{N}_V \quad \text{and} \quad I_V(u_k) \to d$$

is bounded in $H_0^1(\Omega)$ *.*

Proof. First of all note that $d \geq 0$, since

$$I_V(u_k) = I_V(u_k) - I_V'(u_k)u_k = \int_\Omega \frac{1}{2} f(u_k)u_k - F(u_k) \geq 0.$$

Now fix $D > d$. Assume, by contradiction, that $\|u_k\| \to \infty$ and set $v_k := t_k u_k$ with $t_k = \frac{2\sqrt{D}}{\|u_k\|}$. By Lemma 1.2.1 (c), for k large enough we have that

$$D \geq I_V(u_k) \geq I_V(v_k) = \frac{1}{2} t^2 \|u_k\|^2 - \int_\Omega F(v_k) = 2D - \int_\Omega F(v_k).$$

By using hypothesis (f_2), we get that

$$D \leq \int_\Omega F(v_k) \leq c(|v_k|_{p_1+1}^{p_1+1} + |v_k|_{p_2+1}^{p_2+1}).$$

As $D > d \geq 0$ and (v_k) is bounded in $H_0^1(\Omega) \subset H^1(\mathbb{R}^N)$, this lower bound, together with Lions lemma [[48],Lemma 1.21], implies that there exist $\delta > 0$ and a sequence (y_k) in $\mathbb{R}^N$ such that

$$\int_{B_1(y_k)} v_k^2 = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} v_k^2 \geq \delta.$$

Set $\tilde{u}_k(x) := u_k(x + y_k)$ and $\tilde{v}_k(x) := v_k(x + y_k)$. After passing to a subsequence $\tilde{v}_k \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$, $\tilde{v}_k \to v$ in $L^2_{loc}(\mathbb{R}^N)$ and $\tilde{v}_k(x) \to v(x)$ a.e. in $\mathbb{R}^N$. Therefore,

$$\int_{B_1(0)} v^2 = \lim_{k \to \infty} \int_{B_1(0)} \tilde{v}_k^2 = \lim_{k \to \infty} \int_{B_1(y_k)} v_k^2 \geq \delta.$$

Hence, $v \neq 0$ and there exists a subset Λ of positive measure in $B_1(0)$ such that $v(x) \neq 0$ for every $x \in \Lambda$. It follows that $|\tilde{u}_k(x)| \to \infty$ for every $x \in \Lambda$. Property (f_5) implies that $\frac{1}{2} f(u)u - F(u) \geq 0$ if $u \in \mathbb{R} \setminus \{0\}$. So, from property (f_3) and Fatou's lemma, we conclude that

$$D > \lim_{k \to \infty} I_V(u_k) = \lim_{k \to \infty} \int_\Omega [\frac{1}{2} f(u_k)u_k - F(u_k)]$$

$$= \lim_{k \to \infty} \int_{\mathbb{R}^N} [\frac{1}{2} f(u_k)u_k - F(u_k)] = \lim_{k \to \infty} \int_{\mathbb{R}^N} [\frac{1}{2} f(\tilde{u}_k)\tilde{u}_k - F(\tilde{u}_k)]$$

$$\geq \liminf_{k\to\infty}\int_\Lambda [\frac{1}{2}f(\tilde{u}_k)\tilde{u}_k - F(\tilde{u}_k)] \geq \int_\Lambda \liminf_{k\to\infty}[\frac{1}{2}f(\tilde{u}_k)\tilde{u}_k - F(\tilde{u}_k)] = \infty$$

This is a contradiction. □

Lemma 1.3.2 $c_V, c_\infty > 0$.

Proof. Let $u_k \in \mathcal{N}_V$ be such that $I_V(u_k) \to c_V$. By Lemma 1.3.1, after passing to a subsequence, we have that (u_k) is bounded in $H_0^1(\Omega)$. From Lemma 1.2.1(a) and by property (f_2) we obtain

$$0 < \varrho^2 \leq \|u_k\|_\Omega^2 = \int_\Omega f(u_k)u_k \leq c(|u_k|_{p_1+1}^{p_1+1} + |u_k|_{p_2+1}^{p_2+1}).$$

This inequality, together with Lions lemma (considering the extention of u_k to $H^1(\mathbb{R}^N)$), implies that there exist $\delta > 0$ and a sequence (y_k) in $\mathbb{R}^N$ such that

$$\int_{B_1(y_k)} u_k^2 = \sup_{y\in\mathbb{R}^N}\int_{B_1(y)} u_k^2 \geq \delta.$$

Set $\tilde{u}_k(x) := u_k(x+y_k)$ After passing to a subsequence $\tilde{u}_k \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $\tilde{u}_k \to u$ in $L^2_{loc}(\mathbb{R}^N)$ and $\tilde{u}_k(x) \to u(x)$ a.e. in $\mathbb{R}^N$. Therefore,

$$\int_{B_1(0)} u^2 = \lim_{k\to\infty}\int_{B_1(0)} \tilde{u}_k^2 = \lim_{k\to\infty}\int_{B_1(y_k)} u_k^2 \geq \delta.$$

Hence, $u \neq 0$ and there exists a subset Λ of positive measure in $B_1(0)$ such that $u(x) \neq 0$ for every $x \in \Lambda$. Property (f_5) implies that $\frac{1}{2}f(u)u - F(u) > 0$ if $u \in \mathbb{R}\setminus\{0\}$. So, from the Fatou's lemma, we conclude that

$$c_V = \lim_{k\to\infty} I_V(u_k) = \lim_{k\to\infty}\int_\Omega [\frac{1}{2}f(u_k)u_k - F(u_k)]$$

$$= \lim_{k\to\infty}\int_{\mathbb{R}^N}[\frac{1}{2}f(u_k)u_k - F(u_k)] = \lim_{k\to\infty}\int_{\mathbb{R}^N}[\frac{1}{2}f(\tilde{u}_k)\tilde{u}_k - F(\tilde{u}_k)]$$

$$\geq \liminf_{k\to\infty}\int_\Lambda[\frac{1}{2}f(\tilde{u}_k)\tilde{u}_k - F(\tilde{u}_k)] \geq \int_\Lambda \liminf_{k\to\infty}[\frac{1}{2}f(\tilde{u}_k)\tilde{u}_k - F(\tilde{u}_k)] = \int_\Lambda[\frac{1}{2}f(u)u - F(u)] > 0$$

as claimed. By repeating this argument we obtain $c_\infty > 0$. □

Lemma 1.3.3 *If u is a solution of (P_V) with $I_V(u) \in [c_V, 2c_V)$, then u does not change sign.*

Proof. If u is a solution of P_V then

$$0 = I'_V(u)u^{\pm} = J_V(u^{\pm}),$$

where $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$ and so $u^{\pm} \in \mathcal{N}_V$. Now if $u^+ \neq 0$ and $u^- \neq 0$ then

$$I_V(u) = I_V(u^+) + I_V(u^-) \geq 2c_V.$$

This proves the lemma. □

Note that $\nabla_{\mathcal{N}_V} I_V(u)$ is the orthogonal projection of $\nabla I_V(u)$ onto the tangent space of $\mathcal{N}_V$ at u that define by $T_u(\mathcal{N}_V) := \{v \in H_0^1(\Omega);\quad J'_V(u)v = 0\}$. Recall that a sequence (u_k) in $H_0^1(\Omega)$ is said to be a $(PS)_d$-sequence for I_V on $\mathcal{N}_V$ if $I_V(u_k) \to d$ and $\|\nabla_{\mathcal{N}_V} I_V(u_k)\| \to 0$. The functional I_V satisfies the Palais-Smale condition on $\mathcal{N}_V$ at the level d if every $(PS)_d$-sequence for I_V on $\mathcal{N}_V$ contains a convergent subsequence.

Remark 1.3.4 *We can write $\nabla I_V(u)$, the gradient of I_V at u, as*

$$\nabla I_V(u) = \nabla_{\mathcal{N}_V} I_V(u) + t\nabla J_V(u).$$

Indeed, by the definition $\langle \nabla_{\mathcal{N}_V} I_V(u), v\rangle = \langle \nabla I_V(u), v\rangle$ for all $v \in T_u(\mathcal{N}_V)$ or

$$\langle \nabla_{\mathcal{N}_V} I_V(u) - \nabla I_V(u), v\rangle = 0, \quad \forall v \in T_u(\mathcal{N}_V) := \{v \in H_0^1(\Omega);\quad J'_V(u)v = 0\}.$$

On the other hand $T_u(\mathcal{N}_V)$ is of codimension one and so

$$H_0^1(\Omega) = E = T_u(\mathcal{N}_V) \oplus < J'(u) > .$$

Now by the Hahn-Banach Theorem, there is a continuous linear function $\nabla_{\mathcal{N}_V} I_V(u)$ on E such that

$$\nabla_{\mathcal{N}_V} I_V(u) - \nabla I_V(u) = tJ'_V(u)$$

or

$$\nabla I_V(u) = \nabla_{\mathcal{N}_V} I_V(u) + t\nabla J_V(u),$$

as we want.

As already said in the introduction, we will work on the Nehari manifold. nowadays, Nehari manifold is a classical tool in variational methods because of its useful properties. Next Lemma we shown that $\mathcal{N}_V$ manifold is a natural constraint.

Lemma 1.3.5 *Every $(PS)_d$-sequence (u_k) for I_V restricted the $\mathcal{N}_V$ contains a subsequence which is a $(PS)_d$-sequence for I_V in $H_0^1(\Omega)$.*

Proof. Let (u_k) be $(PS)_d$-sequence for I_V on $\mathcal{N}_V$. By Lemma (1.3.1), after passing to a subsequence, we have that (u_k) is bounded in $H_0^1(\Omega)$. Write

$$\nabla I_V(u_k) = \nabla_{\mathcal{N}_V} I_V(u_k) + t_k \nabla J_V(u_k) \tag{1.3.1}$$

By property (f_4), the Sobolev embedding and Hölder's inequality, for any $v \in H_0^1(\Omega)$,

$$\begin{aligned} \left| \int_\Omega [f'(u_k)u_k - f(u_k)]v \right| &\leq C \int_\Omega (|u_k|^{p_1} + |u_k|^{p_2})|v| \\ &\leq C(|u_k|_{p_1+1}^{p_1} |v|_{p_1+1} + |u_k|_{p_2+1}^{p_2} |v|_{p_2+1}) \\ &\leq C(\|u_k\|_\Omega^{p_1} + \|u_k\|_\Omega^{p_2})\|v\| \leq C\|v\|_\Omega. \end{aligned}$$

Therefore

$$|\langle \nabla J_V(u_k), v\rangle_\Omega| = |2\langle u_k, v\rangle_\Omega - \int_\Omega [f'(u_k)u_k + f(u_k)]v| \leq C\|v\|_\Omega \quad \forall v \in H_0^1(\Omega).$$

This proves that $(\nabla J_V(u_k))$ is bounded.

As $|\nabla J_V(u_k)u_k| \leq \|\nabla J_V(u_k)\|\|u_k\| < C$, after passing to a subsequence, we have that $|J_V'(u_k)u_k| \to \rho \geq 0$. We show that $\rho > 0$.

From Lemma (1.2.1)(a) and by property (f_2) we obtain

$$0 < \varrho^2 \leq \|u_k\|_\Omega^2 = \int_\Omega f(u_k)u_k \leq c(|u_k|_{p_1+1}^{p_1+1} + |u_k|_{p_2+1}^{p_2+1}).$$

This inequality, together with Lions lemma, implies that there exist $\delta > 0$ and a sequence (y_k) in $\mathbb{R}^N$ such that

$$\int_{B_1(y_k)} u_k^2 = \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} u_k^2 \geq \delta.$$

Set $\tilde{u}_k(x) := u_k(x + y_k)$ After passing to a subsequence $\tilde{u}_k \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $\tilde{u}_k \to u$ in $L_{loc}^2(\mathbb{R}^N)$ and $\tilde{u}_k(x) \to u(x)$ a.e. in $\mathbb{R}^N$. Therefore,

$$\int_{B_1(0)} u^2 = \lim_{k\to\infty} \int_{B_1(0)} \tilde{u}_k^2 = \lim_{k\to\infty} \int_{B_1(y_k)} u_k^2 \geq \delta.$$

Hence, $u \neq 0$ and there exists a subset Λ of positive measure in $B_1(0)$ such that $u(x) \neq 0$ for every $x \in \Lambda$. Property (f_5) implies that $f'(u(x))(u(x))^2 - f(u(x))u(x) > 0$ if $u(x) \neq 0$.

So, from the Fatou's lemma, we conclude that

$$\rho = \lim_{k\to\infty} |\nabla J_V(u_k)u_k| = \lim_{k\to\infty} \left|2\|u_k\|_\Omega^2 - \int_\Omega [f'(u_k)u_k^2 + f(u_k)u_k]\right|$$

$$= \lim_{k\to\infty} \int_\Omega f'(u_k)u_k^2 - f(u_k)u_k$$

$$= \lim_{k\to\infty} \int_{\mathbb{R}^N} [f'(u_k)u_k^2 - f(u_k)u_k] = \lim_{k\to\infty} \int_{\mathbb{R}^N} [f'(\tilde{u}_k)\tilde{u}_k^2 - f(\tilde{u}_k)\tilde{u}_k]$$

$$\geq \liminf_{k\to\infty} \int_\Lambda [f'(\tilde{u}_k)\tilde{u}_k^2 - f(\tilde{u}_k)\tilde{u}_k] \geq \int_\Lambda \liminf_{k\to\infty} [f'(\tilde{u}_k)\tilde{u}_k^2 - f(\tilde{u}_k)\tilde{u}_k] = \int_\Lambda [f'(u)u^2 - f(u)u] > 0.$$

Taking the inner product of (1.3.1) with u_k we obtain

$$0 = I_V'(u_k)u_k = \langle \nabla_{\mathcal{N}_V} I_V(u_k), u_k\rangle + t_k \nabla J_V(u_k)u_k = o_k(1) + t_k \nabla J_V(u_k)u_k$$

and so $t_k \to 0$ and from (1.3.1) we deduce $\nabla I_V(u_k) \to 0$ as $\nabla_{\mathcal{N}_V} I_V(u_k) \to 0$ and this proves the lemma. □

Lemma 1.3.6 *(Splitting) Let (u_k) be a bounded sequence in $H_0^1(\Omega)$ such that*

$$I_V(u_k) \to d \quad \textit{and} \quad I_V'(u_k) \to 0 \quad \textit{in} \;\; H^{-1}(\Omega).$$

Replacing u_k by a subsequence if necessary, there exist a solution u_0 de (P_V), a number $m \in \mathbb{N}$, m functions $w_1, \cdots, w_m$ in $H^1(\mathbb{R}^N)$ and m sequences of points $(y_k^j) \in \mathbb{R}^N$, $1 \leq j \leq m$, satisfying:

a) $u_k \to u_0$ in $H_0^1(\Omega)$ or

b) w_j are nontrivial solutions of the limit problem (P_∞);

c) $|y_k^j| \to +\infty$ and $|y_k^j - y_k^i| \to +\infty$ $i \neq j$;

d) $u_k - \sum_{i=1}^m w_j \to u_0 \quad$ in $\; H^1(\mathbb{R}^N)$.

e) $d = I_V(u_0) + \sum_{i=1}^m I_\infty(w_j)$.

Proof. From Lemma 1.3.1, the sequence (u_k) is bounded so, after passing to a subsequence, we may assume $u_k \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, $u_k \to u_0$ in $L^2_{loc}(\Omega)$ and $u_k \to u_0$ a.e. in Ω.

By the weak continuity of I'_V, $I'_V(u_k) \to I'_V(u_0)$ and so $I'_V(u_0) = 0$.

Now let $u_k^1 := u_k - u_0$ and as we saw earlier define $u_k^1 := 0$ in $\mathbb{R}^N \setminus \Omega$ and so $u_k^1 \subset H^1(\mathbb{R}^N)$ then $u_k^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$ and as above we can assume $u_k^1 \to 0$ a.e. in $\mathbb{R}^N$ and so

$$I_\infty(u_k^1) = I_V(u_k^1) + o(1) \tag{1.3.2}$$

where $o(1) \to 0$ for k large enough. Indeed $I_\infty(u_k^1) = I_V(u_k^1) + \int_{\mathbb{R}^N} (V(x) - V_\infty)(u_k^1)^2$, $\lim_{|x| \to +\infty} V(x) = V_\infty$ and $u_k \to u_0$ in $L^2_{loc}(\Omega)$ and so we have,

$$\int_{\mathbb{R}^N} (V(x) - V_\infty)(u_k^1)^2 = \int_{B_D(0)} (V(x) - V_\infty)(u_k^1)^2 + \int_{\mathbb{R}^N \setminus B_D(0)} (V(x) - V_\infty)(u_k^1)^2$$

$$\leq \int_{B_D(0)} C(u_k^1)^2 + \int_{\mathbb{R}^N \setminus B_D(0)} \varepsilon (u_k^1)^2 \leq C\varepsilon.$$

Also

$$I'_\infty(u_k^1) = I'_V(u_k^1) + o(1) \quad \text{in} \quad H^{-1}(\Omega) \tag{1.3.3}$$

since for any $\varphi \in C^\infty(\Omega)$ with $\|\varphi\| = 1$, $\langle I'_\infty(u_k^1) - I'_V(u_k^1), \varphi \rangle = o(1)$ without dependence to φ, because like as above by $\lim_{|x| \to +\infty} V(x) = V_\infty$, $u_k^1 \to 0$ in $L^2_{loc}(\Omega)$ and Holder's inequality we have

$$\langle I'_\infty(u_k^1) - I'_V(u_k^1), \varphi \rangle = \int_{\mathbb{R}^N} (V(x) - V_\infty) u_k^1 \varphi$$

$$= \int_{B_D(0)} (V(x) - V_\infty) u_k^1 \varphi + \int_{\mathbb{R}^N \setminus B_D(0)} (V(x) - V_\infty) u_k^1 \varphi$$

$$\leq \int_{B_D(0)} C u_k^1 \varphi + \int_{\mathbb{R}^N \setminus B_D(0)} \varepsilon u_k^1 \varphi \leq C \|u_k^1\|_{L^2(B_D(0))} \|\varphi\| + \varepsilon C \|u_k^1\| \|\varphi\| = o(1).$$

Now we claim that

$$I_V(u_k^1) = I_V(u_k) - I_V(u_0) + o(1), \tag{1.3.4}$$

and

$$I'_V(u_k^1) = o(1), \tag{1.3.5}$$

so by replacing (1.3.4) in (1.3.2) as $I_V(u_k) \to d$ we have

$$I_\infty(u_k^1) \to d - I_V(u_0). \tag{1.3.6}$$

moreover by replacing (1.3.5) in (1.3.3) we have

$$I'_\infty(u_k^1) \to 0 \quad \text{in} \quad H^{-1}(\Omega). \tag{1.3.7}$$

To show the claim, for (1.3.4) note that

$$I_V(u_k) = \frac{1}{2}\|u_k^1 + u_0\|_\Omega^2 - \int_\Omega F(u_k^1 + u_0) = I_V(u_k^1) + I_V(u_0)$$

$$+\langle u_k^1, u_0\rangle_\Omega - \int_\Omega F(u_k^1 + u_0) - F(u_k^1) - F(u_0),$$

since $u_k^1 \rightharpoonup 0$ we have $\langle u_k^1, u_0\rangle_\Omega \to 0$ as $k \to \infty$ and so to prove the claim it is enough to show

$$\int_\Omega F(u_k^1 + u_0) - F(u_k^1) - F(u_0) \to 0. \tag{1.3.8}$$

Since $u_0 \in H_0^1(\Omega)$, given $\varepsilon > 0$ we can choose $B_D(0)$ such that $\int_{\mathbb{R}^N \setminus B_D(0)} |\nabla u_0|^2 + u_0^2 \leq \varepsilon$. Now, by the Mean Value Theorem and (f_2) we have

$$\int_{\mathbb{R}^N \setminus B_D(0)} F(u_k^1 + u_0) - F(u_k^1) - F(u_0)$$

$$\leq \int_{\mathbb{R}^N \setminus B_D(0)} f(u_k^1 + \theta u_0)u_0 + C(|u_0|_{\mathbb{R}^N \setminus B_D}^{p_1+1} + |u_0|_{\mathbb{R}^N \setminus B_D}^{p_1+1})$$

and by the boundedness of u_k^1, Hölder's inequality and Sobolev embedding

$$\int_{\mathbb{R}^N \setminus B_D(0)} F(u_k^1 + u_0) - F(u_k^1) - F(u_0) \leq C\varepsilon. \tag{1.3.9}$$

On the other hand, $u_k^1 \to 0$ in $B_D(0)$ and so since F is continuous

$$\int_{B_D(0)} F(u_k^1 + u_0) - F(u_k^1) - F(u_0) \to 0, \tag{1.3.10}$$

hence (1.3.9) and (1.3.10) yields (1.3.8) and this proves (1.3.4) as we want.

In order to prove claim (1.3.5) for any $\varphi \in C^\infty(\Omega)$ with $\|\varphi\| = 1$ we have

$$\langle I'_V(u_k^1), \varphi\rangle = \langle I'_V(u_k), \varphi\rangle + \langle I'_V(u_0), \varphi\rangle + \int_\Omega f(u_k^1 + u_0) - f(u_k^1) - f(u_0)\varphi$$

since $I'_V(u_k) \to 0$ and $I'_V(u_0) = 0$ it is enough to show

$$\int_\Omega [f(u_k^1 + u_0) - f(u_k^1) - f(u_0)]\varphi \to 0 \tag{1.3.11}$$

without dependence to φ and arguing as in the proof of (1.3.8), we obtain (1.3.11). Now let

$$\delta := \lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_k^1|^2$$

if $\delta = 0$, Lions' lemma implies that $u_k^1 \to 0$ in $L^p(\Omega)$, $2 \le p < 2^*$. Since $I'_\infty(u_k^1) \to 0$ it follows that $u_k^1 \to 0$ in $H^1(\Omega)$ and the proof is complete.
If $\delta > 0$, we may assume the existence of $(y_k^1) \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_k^1)} |u_k^1|^2 > \frac{\delta}{2}.$$

Let us define $w_k^1 := u_k^1(\cdot - y_k^1)$. We may assume that $w_k^1 \rightharpoonup w_1$ in $H^1(\mathbb{R}^N)$. By the weak continuity of I'_∞, w_1 is solution of P_∞ and $w_k^1(x) \to w_1(x)$ a.e. on $\mathbb{R}^N$. Since

$$\int_{B_1(0)} |w_k^1|^2 > \frac{\delta}{2}$$

it follows that

$$\int_{B_1(0)} |w_1|^2 \ge \frac{\delta}{2}$$

so $w_1 \neq 0$ and (y_k^1) is unbounded since $u_k^1 \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$.We may assume that $|y_k^1| \to \infty$.
Define $u_k^2 := u_k^1 - w_1(\cdot - y_k^1)$ then u_k^2 satisfies as above

$$I_\infty(u_k^2) \to d - I_V(u_0) - I_\infty(w_1)$$

and

$$I'_\infty(u_k^2) \to 0 \text{ in } H^{-1}(\Omega).$$

Any nontrivial critical point u of I_∞ satisfies $I_\infty(u) \ge c_\infty > 0$. Iterating the above procedure we construct sequences w_i and (y_k^j). Since for every i, $I_\infty(w_i) \ge c_\infty$, the iteration must terminate at some finite index m.

□

Lemma 1.3.7 *Problem (P_∞) does not have a solution u such that $I_\infty(u) \in (c_\infty, 2c_\infty)$*

Proof. Under our assumptions on f including that f is odd, the limit problem (P_∞) has a positive solution w with $I_\infty(w) = c_\infty$ [13]. If u is a solution of P_∞ such that $I_\infty(u) \in [c_\infty, 2c_\infty)$ then, by Lemma (1.3.3), u does not change sign and, by [15], it is radially symmetric. By assumption (U) problem (P_∞) has a unique positive solution and therefore $u = \pm w$, up to a translation. Hence, $I_\infty(u) = c_\infty$. □

Corollary 1.3.8 *(Compactness) If c_V is not attained, then $c_V \geq c_\infty$ and I_V satisfies the Palais-Smale condition on $\mathcal{N}_V$ at every level $d \in (c_\infty, 2c_\infty)$.*

Proof. Let (u_k) be a $(PS)_d$-sequence for I_V on $\mathcal{N}_V$. By Lemmas 1.3.1 and Lemmas 1.3.5, after passing to a subsequence, we have that (u_k) is a bounded $(PS)_d$-sequence for I_V. By the definition $c_V := \inf_{u\in\mathcal{N}_V} I_V(u)$, there exists $(u_j) \in \mathcal{N}_V$ such that $I_V(u_j) \to c_V$. Now by the Ekeland variational principle there exists $(\tilde{u}_j) \in \mathcal{N}_V$ such that $I_V(\tilde{u}_j) \to c_V$ and $I'_V(\tilde{u}_j) \to 0$ (Theorem 8.5 [48]). Now by the Splitting lemma if $d = c_V$ is not attained, we have $c_V = I_V(u_0) + \sum_{i=1}^{m} I_\infty(w_j)$ and so $c_V \geq c_\infty$. If $d \in (c_\infty, 2c_\infty)$ and (u_k) does not have a convergent subsequence then, by the Splitting lemma,

$$2c_\infty > d = I_V(u_0) + \sum_{i=1}^{m} I_\infty(w_j) \geq \begin{cases} mc_\infty & \text{if } u_0 = 0 \\ c_V + mc_\infty \geq (m+1)c_\infty & \text{if } u_0 \neq 0 \end{cases} \quad (1.3.12)$$

then in both cases, $m < 2$ and so $m = 1$. The hypothesis $2c_\infty > d \geq (m+1)c_\infty$ implies that it is not possible $m = 1$ and $u_0 \neq 0$, there for $u_0 = 0$, that follows $I_V(u_n) \to I_\infty(w_1) = d$. But by (U) the solution is unique and so $w_1 = w$ that yields there exists a solution w of P_∞ with $d = I_\infty(w)$, which contradicts Lemma 1.3.7. Hence, I_V satisfies the Palais-Smale condition on $\mathcal{N}_V$ at every $d \in (c_\infty, 2c_\infty)$. □

Remark 1.3.9 *If $0 < V_0 < V(x) < V_\infty$, then $c_V < c_\infty$ and by Lemma 1.3.6, $u_k \to u_0$ in $H_0^1(\Omega)$. There for, $I'(u_0) = 0$ and $I(u_0) = c_v > 0$, here u_0 is a solution of (P_V) and c_V is attained. If $V(x) = V_\infty$, then by [24] and [37] c_V is not attained.*

1.4 Existence of a positive bound state solution

For $R > 0$, $|y_0| = 1$ and $y \in \partial B_2(y_0)$ let

$$\varepsilon_R := \int_{\mathbb{R}^N} f(w_0^R) w_y^R = \int_{\mathbb{R}^N} f(w_y^R) w_0^R.$$

Remark 1.4.1 *Note that in principle $\varepsilon_R = \varepsilon_R(y)$ is dependent on y, but we are going to show that the estimates on ε_R are independent of y.*

Lemma 1.4.2 *There exists $C > 0$ such that*

$$\varepsilon_R \leq C R^{-\frac{N-1}{2}} e^{-2\sqrt{V_\infty} R} \tag{1.4.1}$$

for all $y \in \partial B_2(y_0)$ and $R > 0$ sufficiently large.

Proof. From property (f_2), performing a change of variable, we have that

$$\varepsilon_R \leq C \Big(\int_{\mathbb{R}^N} |w(x)|^{p_1} |w(x - R(y-y_0))| + \int_{\mathbb{R}^N} |w(x)|^{p_2} |w(x - R(y-y_0))| \Big)$$

As $p_2 \geq p_1 > 1$, using estimates (1.2.1), Lemma 1.2.3 and Lemma 1.2.6 with $p = p_1$ and $q = 1$ we obtain that

$$\int_{\mathbb{R}^N} |w(x)|^{p_1} |w(x - R(y-y_0))| \leq C R^{-\frac{N-1}{2}} e^{-2\sqrt{V_\infty} R}$$

and

$$\int_{\mathbb{R}^N} |w(x)|^{p_2} |w(x - R(y-y_0))| dx \leq C R^{-\frac{N-1}{2}} e^{-2\sqrt{V_\infty} R}$$

so the lemma is proved. □

Note that above lemma implies that

$$\varepsilon_R \to 0 \quad \text{as } R \to \infty \quad \text{uniformly in} \quad y \in \partial B_2(y_0).$$

Lemma 1.4.3 *There exists $C > 0$ such that for all $s, t \geq \frac{1}{2}$, $y \in \partial B_2(y_0)$ and R large enough,*

$$\int_{\mathbb{R}^N} f(s w_0^R) t w_y^R \geq C R^{-\frac{N-1}{2}} e^{-2\sqrt{V_\infty} R}.$$

Proof. For $|x| < 1$ and R large enough we have

$$1 + |x| < 1 + |x - R(y - y_0)| < 1 + |x| + R|(y - y_0)| < 4R. \tag{1.4.2}$$

Now by (f_5) (1.4.2) and the decay estimates (1.2.1) there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} f(sw_0^R)tw_y^R = st\int_{\mathbb{R}^N}\left[\frac{f(sw_0^R)}{sw_0^R}\right]w_0^R w_y^R$$

$$\geq \frac{1}{4}\int_{\mathbb{R}^N}\left[\frac{f(\frac{1}{2}w_0^R)}{\frac{1}{2}w_0^R}\right]w_0^R w_y^R$$

$$\geq \frac{1}{4}\int_{B_1(Ry_0)}\left[\frac{f(\frac{1}{2}w_0^R)}{\frac{1}{2}w_0^R}\right]w_0^R w_y^R$$

$$\geq \frac{1}{4}\left[\min_{x\in B_1(0)}\frac{f(\frac{1}{2}w(x))}{\frac{1}{2}w(x)}\right]\int_{B_1(0)} w(x)w(x-R(y-y_0))$$

$$\geq C\int_{B_1(0)}(1+|x|)^{-\frac{N-1}{2}}e^{-\sqrt{V_\infty}|x|}(1+|x-R(y-y_0)|)^{-\frac{N-1}{2}}e^{-\sqrt{V_\infty}|x-R(y-y_0)|}$$

$$\geq CR^{-\frac{N-1}{2}}e^{-2\sqrt{V_\infty}R}.$$

□

Remark 1.4.4 *If we set $s, t = 1$ in above lemma we have*

$$\varepsilon_R \geq CR^{-\frac{N-1}{2}}e^{-2\sqrt{V_\infty}R}. \tag{1.4.3}$$

Lemma 1.4.5 *For every $b > 1$ there is a constant C, such that*

$$\left|\int_\Omega [sf(w_0^R\psi) - f(sw_0^R\psi)]w_y^R\psi\right| \leq C|s-1|\varepsilon_R,$$

for all $s \in [0, b]$, $y \in \partial B_2(y_0)$ and R large enough.

Proof. Fix $u \in \mathbb{R}$ and consider the function $g(s) := sf(u) - f(su)$. By property (f_2),

$$g'(s) := f(u) - f'(su)u \leq |f(u)| + (s^{p_1-1}|u|^{p_1} + s^{p_2-1}|u|^{p_2})$$

$$\leq |f(u)| + C(|u|^{p_1} + |u|^{p_2}) \qquad \forall s \in [0, 1],$$

hence, by the Mean Value Theorem,

$$|sf(u) - f(su)| = |g(u) - g(1)| = |g'(t)||s-1|$$

$$\leq [|f(u)| + C(|u|^{p_1} + |u|^{p_2})]|s-1|.$$

It follows that

$$\int_\Omega |sf(w_0^R\psi) - f(sw_0^R\psi)|w_y^R\psi$$

$$\leq |s-1|[\int_\Omega f(w_0^R\psi)w_y^R\psi + C\int_\Omega (|w_0^R\psi|^{p_1} + |w_0^R\psi|^{p_2}))w_y^R\psi],$$

$$\leq |s-1|C\int_{\mathbb{R}^N} (|w_0^R|^{p_1} w_y^R(\psi)^{p_1+1} + |w_0^R|^{p_2} w_y^R(\psi)^{p_2+1}).$$

Now applying Lemma 1.2.6 and being $|\psi| \leq 1$ we have

$$\int_{\mathbb{R}^N} |sf(w_0^R\psi) - f(sw_0^R\psi)|w_y^R\psi \leq |s-1|O(\varepsilon_R) \leq C|s-1|\ \varepsilon_R$$

for all $s \in [0,b]$, $y \in \partial B_2(y_0)$ as claimed. □

Proposition 1.4.6 *There exists $R_1 > 0$ and, for each $R > R_1$, a number $\eta = \eta_R > 0$, $\eta_R = o_R(1)$ such that*

$$I_V(T^R_{\lambda,y} U^R_{\lambda,y}) \leq 2c_\infty - \eta,$$

for all $\lambda \in [0,1]$, $y \in \partial B_2(y_0)$.

Proof. Let us denote, for simplicity,

$$s := T^R_{\lambda,y}\lambda \quad , \quad t := T^R_{\lambda,y}(1-\lambda).$$

Recall that, by Lemma 1.2.10, $s,t \in (0,T_0)$ if R is large enough.
We have that

$$I_V(sw_0^R\psi + tw_y^R\psi)$$

$$= \frac{s^2}{2}\int_\Omega |\nabla(w_0^R\psi)|^2 + \frac{s^2}{2}\int_\Omega V(x)(w_0^R\psi)^2 + \frac{t^2}{2}\int_\Omega |\nabla(w_y^R\psi)|^2 + \frac{t^2}{2}\int_\Omega V(x)(w_y^R\psi)^2$$

$$+st\int_\Omega (\nabla w_0^R\psi)\nabla(w_y^R\psi) + st\int_\Omega V(x)w_0^R\psi w_y^R\psi - \int_\Omega F(sw_0^R\psi + tw_y^R\psi)$$

$$= \frac{s^2}{2}\int_\Omega |\nabla(w_0^R\psi)|^2 + \frac{s^2}{2}\int_\Omega V_\infty(w_0^R\psi)^2 - \int_\Omega F(sw_0^R\psi) \tag{1.4.4}$$

$$+\frac{t^2}{2}\int_\Omega |\nabla(w_y^R\psi)|^2 + \frac{t^2}{2}\int_\Omega V_\infty(w_y^R\psi)^2 - \int_\Omega F(tw_y^R) \tag{1.4.5}$$

$$+\frac{s^2}{2}\int_\Omega (V(x)-V_\infty)(w_0^R\psi)^2+\frac{t^2}{2}\int_\Omega (V(x)-V_\infty)(w_y^R\psi)^2 \tag{1.4.6}$$

$$+st\int_\Omega \nabla(w_0^R\psi)\nabla(w_y^R\psi)+st\int_\Omega V_\infty w_0^R\psi w_y^R\psi \tag{1.4.7}$$

$$+st\int_\Omega (V(x)-V_\infty)w_0^R\psi w_y^R\psi \tag{1.4.8}$$

$$-\int_\Omega F(sw_0^R\psi+tw_y^R\psi)-F(sw_0^R\psi)-F(tw_y^R\psi)-f(sw_0^R\psi)tw_y^R\psi-f(tw_y^R\psi)sw_0^R\psi \tag{1.4.9}$$

$$-\int_\Omega f(sw_0^R\psi)tw_y^R\psi-\int_\Omega f(tw_y^R\psi)sw_0^R\psi. \tag{1.4.10}$$

The sum in line (1.4.4) is equal to $I_\infty(sw_0^R)+o(\varepsilon_R)$. Indeed,

$$(1.4.4)=I_\infty(sw_0^R)+\frac{s^2}{2}\int_{B_{2k}(0)}|\nabla(w_0^R\psi)|^2-|\nabla w_0^R|^2$$

$$+\frac{s^2}{2}\int_{B_{2k}(0)} V_\infty(w_0^R\psi)^2-V_\infty(w_0^R)^2-\int_{B_{2k}(0)} F(sw_0^R)-F(sw_0^R\psi).$$

But by Lemma 1.2.5 with $q=2$ and (1.2.15) we have

$$\frac{s^2}{2}\int_{B_{2k}(0)}|\nabla(w_0^R\psi)|^2-|\nabla w_0^R|^2+\frac{s^2V_\infty}{2}\int_{B_{2k}(0)}(w_0^R\psi)^2-(w_0^R)^2=o(\varepsilon_R).$$

On the other hand, the Mean Value Theorem, (f_2) and Lemma 1.2.5 yield

$$\int_{B_{2K}(0)} F(sw_0^R)-F(sw_0^R\psi)=\int_{B_{2K}(0)} f(sw_0^R+A(x)sw_0^R\psi)(sw_0^R-sw_0^R\psi)$$

$$\le C\int_{B_{2K}(0)}\left(|sw_0^R|^{p_1}+|sw_0^R|^{p_2}\right)sw_0^R=C\int_{B_{2K}(0)}\left(|sw_0^R|^{p_1+1}+|sw_0^R|^{p_2+1}\right)=o(\varepsilon_R),$$

thus

$$(1.4.4)=I_\infty(sw_0^R)+o(\varepsilon_R).$$

Since w_0^R is least energy solution of the limit problem (P_∞) and by Lemma 1.2.1 (c) we have $I_\infty(sw_0^R) \le c_\infty$ and similarly for the sum in line (1.4.5), so

$$(1.4.4) + (1.4.5) \le 2c_\infty + o(\varepsilon_R).$$

By Lemma 1.2.7 we have

$$(1.4.6) + (1.4.8)) = o(\varepsilon_R).$$

As to (1.4.9), by Lemma 1.2.2 there is $0 < \nu < p_1 - 1$, now by Lemma 1.2.6 we have

$$-\int_{\mathbb{R}^N} F(sw_0^R\psi + tw_y^R\psi) - F(sw_0^R\psi) - F(tw_y^R\psi) - f(sw_0^R\psi)tw_y^R\psi - f(tw_y^R\psi)sw_0^R\psi$$

$$\le C(st)^{1+\frac{\nu}{2}} \int_{\mathbb{R}^N} (w_y^R\psi w_0^R\psi)^{1+\frac{\nu}{2}} \le C(st)^{1+\frac{\nu}{2}} \int_{\mathbb{R}^N} (w_y^R w_0^R)^{1+\frac{\nu}{2}} \le CR^{-\frac{N-2}{2}(1+\frac{\nu}{2})} e^{-2(1+\frac{\nu}{2})\sqrt{V_\infty}R}$$

so we have shown that

$$(1.4.9) \le o(\varepsilon_R).$$

We write the sum of the remaining terms as

$$(1.4.7) + (1.4.10) = \frac{t}{2}\int_\Omega [sf(w_0^R\psi) - f(sw_0^R\psi)]w_y^R\psi + \frac{s}{2}\int_\Omega [tf(w_0^R\psi) - f(tw_0^R\psi)]w_y^R\psi$$

$$-\frac{1}{2}\int_\Omega f(sw_0^R\psi)tw_y^R\psi - \frac{1}{2}\int_\Omega f(tw_y^R\psi)sw_0^R\psi.$$

By Lemma 1.4.5 there is a constant $C > 0$ such that

$$\frac{t}{2}\int_\Omega [sf(w_0^R\psi) - f(sw_0^R\psi)]w_y^R\psi + \frac{s}{2}\int_\Omega [tf(w_0^R\psi) - f(tw_0^R\psi)]w_y^R\psi \le C(|s-1| + |t-1|)\varepsilon_R$$

for all $s, t \in [0, T_0], y \in \partial B_2(y_0)$ and R large enough. Moreover like as the sum (1.4.4) we have

$$\frac{1}{2}\int_\Omega f(sw_0^R\psi)tw_y^R\psi + \frac{1}{2}\int_\Omega f(tw_y^R\psi)sw_0^R\psi$$

$$= \frac{1}{2}\int_{\mathbb{R}^N} f(sw_0^R)tw_y^R + \frac{1}{2}\int_{\mathbb{R}^N} f(tw_y^R)sw_0^R + o(\varepsilon_R)$$

and by Lemma 1.4.3, there is a constant $C_0 > 0$ such that

$$\frac{1}{2}\int_{\mathbb{R}^N} f(sw_0^R)tw_y^R + \frac{1}{2}\int_{\mathbb{R}^N} f(tw_y^R)sw_0^R\psi \ge C_0\varepsilon_R$$

for all $s, t \ge \frac{1}{2}$, $y \in \partial B_2(y_0)$ and R large enough. By Lemma 1.2.10, if $\lambda = 1/2$, then $s, t \to 1$ as $R \to \infty$. So taking $R_0 > 0$ sufficiently large and $\delta \in (0, 1/2)$ sufficiently small such that $C(|s-1|+|t-1|)\le \frac{C_0}{2}$, we have

$$(1.4.7)+(1.4.10)\le -\frac{C_0}{2}\varepsilon_R+o(\varepsilon_R)$$

for all $\lambda\in[\frac{1}{2}-\delta,\frac{1}{2}+\delta]$, $y\in\partial B_2(y_0)$ and $R>R_0$. Summing up, we have proved that

$$I_V(sw_0^R+tw_y^R)\le 2c_\infty-\frac{C_0}{2}\varepsilon_R+o(\varepsilon_R) \tag{1.4.11}$$

for all $\lambda\in[\frac{1}{2}-\delta,\frac{1}{2}+\delta]$, $y\in\partial B_2(y_0)$ and $R>R_0$.
On the other hand, for all $\lambda\in[0,\frac{1}{2}-\delta]\cup[\frac{1}{2}+\delta,1]$, $y\in\partial B_2(y_0)$ and R sufficiently large, since if $T^R_{\lambda,y}\le 2$ then $s=T^R_{\lambda,y}\lambda\in[0,1-2\delta]$ or $t=T^R_{\lambda,y}(1-\lambda)\in[1,1-2\delta]$ and if $T^R_{\lambda,y}\ge 2$ then $s=T^R_{\lambda,y}\lambda\in[1+2\delta,\infty]$ or $t=T^R_{\lambda,y}(1-\lambda)\in[1+2\delta,\infty]$, in fact one of s or t is in $[0,1-2\delta]\cup[1+2\delta,\infty]$ and so by Lemma 1.2.1(c) applied to V_∞, there exists $\gamma\in(0,c_\infty)$ such that

$$I_\infty(rw_0^R)\le c_\infty-\gamma \qquad \forall r\in[0,1-2\delta]\cup[1+2\delta,\infty]$$

also with our previous estimates we have (1.4.6)+...+(1.4.10)= $O(\varepsilon_R)$, and so

$$I_V(sw_0^R+tw_y^R)\le 2c_\infty-\gamma+O(\varepsilon_R). \tag{1.4.12}$$

Inequalities (1.4.11) and (1.4.12), together, yield the statement of the proposition. □

Lemma 1.4.7 *For any $\delta>0$, there exists $R_2>0$ such that*

$$I_V(T^R_{\lambda,y}U^R_{\lambda,y})<c_\infty+\delta,$$

for $\lambda=0$ and every $y\in\partial B_2(y_0)$ and $R\ge R_2$. In particular, $c_V\le c_\infty$.

Proof. By Lemma 1.2.10, $T^R_{\lambda,y}$ is bounded uniformly in λ,y and R. As w_y^R is a ground state of problem (P_∞), using Lemma 1.2.1(c) and Lemma 1.2.7, we obtain

$$\begin{aligned}I_V(T^R_{0,y}U^R_{0,y}) &= I_\infty(T^R_{0,y}w_y^R\psi)+(T^R_{0,y})^2\int_\Omega (V(x)-V_\infty)(w_y^R\psi)^2\\ &\le I_\infty(T^R_{0,y}w_y^R)+o(\varepsilon_R)+(T^R_{0,y})^2\int_{\mathbb{R}^N}(V(x)-V_\infty)(w_y^R)^2\\ &\le \max_{s>0} I_\infty(sw_y^R)+o(1)\le c_\infty+o_R(1),\end{aligned}$$

where $o_R(1)\to 0$ as $R\to\infty$, uniformly in $y\in\partial B_2(y_0)$. □

Let $\beta: L^2(\mathbb{R}^N)\setminus\{0\}\to\mathbb{R}^N$ be a barycenter map, i.e. a continuous map such that, for every $u \in L^2(\mathbb{R}^N)$, every $y \in \mathbb{R}^N$ and every linear isometry A of $\mathbb{R}^N$,

$$\beta(u(\cdot - y)) = \beta(u) + y \quad \text{and} \quad \beta(u \circ A^{-1}) = A(\beta(u)). \tag{1.4.13}$$

Note that $\beta(u) = 0$ if u is radial. Barycenter maps have been constructed in [7].

Lemma 1.4.8 *If c_V is not attained then $c_V = c_\infty$ and there exists $\delta > 0$ such that*

$$\beta(u) \neq 0, \qquad \forall u \in \mathcal{N}_V \cap I_V^{c_\infty + \delta}$$

where $I_V^c = \{u \in H_0^1(\Omega), \quad I_V(u) \leq c\}$.

Proof. If c_V is not attained, Corollary 1.3.8 and Lemma 1.4.7 imply that $c_V = c_\infty$. Arguing by contradiction, assume that for each $k \in N$ there exists $v_k \in \mathcal{N}_V$ such that $I_V(v_k) < c_V + \frac{1}{k}$ and $\beta(v_k) = 0$. By Ekeland variational principle [29], there exists a $(PS)_d$-sequence (u_k) for I_V on $\mathcal{N}_V$ at the level $d = c_V$ such that $\|u_k - v_k\| \to 0$ [[48], Theorem 8.5]. By Lemmas 1.3.5 and 1.3.1, after passing to a subsequence, we have that (u_k) is a bounded $(PS)_d$-sequence for I_V. As c_V is not attained, Lemma 1.3.6 (splitting) implies that there exists a sequence (y_k) in $\mathbb{R}^N$ such that $|y_k| \to \infty$ and $\|u_k - w(\cdot - y_k)\| \to 0$, where w is the (positive or negative) radial ground state of (P_∞). Setting $\tilde{v}_k(x) := v_k(x + y_k)$, and using properties (1.4.13) and the continuity of the barycenter, we conclude that

$$-y_k = \beta(v_k) - y_k = \beta(\tilde{v}_k) \to \beta(w) = 0$$

this is a contradiction. □

We have constructed all the tools in order to apply a topological argument analogous to that found in [24] and prove the main result. For the sake of completeness we recall the argument and prove theorem 1.1.4.

Proof of Theorem 1.1.4 : If c_V is attained by I_V at some $u \in \mathcal{N}_V$ then, by Lemma 1.3.2, u is a nontrivial solution of problem (P_V). So assume that c_V is not attained. Then, by Lemma 1.4.8, $c_V = c_\infty$. We will show that I_V has a critical value in $(c_\infty, 2c_\infty)$. By Lemma 1.4.8, we may fix $\delta \in (0, \frac{c_\infty}{4})$ such that

$$\beta(u) \neq 0, \qquad \forall u \in \mathcal{N}_V \cap I_V^{c_\infty + \delta}.$$

Proposition 1.4.6 and Lemma 1.4.7 allow us to choose $\eta \in (0, \frac{c_\infty}{4})$ and $R > 0$ such that

$$I_V(T^R_{\lambda,y}U^R_{\lambda,y}) \leq \begin{cases} 2c_\infty - \eta & \text{for all } \lambda \in [0,1] \text{ and all } y \in \partial B_2(y_0) \\ c_\infty + \delta & \text{for } \lambda = 0 \text{ and all } y \in \partial B_2(y_0). \end{cases}$$

Define $\alpha : B_2(y_0) \to \mathcal{N}_V \cap I_V^{2c_\infty - \eta}$ by

$$\alpha((1-\lambda)y_0 + \lambda y) := T^R_{\lambda,y}U^R_{\lambda,y} \quad \text{with} \;\; \lambda \in [0,1], \;\; y \in \partial B_2(y_0).$$

Arguing by contradiction, assume that I_V does not have a critical value in $(c_\infty, 2c_\infty)$. As, by Corollary 1.3.8, I_V satisfies the Palais-Smale condition on $\mathcal{N}_V$ at every level in $(c_\infty, 2c_\infty)$, there exists $\varepsilon > 0$ such that

$$\|\nabla_{\mathcal{N}_V} I_V(u)\| \geq \varepsilon, \qquad \forall u \in \mathcal{N}_V \cap I_V^{-1}[c_\infty + \delta, 2c_\infty - \eta].$$

Hence, the negative gradient flow of I_V on $\mathcal{N}_V$, which exists since $\mathcal{N}_V$ is a manifold of class C^1, yields a continuous function

$$\rho : \mathcal{N}_V \cap I_V^{2c_\infty - \eta} \to \mathcal{N}_V \cap I_V^{c_\infty + \delta}$$

such that $\rho(u) = u$ for all $u \in \mathcal{N}_V \cap I_V^{c_\infty+\delta}$ (see [2]). Now we define $\Gamma(x) := (\beta \circ \rho \circ \alpha \circ \tau)(x)$, where $\tau(x) = x + y_0$. By Lemma 1.4.8 $\Gamma(x) \neq 0$ and so the function $\tilde{h} : B_2(0) \to \partial B_2(0)$ given by

$$\tilde{h} := 2\frac{\Gamma(x)}{|\Gamma(x)|}$$

is well defined and continuous. Moreover, if $y \in \partial B_2(y_0)$, then

$$\alpha(y) = T^R_{0,y}U^R_{0,y} = T^R_{0,y}w^R_y \in \mathcal{N}_\Omega \cap I_\Omega^{c_\infty+\delta}$$

and hence

$$(\beta \circ \rho \circ \alpha)(y) = \beta(T^R_{0,y}w^R_y) = y.$$

Therefore, $h(x) = \frac{\Gamma(x)}{2}\tilde{h}(x) - y_0 = x$ for every $x \in \partial B_2(0)$ and since by Brouwer Fixed Point Theorem such a map does not exist, I_V must have a critical point $u \in \mathcal{N}_V$ with $I_V(u) \in (c_\infty, 2c_\infty)$. By Lemma 1.3.3, u does not change sign and, since f is odd, $-u$ is also a solution of (P_V). This proves that problem (P_V) has a positive solution.

Chapter 2

Zero mass limit problem

We consider the Null Mass nonlinear field equation

$$\begin{cases} -\Delta u = f(u) \quad \text{in} \quad \Omega \\ \quad u > 0 \\ \quad u \mid_{\partial\Omega} = 0 \end{cases} \tag{$\mathcal{P}$}$$

where $\mathbb{R}^N \setminus \Omega$ is regular bounded domain and like in chapter 1 there is no restriction on its size, nor any symmetry assumption. The nonlinear term f is under the double power growth condition.

2.1 Introduction

We consider the problem

$$\begin{cases} -\Delta u = f(u) \quad \text{in} \quad \Omega, \\ \quad u > 0, \\ \quad u \mid_{\partial\Omega} = 0, \end{cases} \tag{$\mathcal{P}$}$$

where $N \geq 3$, $\mathbb{R}^N \setminus \Omega \subseteq B_k(0)$ is a regular bounded domain and the conditions that we consider on the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is odd and of class $C^1(\mathbb{R}, \mathbb{R})$ such that:

(f_1) Let $F(s) := \displaystyle\int_0^s f(t)dt$, then $0 < \mu F(s) \leq f(s)s < f'(s)s^2$ for any $s \neq 0$ and for some $\mu > 2$;

(f_2) $F(0) = f(0) = f'(0) = 0$. There exist $C_1 > 0$ and $2 < p < 2^* < q$ such that

$$\begin{cases} |f^{(k)}(s)| \leq C|s|^{p-(k+1)} & \text{for } |s| \geq 1 \\ |f^{(k)}(s)| \leq C|s|^{q-(k+1)} & \text{for } |s| \leq 1 \end{cases}$$

for $k \in \{0, 1\}$, $s \in \mathbb{R}$.

Remark 2.1.1 *It is straightforward from* (f_1) *that*

$$F(s) \geq C|s|^{\mu} \quad , \quad \textit{for all} \ \ |s| \geq 1, \tag{2.1.1}$$

and by (f_2) *we can write*

$$|f^{(k)}(s)| \leq C|s|^{2^*-(k+1)} \quad , \quad \textit{for all} \ \ s \in \mathbb{R}. \tag{2.1.2}$$

Moreover, since $\mu F(s) \leq f(s)s$*, then* $C_1|s|^{\mu} \leq C_2|s|^{p}$ *and so* $\mu \leq p$*.*

A model nonlinear term which satisfies all assumptions is

$$f(u) = \begin{cases} u^q & \text{if} \quad u \leq 1 \\ a + bu + cu^p & \text{if} \quad u \geq 1 \end{cases}$$

with a choice of constants for which f belongs to C^1.

The energy functional associated with problem $(\mathcal{P})$ is

$$I_{\Omega}(u) = \frac{1}{2}\|u\|_{\Omega}^2 - \int_{\Omega} F(u)dx, \text{ with } u \in \mathcal{D}^{1,2}(\Omega).$$

The main result in this paper is the following theorem.

Theorem 2.1.2 *Assume that the positive solution in the whole of* $\mathbb{R}^N$ *is unique. Then, under assumptions* $(f_1)-(f_2)$ *, problem* $(\mathcal{P})$ *has a positive classical solution* $u \in \mathcal{D}^{1,2}(\Omega)$*.*

Remark 2.1.3 *Note that the assumption of uniqueness of positive solution in the whole of* $\mathbb{R}^N$*:*

$$\begin{cases} -\Delta u = f(u) \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \end{cases} \tag{$\mathcal{P}_{\mathbb{R}^N}$}$$

is a natural one. For instance, L. A. Caffarelli, B. Gidas and J. Spruck [18] proved that the functions

$$u_{\gamma,x_0}(x) = (N(N-2)\gamma)^{\frac{N-2}{4}}(\gamma + |x + x_0|)^{\frac{2-N}{2}}$$

are the only positive solutions of $(\mathcal{P}_{\mathbb{R}^N})$ *with* $f(u) = u^{2^*-1}$ *for some real number* $\gamma > 0$ *and point* $x_0 \in \mathbb{R}^N$.
For other non-linearities $f(u)$ *for which the uniqueness of positive solution holds see [25].*

Remark 2.1.4 *We may assume in Theorem 2.1.2 that the critical value (ground level)* c *of the functional* $I_{\mathbb{R}^N}$ *is isolated with radius* $r \geq c$, *rather than assuming the uniqueness of positive solution of* $\mathcal{P}_{\mathbb{R}^N}$.

This chapter is organized as follows. In section 2, we formulate the Orlicz space, the variational setting and present some preliminary results. Section 3 is dedicated to compactness condition. In section 4, applying a topological argument, which involves the barycenter map, we show that the energy functional associated with problem $(\mathcal{P})$ has a positive critical value.

2.2 Preliminary results

We will use the following notation:

$$\langle u, v\rangle_\Omega = \int_\Omega \nabla u \cdot \nabla v dx \quad , \quad \|u\|_\Omega^2 = \int_\Omega |\nabla u|^2 dx,$$

and we denote by $\mathcal{D}^{1,2}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|.\|_\Omega$ or $\|.\|_{\mathcal{D}^{1,2}(\Omega)}$.

Likewise we write

$$\langle u, v\rangle_{\mathbb{R}^N} = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v dx \quad , \quad \|u\|_{\mathbb{R}^N}^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx$$

and also denote by $\mathcal{D}^{1,2}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|.\|_{\mathbb{R}^N}$ or $\|.\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$. Note that, there is a canonical isometry from $\mathcal{D}^{1,2}(\Omega)$ into $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Indeed, consider the extension by 0 outside Ω. The energy functional associated with problem $(\mathcal{P})$ is

$$I_\Omega(u) = \frac{1}{2}\|u\|_\Omega^2 - \int_\Omega F(u)dx, \text{ with } u \in \mathcal{D}^{1,2}(\Omega).$$

Set

$$J_\Omega(u) = I'_\Omega(u)u = \|u\|_\Omega^2 - \int_\Omega f(u)u dx,$$

$$\mathcal{N}_\Omega := \{u \in \mathcal{D}^{1,2}(\Omega)\backslash\{0\} : J_\Omega(u) = 0\},$$

and

$$c_\Omega := \inf_{u \in \mathcal{N}_\Omega} I(u).$$

The variational approach to solve this problem requires the study of the problem $(\mathcal{P}_{\mathbb{R}^N})$ in the whole $\mathbb{R}^N$ with the functional

$$I_{\mathbb{R}^N}(u) = \frac{1}{2}\|u\|^2_{\mathbb{R}^N} - \int_{\mathbb{R}^N} F(u)dx, \text{ with } u \in \mathcal{D}^{1,2}(\mathbb{R}^N),$$

and in the same way

$$J_{\mathbb{R}^N}(u) = I'_{\mathbb{R}^N}(u)u = \|u\|^2_{\mathbb{R}^N} - \int_{\mathbb{R}^N} f(u)udx,$$

$$\mathcal{N}_{\mathbb{R}^N} := \{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\backslash\{0\} : J_{\mathbb{R}^N}(u) = 0\},$$

and

$$c := \inf_{u \in \mathcal{N}_{\mathbb{R}^N}} I_{\mathbb{R}^N}(u).$$

Let w be a positive radial solution of $(\mathcal{P}_{\mathbb{R}^N})$ which is well known to exist by [13] and by [47] there are positive constants C_1 , C_2 and C_3 such that

$$C_1(1 + |x|)^{-(N-2)} \leq |w(x)| \leq C_2(1 + |x|)^{-(N-2)}, \ \forall x \in \mathbb{R}^{\mathbb{N}}, \tag{2.2.1}$$

and

$$|\nabla w(x)| \leq C_3(1 + |x|)^{-(N-2)}, \ \forall x \in \mathbb{R}^{\mathbb{N}}. \tag{2.2.2}$$

Given $1 \leq p < q$, now we consider the space $L^p + L^q$ made up of functions $v : \Omega \to \mathbb{R}$ such that

$$v = v_1 + v_2 \quad \text{with} \quad v_1 \in L^p(\Omega), v_2 \in L^q(\Omega).$$

$L^p + L^q$ is a Banach space with the norm

$$\|v\|_{L^p+L^q} = \inf\{\|v_1\|_{L^p} + \|v_2\|_{L^q} : v = v_1 + v_2\} \tag{2.2.3}$$

and with equivalent norm

$$\|v\|_{L^p+L^q} = \sup_{\phi \neq 0} \frac{\int v(x)\phi(x)dx}{\|\phi\|_{L^{p'}} + \|\phi\|_{L^{q'}}}, \tag{2.2.4}$$

we obtain $L^p + L^q = (L^{p'} + L^{q'})'$ (see [11] and [4]).

Remark 2.2.1 *V. Benci and D. Fortunato in [11] showed that* $L^{2^*} \subset L^p + L^q$ *when* $2 < p < 2^* < q$. *Then, by the Sobolev inequality, we get the continuous embedding:*

$$\mathcal{D}^{1,2}(\Omega) \subset L^p + L^q.$$

Now we present a fundamental lemmas which its proof may be found in [12] and which will be systematically used in the forthcoming arguments.

Lemma 2.2.2 *The functional* $\mathcal{F} : L^p + L^q \to \mathbb{R}$ *defined by*

$$\mathcal{F}(u) := \int_\Omega F(u)dx,$$

is of class C^2 *and we have*

$$\mathcal{F}'(u)v = \int_\Omega f(u)vdx,$$

$$\mathcal{F}''(u)vw := \int_\Omega f'(u)vwdx.$$

Proof. [Lemma 2.6 [12] and the Appendix] □

Remark 2.2.3 *Lemma 2.2.2 ensures that the functional*

$$I_\Omega(u) = \frac{1}{2}\|u\|_\Omega^2 - \int_\Omega F(u)dx, \text{ with } u \in \mathcal{D}^{1,2}(\Omega).$$

is well defined, of class C^2 *and any critical point of* I_Ω *is a weak solution of* $(\mathcal{P})$.

Lemma 2.2.4 *(a)* $\mathcal{N}_{\mathbb{R}^N}$ *is a closed* C^1 *manifold;*

(b) given $u \neq 0$; *there exists a unique number* $t = t(u) > 0$ *such that* $ut(u) \in \mathcal{N}_{\mathbb{R}^N}$ *and* $I_{\mathbb{R}^N}(t(u)u)$ *is the maximum for* $I_{\mathbb{R}^N}(tu)$ *when* $t \geq 0$;

(c) the dependence of $t(u)$ *on* u *is of class* C^1;

(d) $\inf_{u\in\mathcal{N}_{\mathbb{R}^N}} \|u\|_{\mathbb{R}^N} = \rho > 0$.

Proof. Item (a) follows from (f_1) and Lemma 2.2.2 for $u \in \mathcal{N}_{\mathbb{R}^N}$

$$I'_{\mathbb{R}^N}(u)u = \int_{\mathbb{R}^N} 2|\nabla u|^2 - f(u)u - f'(u)u^2dx = \int_{\mathbb{R}^N} |\nabla u|^2 - f'(u)u^2dx = \int_{\mathbb{R}^N} f(u)u - f'(u)u^2dx < 0$$

and $\mathcal{N}_{\mathbb{R}^N} = J_{\mathbb{R}^N}^{-1}(\{0\})$ is closed subset of $\mathcal{D}^{1,2}(\mathbb{R}^N)$.
(b) Given $u \neq 0$, if we set

$$g_u(t) := \int_{\mathbb{R}^N} \frac{1}{2} t^2 |\nabla u|^2 - F(tu)dx \quad \text{for} \quad t \geq 0$$

we have

$$g_u'(t) = \int_{\mathbb{R}^N} t|\nabla u|^2 - f(tu)udx \quad , \quad g_u''(t) = \int_{\mathbb{R}^N} |\nabla u|^2 - f'(tu)u^2 dx.$$

By (f_1) we see that if $\bar{t} > 0$ is a critical point of g_u, then $g_u''(\bar{t}) < 0$ so $\bar{t}$ is a point of maximum for g. Furthermore, $0 = g_u(0) = g_u'(0)$ and $g_u''(0) > 0$, and hence 0 is a point of minimum for g_u. By (2.1.1) and $F(u) > 0$ in (f_1) we obtain

$$\begin{aligned} g_u(t) &\leq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - C \int_{t|u|<1} F(tu)dx - Ct^\mu \int_{t|u|>1} |u|^\mu d \\ &\leq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - c_0 t^\mu \int_{t|u|>1} |u|^\mu dx. \end{aligned}$$

Since $u \neq 0$, then there exists $\Lambda \subset \mathbb{R}^N$ with $|\Lambda| > 0$ (Lebesgue positive measure) such that $|u\,|_\Lambda\,| < 0$. By Monotone Convergence Theorem, $g_u(t) \to -\infty$ as $t \to \infty$. At this point we have the claim.
(c) We define the operator $K : \mathbb{R}^+ \times \mathcal{D}^{1,2}(\Omega) \to \mathbb{R}$ by

$$K(t,u) = t\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} f(tu)udx.$$

By Lemma 2.2.2, K is of class C^1 and if (t_0, u_0) is such that $K(t_0, u_0) = 0$ and $t_0 \neq 0$, then by (f_1)

$$K_t'(t_0, u_0) = \int_{\mathbb{R}^N} |\nabla u_0|^2 - f'(t_0 u_0)u_0^2 dx = \int_{\mathbb{R}^N} \frac{f(t_0 u_0)u_0}{t_0} - f'(t_0 u_0)u_0^2 dx < 0.$$

By the Implicit Function Theorem, we get that $u \to t(u)$ is of class C^1 and

$$t'(u_0)[\varphi] = \frac{t_0^2 \displaystyle\int_{\mathbb{R}^N} 2t_0 \nabla u_0 \nabla \varphi - f(t_0 u_0)\varphi - f'(t_0 u_0)t_0 u_0 \varphi dx}{\displaystyle\int_{\mathbb{R}^N} f'(t_0 u_0)(t_0 u_0)^2 - f(t_0 u_0)t_0 u_0 dx}$$

where $t_0 = t(u_0)$.
(d) By contradiction, suppose that the minimizing sequence (u_n) converges to 0. We set

$u_n = t_n v_n$ with $\|v_n\|_{\mathbb{R}^N} = 1$. Since $u_n \in \mathcal{N}_{\mathbb{R}^N}$ and (t_n) converges to 0, we have:

$$t_n = \int_{\mathbb{R}^N} f(t_n v_n) v_n \leq C t_n^{2^*-1} \int_{\mathbb{R}^N} |v_n|^{2^*}.$$

Hence, we get

$$1 \leq C t_n^{2^*-2} \int_{\mathbb{R}^N} |v_n|^{2^*}$$

which yields a contradiction if $t_n \to 0$.

□

Remark 2.2.5 *Similarly, by substituting $\mathbb{R}^N$ with Ω, Lemma 2.2.4 holds also for $\mathcal{N}_\Omega$.*

Remark 2.2.6 *If $u \neq 0$ is a critical point of the functional I_Ω on $\mathcal{N}_\Omega$, then u is a critical point of I_Ω. Indeed, consider $u \in \mathcal{N}_\Omega$ and use (f_1) in order to obtain*

$$\langle J'_\Omega u, u\rangle_\Omega = 2\|u\|^2_\Omega - \int_\Omega f'(u)u^2 + f(u)u \leq \int_\Omega \left(\frac{f(u)}{u} - f'(u)\right)u^2 < 0.$$

Now, suppose that $u \in \mathcal{N}_\Omega$ is a constrained critical point of I_Ω, then there exists a real number ϑ such that $I'_\Omega(u) - \vartheta J'_\Omega(u) = 0$; taking u as test function one gets $\vartheta\langle J'_\Omega u, u\rangle = 0$ then yields $\vartheta = 0$, i.e. u is a free critical point.

Lemma 2.2.7 $c_\Omega = c > 0$.

Proof. We have $c \leq c_\Omega$, because we consider $\mathcal{N}_\Omega \subset \mathcal{N}_{\mathbb{R}^N}$ (indeed $u \in \mathcal{D}^{1,2}(\Omega)$ can be extended by zero outside Ω). On the other hand, by Lemma 2.4.5 in the following section 4 we have $c_\Omega \leq c$ and so $c_\Omega = c$.
Now we show that $c > 0$. Let $(u_n) \subset \mathcal{N}_{\mathbb{R}^N}$ be a minimizing sequence of c, by (f_1) we have

$$\begin{aligned}(\frac{1}{2} - \frac{1}{\mu})\int_{\mathbb{R}^N} |\nabla u_n|^2 dx &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 - \frac{1}{\mu}\int_{\mathbb{R}^N} f(u_n)u_n \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 - \int_{\mathbb{R}^N} F(u_n) \\ &= I_{\mathbb{R}^N}(u_n). \end{aligned} \tag{2.2.5}$$

Now suppose by contradiction that $c = 0$. Then the minimizing sequence (u_n) is such that $(I_{\mathbb{R}^N}(u_n))$ goes to zero, hence by (2.2.5) (u_n) converges to zero in $\mathcal{D}^{1,2}(\Omega)$. This is absurd by Lemma 2.2.4 (a) and (d). □

Remark 2.2.8 *The existence of a ground state solution w of $(\mathcal{P}_{\mathbb{R}^N})$ was proved by H. Beresticky and P. L. Lions [13] under very general assumptions on f and a minimizer w of $c := \inf_{u\in\mathcal{N}_{\mathbb{R}^N}} I_{\mathbb{R}^N}(u)$ is such that it is a positive spherically symmetric about the origin; in other words, c is attained.*

Lemma 2.2.9 *Problem $(\mathcal{P})$ has no ground state, in other words, c_Ω is not attained.*

Proof. We proved in the previous lemma that $c_\Omega = c > 0$. At this point, we suppose by contradiction, that there exists $\overline{u} \in \mathcal{N}_\Omega$ such that $I_\Omega(\overline{u}) = c_\Omega$. Setting $\overline{u} = 0$ in $\mathbb{R}^N \setminus \Omega$, $\overline{u}$ can be regarded as an element of $\mathcal{N}_{\mathbb{R}^N}$. We can assume $\overline{u} \geq 0$ since if $\overline{u} \in \mathcal{N}_{\mathbb{R}^N}$ then $|\overline{u}| \in \mathcal{N}_{\mathbb{R}^N}$ and $I_{\mathbb{R}^N}(|\overline{u}|) = I_{\mathbb{R}^N}(\overline{u}^+ + \overline{u}^-) = I_{\mathbb{R}^N}(\overline{u}^+) + I_{\mathbb{R}^N}(\overline{u}^-) = I_{\mathbb{R}^N}(\overline{u}) = c$. Hence u is a minimizer of $I_{\mathbb{R}^N}$ on $\mathcal{N}_{\mathbb{R}^N}$ and a solution of $(\mathcal{P}_{\mathbb{R}^N})$ in $\mathbb{R}^N$. Now by Brezis-Kato theorem we see that $u \in C^2(\mathbb{R}^N)$ (we show details in the end of this chapter; this can be seen by bootstrap procedure). Then, by the strong maximum principle, $\overline{u}$ is strictly positive in $\mathbb{R}^N$ and so we have a contradiction.

Lemma 2.2.10 *For every $0 < \nu < q - 2$ and $\rho > 0$ there exists $C_\rho > 0$ such that for all $0 \leq u, v \leq \rho$ we have*

$$F(u+v) - F(u) - F(v) - f(u)v - f(v)u \geq -C_\rho(uv)^{1+\frac{\nu}{2}} \tag{2.2.6}$$

Proof. The inequality (2.2.6) is obviously satisfied if $u = 0$ or $v = 0$. By (f_1) the function $f(s)$ is increasing in $s > 0$, which yields for $u, v > 0$

$$F(u+v) - F(u) = \int_u^{u+v} f(w)dw \geq f(u)v.$$

Moreover by (f_2) for every $0 < \nu < q - 2$ it follows

$$f(u) = o(|u|^{1+\nu}) \quad \text{as} \quad |u| \to 0,$$

and so $\tilde{C}_\rho := \sup_{0<u\leq\rho} \frac{f(u)}{u^{1+\nu}} < \infty$. Now if $0 < v \leq u$, we deduce

$$F(u+v) - F(u) - F(v) - f(u)v - f(v)u \geq -F(v) - f(v)u$$

$$= \int_0^v -\frac{f(w)}{w^{1+\nu}} w^{1+\nu} dw - \frac{f(v)}{v^{1+\nu}} uv^{1+\nu} \geq -\tilde{C}_\rho \frac{v^{2+\nu}}{2+\nu} - \tilde{C}_\rho uv^{1+\nu}$$

$$\geq \big[-(\frac{v}{u})^{\frac{\nu}{2}}((\frac{v}{u})^{\frac{\nu}{2}} + \frac{1}{2}(\frac{v}{u})^{1+\frac{\nu}{2}})\big]\tilde{C}_\rho(uv)^{1+\frac{\nu}{2}} \geq -\frac{3}{2}\tilde{C}_\rho(uv)^{1+\frac{\nu}{2}}.$$

Using the symmetry of the expressions with respect to u and v, the same estimate holds for $0 < u \leq v$, and the proof is complete. $\square$

Now let $y_0 \in \mathbb{R}^N$ with $|y_0| = 1$ be fixed and let $B_2(y_0) := \{x \in \mathbb{R}^N : |x - y_0| \leq 2\}$. We write for each $y \in \partial B_2(y_0)$ and $R > 0$

$$w_0^R := w(\cdot - Ry_0) \quad , \quad w_y^R := w(\cdot - Ry).$$

where w is the positive radial solution of $(\mathcal{P}_{\mathbb{R}^N})$.

Lemma 2.2.11 *Let $r > 1$ and $R > 0$ large enough, then*

$$a) \int_{B_{2K}(0)} |w_0^R|^r \leq CR^{-r(N-2)} \quad \textit{and} \quad \int_{B_{2K}(0)} |w_y^R|^r \leq CR^{-r(N-2)}; \tag{2.2.7}$$

$$b) \int_{B_{2K}(0)} |\nabla w_0^R|^r \leq CR^{-r(N-2)} \quad \textit{and} \quad \int_{B_{2K}(0)} |\nabla w_y^R|^r \leq CR^{-r(N-2)}. \tag{2.2.8}$$

Proof. In order to prove the first estimate, note that for $2K < \frac{1}{2}R$

$$\frac{1}{2}R < R - \frac{1}{2}R < |Ry_0| - |x| < |x - Ry_0| < 1 + |x - Ry_0|. \tag{2.2.9}$$

Now by (2.2.9), (2.2.1) and $r > 1$, we have

$$\int_{B_{2K}(0)} |w(x - Ry_0)|^r dx \leq C \int_{B_{2K}(0)} (1 + |x - Ry_0|)^{-r(N-2)} dx \leq CR^{-r(N-2)}.$$

The proofs of the other estimates in (2.2.7) and (2.2.8) are similar. $\square$

Now we are going to obtain a more delicate estimate of the integrals in the whole $\mathbb{R}^N$ that we need this estimate for the proof of important Proposition 2.4.4, the proof of this lemma is inspired by the work of M. Clapp and L. Maia [25].

Lemma 2.2.12 *Let $r > 2^*/2$ and $s \geq 1$ then*

$$\int_{\mathbb{R}^N} (w_0^R)^r (w_y^R)^s \leq CR^{-s(N-2)}, \tag{2.2.10}$$

and

$$\int_{\mathbb{R}^N} (w_y^R)^r (w_0^R)^s \leq CR^{-s(N-2)}. \tag{2.2.11}$$

Proof. In order to prove the first estimate of the integral

$$\int_{\mathbb{R}^N}(w_0^R)^r(w_y^R)^s = \int_{\mathbb{R}^N}(w(x-Ry_0))^r(w(x-Ry))^s dx,$$

we consider the change of variables $x = z + \frac{Ry_0+Ry}{2}$, thus

$$\int_{\mathbb{R}^N}(w(x-Ry_0))^r(w(x-Ry))^s dx = \int_{\mathbb{R}^N}\big(w\big(z-\frac{Ry_0-Ry}{2}\big)\big)^r\big(w\big(z+\frac{Ry_0-Ry}{2}\big)\big)^s dz,$$

$$= \int_{\mathbb{R}^N}(w(z-P_R))^r(w(z+P_R))^s dz = 2\int_{\mathbb{Q}^+}(w(z-P_R))^r(w(z+P_R))^s dz$$

$$= 2\int_{B_1(P_R)}(w(z-P_R))^r(w(z+P_R))^s dz + 2\int_{\mathbb{Q}^+\setminus B_1(P_R)}(w(z+P_R))^r(w(z-P_R))^s dz,$$

by denoting $P_R = \frac{Ry_0-Ry}{2}$, using the symmetry of the integrals and denoting $\mathbb{Q}^+ = \{z \in \mathbb{R}^N : \langle z-P_R, P_R\rangle \geq 0\}$.
Note that for $\xi \in \mathbb{Q}^+$ and R sufficiently large

$$\begin{cases} \text{if } \ |\xi| > 1 \ \text{ then } \ R < 1 + |\xi + 2P_R|, \\ \text{if } \ |\xi| < 1 \ \text{ then } \ 2R < 1 + |\xi + 2P_R|. \end{cases} \tag{2.2.12}$$

Now by another change of variables $\xi = z - P_R$, with (2.2.12) and (2.2.1), we obtain

$$\int_{\mathbb{R}^N}(w_0^R)^r(w_y^R)^s = 2\int_{B_1(0)}(w(\xi))^r(w(\xi+2P_R))^s d\xi + 2\int_{\{\mathbb{Q}^+-P_R\}\setminus B_1(0)}(w(\xi))^r(w(\xi+2P_R))^s d\xi$$

$$\leq C\int_{B_1(0)}(1+|\xi+2P_R|)^{-s(N-2)}d\xi + C\int_{\{\mathbb{Q}^+-P_R\}\setminus B_1(0)}(1+|\xi|)^{-r(N-2)}(1+|\xi+2P_R|)^{-s(N-2)}d\xi$$

$$\leq CR^{-s(N-2)}\int_{B_1(0)}d\xi + CR^{-s(N-2)}\int_{\{\mathbb{Q}^+-P_R\}\setminus B_1(0)}|\xi|^{-r(N-2)}d\xi$$

$$\leq CR^{-s(N-2)},$$

since

$$\int_{\{\mathbb{Q}^+-P_R\}\setminus B_1(0)}|\xi|^{-r(N-2)}d\xi < \int_1^\infty y^{-r(N-2)}y^{N-1}dy$$

and for $r > \frac{2^*}{2}$ we have $-r(N-2)+N-1 < -1$. The proof of estimative (2.2.11) is similar and this completes the proof of this lemma. □

Define for $\lambda \in [0,1]$

$$Z_{\lambda,y}^R := \lambda w_0^R + (1-\lambda)w_y^R$$

and

$$U^R_{\lambda,y} := Z^R_{\lambda,y}\psi \tag{2.2.13}$$

where $\psi \in C^\infty(\mathbb{R}^N)$ is continuous radially symmetric and increasing cutoff function such that

$$\psi(x) = \begin{cases} 0 & |x| \le K, \\ 0 < \psi < 1 & K < |x| < 2K, \\ 1 & |x| \ge 2K. \end{cases}$$

where K is the radius of the smallest sphere $B_K(0)$ that contains $\mathbb{R}^N \backslash \Omega$. We can consider $U^R_{\lambda,y} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ by extending $U^R_{\lambda,y} = 0$ outside Ω.

Lemma 2.2.13 *$U^R_{\lambda,y} - Z^R_{\lambda,y} \to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, as $R \to \infty$.*

Proof. First of all, if $R > 0$ is sufficiently large we claim that

$$\|\nabla w^R_0 - \nabla(\psi w^R_0)\|_{L^2(B_{2K}(0))} \le CR^{-(N-2)} \tag{2.2.14}$$

and

$$\|\nabla w^R_y - \nabla(\psi w^R_y)\|_{L^2(B_{2K}(0))} \le CR^{-(N-2)}. \tag{2.2.15}$$

By the claim we have

$$\|U^R_{\lambda,y} - Z^R_{\lambda,y}\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \le \lambda\|w^R_0 - \psi w^R_0\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)} + (1-\lambda)\|w^R_y - \psi w^R_y\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}$$

$$= \lambda\|\nabla w^R_0 - \nabla(\psi w^R_0)\|_{L^2(B_{2K}(0))} + (1-\lambda)\|w^R_y - \psi w^R_y\|_{L^2(B_{2K}(0))} \le CR^{-(N-2)}$$

and this shows that $U^R_{\lambda,y} - Z^R_{\lambda,y} \to 0$ if $R \to \infty$, which concludes proof of the lemma. Now, in order to complete this proof we have to show the claim. Since $\psi \in C^\infty$, then there exist positive constants C_1 and C_2 such that

$$|\nabla(\psi w^R_0)| = |(\nabla\psi)w^R_0 + (\nabla w^R_0)\psi| \le C_1|w^R_0| + C_2|\nabla w^R_0| \quad \text{in} \quad B_{2K}(0) \tag{2.2.16}$$

and so by Lemma 2.2.11 with $r = 2$ and (2.2.16),

$$\|\nabla w^R_0 - \nabla(\psi w^R_0)\|^2_{L^2(B_{2K}(0))} \le \int_{B_{2K}(0)} \left(C_1|w^R_0| + (C_2+1)|\nabla w^R_0|\right)^2 dx$$

$$\le CR^{-2(N-2)}$$

as claimed. □

Lemma 2.2.14 *For $t>0$, $J_{\mathbb{R}^N}(tU^R_{\lambda,y}) - J_{\mathbb{R}^N}(tZ^R_{\lambda,y}) \to 0$, as $R\to\infty$.*

Proof. By the definition of $J_{\mathbb{R}^N}$ we have

$$|J_{\mathbb{R}^N}(tU^R_{\lambda,y}) - J_{\mathbb{R}^N}(tZ^R_{\lambda,y})|$$

$$= \left| \|tU^R_{\lambda,y}\|^2_{\mathbb{R}^N} - \int_{\mathbb{R}^N} f(tU^R_{\lambda,y})(tU^R_{\lambda,y}) - \|tZ^R_{\lambda,y}\|^2_{\mathbb{R}^N} + \int_{\mathbb{R}^N} f(tZ^R_{\lambda,y})(tZ^R_{\lambda,y}) \right|$$

$$\le \|tU^R_{\lambda,y} - tZ^R_{\lambda,y}\|^2_{\mathbb{R}^N} + \left| \int_{\mathbb{R}^N} f(tZ^R_{\lambda,y})tZ^R_{\lambda,y} - f(tU^R_{\lambda,y})tU^R_{\lambda,y} \right|. \tag{2.2.17}$$

By Lemma 2.2.13 the first term of (2.2.17) is equal to $o_R(1)$ where $o_R(1)\to 0$ as $R\to 0$, so it's enough to show that

$$\left| \int_{\mathbb{R}^N} f(tZ^R_{\lambda,y})(tZ^R_{\lambda,y}) - f(tU^R_{\lambda,y})(tU^R_{\lambda,y}) \right| = \left| \int_{B_{2K}(0)} f(tZ^R_{\lambda,y})(tZ^R_{\lambda,y}) - f(tU^R_{\lambda,y})(tU^R_{\lambda,y}) \right| = o_R(1).$$

For this purpose, (2.1.2), Lemma 2.2.11 and the inequality $(a+b)^p \le 2^p(a^p+b^p)$ yield

$$\left| \int_{B_{2K}(0)} f(tZ^R_{\lambda,y})(tZ^R_{\lambda,y}) - f(tU^R_{\lambda,y})(tU^R_{\lambda,y}) \right| \le \int_{B_{2K}(0)} |tZ^R_{\lambda,y}|^{2^*} + \ |tU^R_{\lambda,y}|^{2^*}$$

$$\le \int_{B_{2K}(0)} |1+\psi^{2^*}||tZ^R_{\lambda,y}|^{2^*} \le C\int_{B_{2K}(0)} |Z^R_{\lambda,y}|^{2^*} \le C\int_{B_{2K}(0)} |\lambda w^R_0 + (1-\lambda)w^R_y|^{2^*}$$

$$\le C\int_{B_{2K}(0)} |w^R_0|^{2^*} + |w^R_y|^{2^*} \le CR^{-2^*(N-2)} = o_R(1)$$

□

Lemma 2.2.15 *a) There exist $R_0>0$, $T_0>2$ and for each $R\ge R_0$, $y\in\partial B_2(y_0)$ and $\lambda\in[0,1]$, a unique $T^R_{\lambda,y}$ such that*

$$T^R_{\lambda,y}U^R_{\lambda,y} \in \mathcal{N}_\Omega,$$

$T^R_{\lambda,y}\in[0,T_0]$ and $T^R_{\lambda,y}$ is a continuous function of the variables λ, y and R.
b) for $\lambda=1/2$ we have $T^R_{\frac{1}{2},y}\to 2$ as $R\to\infty$ uniformly in $y\in\partial B_2(y_0)$.

Proof. By Lemma 2.2.4 for each $R\ge 0$, $y\in\partial B_2(y_0)$ and $\lambda\in[0,1]$ there exists $T^R_{\lambda,y} = t(U^R_{\lambda,y})$ such that $t\in C^1$. Now for such fixed $R>0$, the function $(\lambda,y)\to U^R_{\lambda,y}$ is

continuous and $t(U^R_{\lambda,y})$ is in C^1, since $[0,1]\times\partial B_2(y_0)$ is a compact set in $\mathbb{R}^2$, then there is $T_0(R)=\max\limits_{(\lambda,y)\in[0,1]\times\partial B_2(y_0)} T^R_{\lambda,y}$ such that $T^R_{\lambda,y}U^R_{\lambda,y}\in\mathcal{N}_\Omega$ and $T^R_{\lambda,y}\in[0,T_0(R)]$.
Supose by contradiction that $T_0(R_j)\longrightarrow\infty$ as $R_j\longrightarrow\infty$, since $T_0(R_j)=\max\limits_{(\lambda,y)\in[0,1]\times\partial B_2(y_0)} T^{R_j}_{\lambda,y}$, then $T_0(R_j)=T^{R_j}_{\lambda,y}$ for some (λ,y). Let $u,v>0$, and $r\in(0,\infty)$, using that $\frac{f(s)}{s}$ is increasing by assumption (f_1),

$$J_{\mathbb{R}^N}(ru+rv)=r^2(\|u\|^2_{\mathbb{R}^N}+\|v\|^2_{\mathbb{R}^N}+2\langle u,v\rangle_{\mathbb{R}^N})-\int_{\mathbb{R}^N}\frac{f(ru+rv)}{ru+rv}(ru+rv)^2$$

$$\leq r^2\left(\|u\|^2_{\mathbb{R}^N}-\int_{\mathbb{R}^N}\frac{f(ru)}{ru}u^2+\|v\|^2_{\mathbb{R}^N}-\int_{\mathbb{R}^N}\frac{f(rv)}{rv}v^2+2\langle u,v\rangle_{\mathbb{R}^N}\right). \qquad (2.2.18)$$

Now for $\lambda\in[0,1]$ and $y\in\partial B_2(y_0)$, setting $u:=\lambda w_0^{R_j}$, $v:=(1-\lambda)w_y^{R_j}$, $r=T^{R_j}_{\lambda,y}$ and (2.2.18), we obtain

$$0=J_{\mathbb{R}^N}(T^{R_j}_{\lambda,y}U^{R_j}_{\lambda,y})$$

$$\leq (T^{R_j}_{\lambda,y})^2(\|\lambda w_0^{R_j}\|^2_{\mathbb{R}^N}-\int_{\mathbb{R}^N}\frac{f(T^{R_j}_{\lambda,y}\lambda w_0^{R_j})}{T^{R_j}_{\lambda,y}\lambda w_0^{R_j}}(\lambda w_0^{R_j})^2$$

$$+\|(1-\lambda)w_y^{R_j}\|^2_{\mathbb{R}^N}-\int_{\mathbb{R}^N}\frac{f(T^{R_j}_{\lambda,y}(1-\lambda)w_y^{R_j})}{T^{R_j}_{\lambda,y}(1-\lambda)w_y^{R_j})}\left((1-\lambda)w_y^{R_j}\right)^2+2\langle\lambda w_0^{R_j},(1-\lambda)w_y^{R_j}\rangle_{\mathbb{R}^N}\Big)$$

$$\leq (T^{R_j}_{\lambda,y})^2\left\{\int_{\mathbb{R}^N}\left(\frac{f(w_0^{R_j})}{w_0^{R_j}}-\frac{f(T^{R_j}_{\lambda,y}\lambda w_0^{R_j})}{T^{R_j}_{\lambda,y}\lambda w_0^{R_j}}\right)(\lambda w_0^{R_j})^2\right.$$

$$\left.+\int_{\mathbb{R}^N}\left(\frac{f(w_y^{R_j})}{w_y^{R_j}}-\frac{f(T^{R_j}_{\lambda,y}(1-\lambda)w_y^{R_j})}{T^{R_j}_{\lambda,y}(1-\lambda)w_y^{R_j}}\right)\left((1-\lambda)w_y^{R_j}\right)^2+o_R(1)\right\}.$$

As we are assuming that $T^{R_j}_{\lambda,y}\to\infty$ as $R_j\to\infty$, then we get a contradiction since by (f_1) and the Monotone Convergence Theorem

$$\int_{\mathbb{R}^N}\left(\frac{f(w_0^R)}{w_0^R}-\frac{f(T^R_{\lambda,y}\lambda w_0^R)}{T^R_{\lambda,y}\lambda w_0^R}\right)(\lambda w_0^R)^2<S_0<0$$

and

$$\int_{\mathbb{R}^N}\left(\frac{f(w_y^R)}{w_y^R}-\frac{f(T^R_{\lambda,y}(1-\lambda)w_y^R)}{T^R_{\lambda,y}(1-\lambda)w_y^R}\right)\left((1-\lambda)w_y^R\right)^2<S_0<0,$$

for $R_j>R_0$, $\lambda\in[0,1]$ and $y\in\partial B_2(y_0)$, where S_0 may be taken, for instance, as $S_0:=\frac{f(w_0^R)}{w_0^R}-\frac{f(2w_0^R)}{2w_0^R}$.
In order to prove part (b) let $\varphi(u,v)=f(u+v)-f(u)-f(v)$ from Lemma 2.2.12 we

have

$$\int_{\mathbb{R}^N} |\varphi(w_0^R, w_y^R)(w_0^R + w_y^R)| \leq \int_{\mathbb{R}^N} (w_0^R w_y^R)^q (w_0^R + w_y^R) = o_R(1).$$

and

$$J_{\mathbb{R}^N}(w_0^R + w_y^R) = \|w_0^R + w_y^R\|_{\mathbb{R}^N}^2 - \int_{\mathbb{R}^N} f(w_0^R + w_y^R)(w_0^R + w_y^R)$$

$$= \|w_0^R\|_{\mathbb{R}^N}^2 + \|w_y^R\|_{\mathbb{R}^N}^2 + 2\langle w_0^R, w_y^R\rangle_{\mathbb{R}^N} - \int_{\mathbb{R}^N} f(w_0^R)(w_0^R) - \int_{\mathbb{R}^N} f(w_y^R)(w_y^R) -$$

$$\int_{\mathbb{R}^N} f(w_0^R)(w_y^R) - \int_{\mathbb{R}^N} f(w_y^R)(w_0^R) + \int_{\mathbb{R}^N} \varphi(w_0^R, w_y^R)(w_0^R + w_y^R)$$

$$= J_{\mathbb{R}^N}(w_0^R) + J_{\mathbb{R}^N}(w_y^R) + o_R(1) = o_R(1),$$

because w is a solution of $(\mathcal{P}_{\mathbb{R}^N})$. So by Lemma 2.2.14 we have

$$J_{\mathbb{R}^N}((w_0^R + w_y^R)\psi) = J_{\mathbb{R}^N}(w_0^R + w_y^R) + o_R(1) = o_R(1) \quad \text{as} \quad R \to \infty. \tag{2.2.19}$$

Therefore, by (2.2.19) and $\mathcal{D}^{1,2}(\Omega) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$

$$J_\Omega(2U_{\frac{1}{2},y}^R) = J_\Omega\big((w_0^R + w_y^R)\psi)\big)$$

$$= J_{\mathbb{R}^N}\big((w_0^R + w_y^R)\psi\big) = o_R(1)$$

and so $T_{\frac{1}{2},y}^R \to 2$. Indeed, without loss of generality, suppose by contradiction that $T_{\frac{1}{2},y}^R \to T > 2$. Given $\delta > 1$ such that $2 < 2\delta < T$, there exists $R_0 > 0$ such that $T_{\frac{1}{2},y}^R > 2\delta$ for all $R > R_0$, $y \in \partial B_2(y_0)$. Then by the previous argument, $f(s)/s$ increasing and the translation invariance of integrals

$$0 = J_{\mathbb{R}^N}\left(\frac{T_{\frac{1}{2},y}^R}{2} w_0^R + \frac{T_{\frac{1}{2},y}^R}{2} w_y^R\right) \leq \left\|\frac{T_{\frac{1}{2},y}^R}{2} w_0^R\right\|_{\mathbb{R}^N}^2 - \int_{\mathbb{R}^N} \left(\frac{f(\frac{T_{\frac{1}{2},y}^R}{2} w_0^R)}{\frac{T_{\frac{1}{2},y}^R}{2} w_0^R}\right)\left(\frac{T_{\frac{1}{2},y}^R}{2} w_0^R\right)^2$$

$$+\left\|\frac{T_{\frac{1}{2},y}^R}{2} w_y^R\right\|_{\mathbb{R}^N}^2 - \int_{\mathbb{R}^N} \left(\frac{f(\frac{T_{\frac{1}{2},y}^R}{2} w_y^R)}{\frac{T_{\frac{1}{2},y}^R}{2} w_y^R}\right)\left(\frac{T_{\frac{1}{2},y}^R}{2} w_y^R\right)^2 + o_R(1)$$

$$\leq 2\int_{\mathbb{R}^N} \left(\frac{f(w)}{w} - \frac{f(\delta w)}{\delta w}\right)(\delta w)^2 + o_R(1) < 0$$

and this is contradiction. Likewise if $T_{\frac{1}{2},y}^R \to T < 2$ then $J_{\mathbb{R}^N}(\frac{T}{2} w_0^R + \frac{T}{2} w_y^R) > 0$, and this completes the proof of the lemma. □

2.3 Compactness condition

First we present two fundamental lemmas which will be used in the proof of Splitting Lemma.

Lemma 2.3.1 *(a) If v and u are in a bounded subset of $L^p + L^q$, then $f'(v)u$ is in a bounded subset of $L^{p'} + L^{q'}$;*

(b) f' is a bounded map from $L^p + L^q$ into $L^{p/p-2} + L^{q/q-2}$.

Proof. [Lemma 2.3 [12] and the Appendix] □

Lemma 2.3.2 *Assume that the sequence $\{u_k\}$ converges to u_0 weakly in $\mathcal{D}^{1,2}(\Omega)$. Set $u_k^1 = u_k - u_0$ then it holds:*

(a) $\|u_k^1\|^2_{\mathcal{D}^{1,2}(\Omega)} = \|u_k\|^2_{\mathcal{D}^{1,2}(\Omega)} - \|u_0\|^2_{\mathcal{D}^{1,2}(\Omega)} + o(1)$;

(b) $\displaystyle\int_\Omega f(u_k^1)u_k^1 = \int_\Omega f(u_k)u_k - \int_\Omega f(u_0)u_0 + o(1)$;

(c) $\displaystyle\int_\Omega F(u_k^1) = \int_\Omega F(u_k) - \int_\Omega F(u_0) + o(1)$.

Proof. [Lemma 2.8, [12]] and Lemma 3.6 in [25]. □

Note that $I'_{\mathcal{N}_V} I(u)$ is orthogonal projection of $I'_\Omega(u)$ onto the tangent space of $\mathcal{N}_\Omega$ at u, that is defined by $T_u(\mathcal{N}_\Omega) := \{v \in D\mathcal{D}^{1,2}(\Omega);\quad J'_\Omega(u)v = 0\}$. Recall that a sequence (u_k) in $\mathcal{D}^{1,2}(\Omega)$ is said to be a $(PS)_d$-sequence for I_Ω restricted to $\mathcal{N}_\Omega$ if $I_\Omega(u_k) \to d$ and $\|I'_{\mathcal{N}_\Omega}(u_k)\| \to 0$. The functional I_Ω satisfies the Palais-Smale condition on $\mathcal{N}_\Omega$ at the level d if every $(PS)_d$-sequence for I_Ω on $\mathcal{N}_\Omega$ contains a convergent subsequence.

Now we proceed with the study of Palais Smale sequences of I_Ω. Usually the compactness results depend on P. L. Lion's Lemma [35]. However that lemma does not apply directly if (u_k) is bounded in $\mathcal{D}^{1,2}(\Omega)$. We obtain the following result inspired by the work of A. Azzollini, V. Benci, T. D'Aprile and D. Fortunato [Lemma 2, [3]].

Lemma 2.3.3 *Suppose (u_k) is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and there exists $R > 0$ such that*

$$\lim_{k\to\infty} (\sup_{y\in\mathbb{R}^N} \int_{B(y,R)} |u_k|^2) = 0,$$

then $\displaystyle\int_{\mathbb{R}^N} f(u_k)u_k \to 0.$

Proof. Fix $\varepsilon \in (0,1)$ and for every k consider the new sequence of functions

$$w_k := \begin{cases} |u_k| & |u_k| \geq \varepsilon, \\ |u_k|^{2^*/2}\varepsilon^{-(2^*/2-1)} & |u_k| \leq \varepsilon. \end{cases}$$

It is easy to verify

$$|w_k|^2 \leq |u_k|^2 \quad , \quad |w_k|^2 \leq |u_k|^{2^*}\varepsilon^{-(2^*-2)} \quad , \quad |\nabla w_k|^2 \leq \left(\frac{2^*}{2}\right)^2 |\nabla u_k|^2,$$

since

$$|w_k|^2 = \begin{cases} |u_k|^2 & |u_k| \geq \varepsilon, \\ |u_k|^{2^*}\varepsilon^{-(2^*-2)} \leq |u_k|^2 \frac{|u_k|^{2^*-2}}{\varepsilon^{(2^*-2)}} \leq |u_k|^2 & |u_k| \leq \varepsilon, \end{cases}$$

$$|w_k|^2 = \begin{cases} |u_k|^2 = \frac{|u_k|^{2^*}}{|u_k|^{2^*-2}} \leq \frac{|u_k|^{2^*}}{\varepsilon^{2^*-2}} = |u_k|^{2^*}\varepsilon^{-(2^*-2)} & |u_k| \geq \varepsilon, \\ |u_k|^{2^*}\varepsilon^{-(2^*-2)} & |u_k| \leq \varepsilon, \end{cases}$$

$$\nabla w_k = \begin{cases} \nabla |u_k| & |u_k| \geq \varepsilon, \\ \nabla(|u_k|^{2^*/2}\varepsilon^{-(2^*/2-1)}) = \frac{2^*}{2}\varepsilon^{-(2^*/2-1)}|u_k|^{2^*/2-1}\nabla|u_k| \leq \frac{2^*}{2}\nabla|u_k| & |u_k| \leq \varepsilon. \end{cases}$$

And so

$$\|w_k\|^2_{H^1(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |w_k|^2 + |\nabla w_k|^2$$

$$\leq \int_{\mathbb{R}^N} |u_k|^{2^*}\varepsilon^{-(2^*-2)} + \int_{\mathbb{R}^N} (\frac{2^*}{2})^2 |\nabla u_k|^2 \leq C\varepsilon^{-(2^*-2)},$$

in particular $w_k \in H^1(\mathbb{R}^N)$. We claim that

$$w_k \to 0 \text{ in } L^s(\mathbb{R}^N) \text{ for each } 2 < s < 2^*.$$

Indeed for any $y \in \mathbb{R}^N$ and $s \in (2, 2^*)$, using the Sobolev continuous embedding $H^1(B(y,R)) \hookrightarrow L^{2^*}(B(y,R))$ we have

$$\int_{B(y,R)} |w_k|^s \leq \left(\int_{B(y,R)} |w_k|^2\right)^{(1-\theta)s/2} \left(\int_{B(y,R)} |w_k|^{2^*}\right)^{\theta s/2^*}$$

$$\leq C\left(\int_{B(y,R)} |w_k|^2\right)^{(1-\theta)s/2} \left(\int_{B(y,R)} |w_k|^2 + |\nabla w_k|^2\right)^{\theta s/2},$$

where $\theta = \frac{s-2}{2s}N$. Now suppose $\theta s \geq 2$ i.e. $s \geq \frac{4}{N} + 2 = \overline{s}$, then

$$\int_{B(y,R)} |w_k|^s \leq C\left(\int_{B(y,R)} |w_k|^2\right)^{(1-\theta)s/2} \left(\int_{B(y,R)} |w_k|^2 + |\nabla w_k|^2\right) \|w_k\|^{\theta s-2}_{H^1(\mathbb{R}^N)}.$$

Now, covering $\mathbb{R}^N$ by balls of radius R, in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls, we find

$$\int_{\mathbb{R}^N} |w_k|^s \leq (N+1) \sup_{y\in\mathbb{R}^N} \left(\int_{B(y,R)} |w_k|^2\right)^{(1-\theta)s/2} \|w_k\|^{\theta s}_{H^1(\mathbb{R}^N)}.$$

But $w_k \in H^1(\mathbb{R}^N)$ and so by the assumption of lemma, $w_k \to 0$ in $L^s(\mathbb{R}^N)$ for $s \geq \overline{s}$. If $2 < s < \overline{s}$, $s = 2\theta + \overline{s}(1-\theta)$ for some $\theta \in (0,1)$, hence by the Holder inequality,

$$\|w_k\|^s_{L^{\overline{s}}(\mathbb{R}^N)} \leq \|w_k\|^{\theta}_{L^2(\mathbb{R}^N)} \|w_k\|^{1-\theta}_{L^s(\mathbb{R}^N)}$$

and the claim then follows from the case already established. Now using (f_2) we conclude

$$\int_{\mathbb{R}^N} f(u_k)u_k \leq C\int_{\{|u_k|\geq 1\}} |u_k|^p + C\int_{\{|u_k|\leq 1\}} |u_k|^q$$

$$\leq C\int_{\{|u_k|\geq \varepsilon\}} |u_k|^p - C\int_{\{\varepsilon\leq|u_k|\leq 1\}} |u_k|^p + C\int_{\{\varepsilon\leq|u_k|\leq 1\}} |u_k|^q + C\int_{\{|u_k|\leq \varepsilon\}} |u_k|^q$$

$$\leq C\int_{\{|u_k|\geq \varepsilon\}} |u_k|^p - C\int_{\{\varepsilon\leq|u_k|\leq 1\}} |u_k|^p + C\int_{\{\varepsilon\leq|u_k|\leq 1\}} |u_k|^p + C\int_{\{|u_k|\leq \varepsilon\}} |u_k|^q$$

$$= C\int_{\{|u_k|\geq \varepsilon\}} |u_k|^p + C\int_{\{|u_k|\leq \varepsilon\}} |u_k|^q$$

$$\leq C\int_{\{|u_k|\geq \varepsilon\}} |w_k|^p + C\int_{\{|u_k|\leq \varepsilon\}} |u_k|^{q-2^*}|u_k|^{2^*}$$

$$\leq C\|w_k\|^p_{L^p(\mathbb{R}^N)} + C\varepsilon^{q-2^*}\|u_k\|^{2^*}_{L^{2^*}(\mathbb{R}^N)}$$

by which, since $w_k \to 0$ and $q > 2^*$

$$\int_{\mathbb{R}^N} f(u_k)u_k \leq C\varepsilon^{q-2^*}.$$

Because $\varepsilon \in (0,1)$ is arbitrary we get the conclusion. □

Lemma 2.3.4 *Every $(PS)_d$-sequence (u_k) for I_Ω restricted the $\mathcal{N}_\Omega$ contains a bounded subsequence which is a $(PS)_d$-sequence for I_Ω in $\mathcal{D}^{1,2}(\Omega)$.*

Proof. Let (u_k) be a $(PS)_d$-sequence for I_Ω on $\mathcal{N}_\Omega$, by (2.2.5) with replaced $\mathbb{R}^N$ by Ω and $I_\Omega(u_k) \to d$ we have that (u_k) is bounded. To complete the proof we show that $I'_{\mathcal{N}_\Omega}(u_k) \to 0$ imply

$$I'_\Omega(u_k) \to 0 \quad \text{in} \;\; (\mathcal{D}^{1,2}(\Omega))'. \tag{2.3.1}$$

Write

$$I'_{\Omega}(u_k) = I'_{\mathcal{N}_\Omega}(u_k) + t_k J'_{\Omega}(u_k) \tag{2.3.2}$$

By property (f_2) and Remark 2.1.1, Hölder's inequality, Sobolev inequality and the boundednes of (u_k), for any $v \in \mathcal{D}^{1,2}(\Omega)$,

$$|\int_{\Omega}[f'(u_k)u_k - f(u_k)]v| \leq C\int_{\Omega}(|u_k|^{2^*-1})|v| \leq C\|u_k\|_{L^{2^*}}^{2^*-1}\|v\|_{L^{2^*}} \leq C\|v\|_{\Omega}.$$

Therefore

$$|\langle J'_{\Omega}(u_k), v\rangle_{\Omega}| = |2\langle u_k, v\rangle_{\Omega} - \int_{\Omega}[f'(u_k)u_k + f(u_k)]v| \leq C\|v\| \ , \quad \forall v \in \mathcal{D}^{1,2}(\Omega).$$

This proves that$(J'_{\Omega}(u_k))$ is bounded in $(\mathcal{D}^{1,2}(\Omega))'$.

As $|J'_{\Omega}(u_k)u_k| \leq \|J'_{\Omega}(u_k)\|\|u_k\|_{\Omega} < C$, after passing to a subsequence, we have that $|J'_{\Omega}(u_k)u_k| \to \varrho \geq 0$. We will show that $\varrho > 0$. From Lemma 2.2.4 (d) and $u_k \in \mathcal{N}_V$, we have

$$0 < \rho^2 \quad \leq \quad \|u_k\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} f(u_k)u_k, \tag{2.3.3}$$

then by Lemma 2.3.3 there is $\delta > 0$ such that

$$\sup_{y\in\mathbb{R}^N}\int_{B(y,R)}|u_k|^2 > \delta,$$

and so there exists a sequence (y_k) such that

$$\int_{B(y_k,R)}|u_k|^2 \geq \delta. \tag{2.3.4}$$

Now consider $\tilde{u}_k = u_k(\cdot - y_k)$, which is bounded and passing to a subsequence, $\tilde{u}_k \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $\tilde{u}_k \to u$ in $L^2_{loc}(\mathbb{R}^N)$. We claim that $u \not\equiv 0$. Indeed if $\|\tilde{u}_k\|_{L^2(B(0,R))} \to 0$ as $k \to \infty$ we have a contradiction with (2.3.4). Hence, $u \not\equiv 0$ and there exists a subset Λ of positive measure such that $u(x) \not\equiv 0$ for every $x \in \Lambda$. Property (f_1) implies that $f'(s)s^2 - f(s)s > 0$ if $s \neq 0$. So, from Fatou's lemma, we conclude that

$$\varrho = \liminf_{k\to\infty}|J'(u_k)u_k| = \liminf_{k\to\infty}\left\{2\|u_k\|^2 - \int_{\Omega}[f'(u_k)u_k^2 + f(u_k)u_k]\right\}$$

$$= \liminf_{k\to\infty}\int_{\Omega}[f'(u_k)u_k^2 - f(u_k)u_k]$$

$$= \liminf_{k\to\infty} \int_{\mathbb{R}^N} [f'(u_k)u_k^2 - f(u_k)u_k] = \liminf_{k\to\infty} \int_{\mathbb{R}^N} [f'(\tilde{u}_k)\tilde{u}_k^2 - f(\tilde{u}_k)\tilde{u}_k]$$

$$\geq \liminf_{k\to\infty} \int_\Lambda [f'(\tilde{u}_k)\tilde{u}_k^2 - f(\tilde{u}_k)\tilde{u}_k] \geq \int_\Lambda \liminf_{k\to\infty} [f'(\tilde{u}_k)\tilde{u}_k^2 - f(\tilde{u}_k)\tilde{u}_k] = \int_\Lambda [f'(u)u^2 - f(u)u] > 0,$$

and the claim that $\varrho > 0$ is proved. Taking the inner product of (2.2.7) with u_k we obtain

$$0 = I'_\Omega(u_k)u_k = \langle I'_{\mathcal{N}_\Omega}(u_k), u_k\rangle_\Omega + t_k J'_{\mathcal{N}_\Omega}(u_k)u_k = o_k(1) + t_k J'_{\mathcal{N}_\Omega}(u_k)u_k,$$

so $t_k \to 0$ and from (2.2.7) we deduce $I'_\Omega(u_k) \to 0$ as $I'_{\mathcal{N}_\Omega}(u_k) \to 0$ and this completes the proof of the lemma.

Lemma 2.3.5 *(Splitting) Let (u_k) be a sequence in $\mathcal{N}_\Omega$ such that*

$$I_\Omega(u_k) \to d \quad \textit{and} \quad I'_{\mathcal{N}_\Omega}(u_k) \to 0 \quad \textit{in} \ \ (\mathcal{D}^{1,2}(\Omega))'.$$

Replacing u_k by a subsequence if necessary, there exist a solution u_0 of $(\mathcal{P})$, a number $m \in \mathbb{N}$, m function $w_1, \cdots, w_m$ in $D^{1,2}(\mathbb{R}^N)$ and m sequences of points $(y_k^j) \in \mathbb{R}^N$, $1 \leq j \leq m$, satisfying:

a) $u_k \to u_0$ in $\mathcal{D}^{1,2}(\Omega)$ or

b) w_j are nontrivial solutions of $(\mathcal{P}_{\mathbb{R}^N})$;

c) $|y_k^j| \to +\infty$ e $|y_k^j - y_k^i| \to +\infty$ $i \neq j$;

d) $u_k - \sum_{i=1}^m w_j(\cdot - y_k^j) \to u_0 \quad$ *in* $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

e) $d = I_\Omega(u_0) + \sum_{i=1}^m I_{\mathbb{R}^N}(w_j)$.

Proof. By Lemma 2.3.4 (u_k) is bounded and we can extract a subsequence, which converges to u_0 weakly in $\mathcal{D}^{1,2}(\Omega)$. We verify that u_0 solves $(\mathcal{P})$. Indeed, by Lemma 2.3.4 for $\varphi \in C_0^\infty(\Omega)$, we have

$$I'_\Omega(u_k)\varphi = \int_\Omega \nabla u_k \nabla\varphi dx - \int_\Omega f(u_k)\varphi dx \to 0 \ \text{ as } \ k \to \infty. \qquad (2.3.5)$$

By (b) of Lemma 2.3.1 and the fact that for $p < 2^*$, $(u_k) \to u_0$ strongly in $L^p(\Gamma)$ where Γ is a bounded subset of Ω, and using the mean value theorem

$$f(u_k(x)) - f(u_0(x)) = f'(u_k(x) + \theta(x)u_0(x))(u_k(x) - u_0(x)) \quad \text{with} \quad 0 < \theta(x) < 1\ ,$$

(f_2) and (2.1.2) we get

$$\int_\Omega |f(u_k) - f(u_0)|\varphi dx \leq \int_{supp\varphi} (|u_k| + |u_0|)^{2^*-2}(u_k - u_0)\varphi dx \to 0 \text{ as } k \to \infty,$$

and so

$$\int_\Omega \nabla u_k \nabla\varphi dx - \int_\Omega f(u_k)\varphi dx \to \int_\Omega \nabla u_0 \nabla\varphi dx - \int_\Omega f(u_0)\varphi dx \text{ as } k \to \infty. \quad (2.3.6)$$

By (2.3.5) and (2.3.6), u_0 solves $(\mathcal{P})$ and immediately $u_0 \in \mathcal{N}_\Omega$. Now set $u_k^1 = u_k - u_0$ and define $u_k^1 = 0$ in $\mathbb{R}^N \setminus \Omega$, so u_k^1 converges to 0 weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and as we will see in Remark 2.3.6, $I'_\Omega(u_k^1)u_k^1 \to 0$ and so

$$I'_\Omega(u_k^1)u_k^1 = \int_\Omega |\nabla u_k^1|^2 - \int_\Omega f(u_k^1)u_k^1 \to 0 \text{ as } k \to \infty. \quad (2.3.7)$$

By (a) and (b) of Lemma 2.3.2, we have

$$\|u_k^1\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \|u_k\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} - \|u_0\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} + o(1) \quad (2.3.8)$$

$$I_{\mathbb{R}^N}(u_k^1) = I_\Omega(u_k) - I_\Omega(u_0) + o(1). \quad (2.3.9)$$

Assume $u_k^1 \not\to 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (otherwise we have the claim), by (2.3.7)

$$0 < \eta \leq \|u_k^1\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} f(u_k^1)u_k^1 + o(1). \quad (2.3.10)$$

Then arguing as in Lemma 2.3.4, there is (y_k) and $\delta > 0$ such that

$$\int_{B(y_k,R)} |u_k^1|^2 > \delta. \quad (2.3.11)$$

Now consider $\tilde{u}_k = u_k^1(\cdot - y_k^1)$, which is bounded, so passing to a subsequence there is $\tilde{u}_k \rightharpoonup u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and $\tilde{u}_k \to u$ in $L^2_{loc}(\mathbb{R}^N)$. We claim that $u \not\equiv 0$. Indeed if $\|\tilde{u}_k\|_{L^p(B(0,R))} \to 0$ as $k \to \infty$ this contradicts (2.3.11) and the claim is proved. Hence by the boundedness of u_k^1, there exists $w_1 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $u_k^1(x - y_k^1) \to w_1 \neq 0$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and the sequence $(y_k^1) \in \mathbb{R}^N$ with $y_k^1 \to \infty$ as $k \to \infty$, since if (y_k^1) were bounded, by passing to subsequence, we should find y^1 that $y_k^1 \to y^1$ and

$$\int_{B(y^1,R)} |u_k^1|^2 > \delta \quad (2.3.12)$$

and as above u_k^1 is bounded, so passing to a subsequence there is u^1 such that $u_k^1 \rightharpoonup u^1$ in $\mathcal{D}^{1,2}(B(y^1, R))$ and $u^1 \not\equiv 0$, which is contradictory with u_k^1 converging weakly to 0 in

$\mathcal{D}^{1,2}(\mathbb{R}^N)$. Moreover w_1 is a weak solution of $(\mathcal{P}_{\mathbb{R}^N})$. The proof of this is Remark 2.3.6, which is stated in what follows. Define $u_k^2 := u_k^1 - w_1(\cdot - y_k^1)$ then by arguing as before u_k^2 satisfies

$$I_{\mathbb{R}^N}(u_k^2) \to d - I_\Omega(u_0) - I_{\mathbb{R}^N}(w_1)$$

and if $u_k^2 \not\to 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (otherwise we have the claim) then there exists a sequence $\{y_k^2\} \in \mathbb{R}^N$ with $\{y_k^2\} \to \infty$ as $k \to \infty$ and $u_k^2(x - y_k^1) \to w_2 \neq 0$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, such that w_2 is a weak solution of $(\mathcal{P}_{\mathbb{R}^N})$. Moreover any nontrivial critical point u of $I_{\mathbb{R}^N}$ satisfies $I_{\mathbb{R}^N}(u) \geq c > 0$, so iterating the above procedure we construct sequences w_i and (y_k^j). Since for every i, $I_{\mathbb{R}^N}(w_i) \geq c$, the iteration must terminate at some finite index m.

Remark 2.3.6 *We prove that w_1 is a weak solution of $(\mathcal{P}_{\mathbb{R}^N})$*

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, using the mean value theorem and (f_2), by (b) of Lemma 2.3.1 we have

$$\int_{\mathbb{R}^N} \nabla u_k^1(x - y_k^1)\nabla\varphi - f(u_k^1(x - y_k^1))\varphi \, dx$$

$$= \int_{\mathbb{R}^N} \nabla u_k^1(z)\nabla\varphi(z + y_k^1) - f(u_k^1(z))\varphi(z + y_k^1) \, dz$$

$$= \int_{\mathbb{R}^N} [f(u_k) - f(u_0) - f(u_k^1)]\varphi(z + y_k^1) \, dz + o(1)$$

$$\leq \int_{B_R} [f(u_0 + u_k^1) - f(u_0)]\varphi(z + y_k^1) \, dz + \int_{\mathbb{R}^N \setminus B_R} [f(u_0 + u_k^1) - f(u_k^1)]\varphi(z + y_k^1) \, dz$$

$$- \int_{B_R} f(u_k^1)\varphi(z + y_k^1) \, dz - \int_{\mathbb{R}^N \setminus B_R} f(u_0)\varphi(z + y_k^1) \, dz + o(1)$$

$$\leq C\|(|u_0|^{2^*-2} + |u_k^1|^{2^*-2})\varphi(\cdot + y_k^1)\|_{L^{p'}(\mathbb{R}^N)} a_{k,R}$$

$$+ C\|(|u_0|^{2^*-2} + |u_k^1|^{2^*-2})\varphi(\cdot + y_k^1)\|_{L^{p'} \cap L^{q'}(\mathbb{R}^N)} b_R + o(1)$$

where $a_{k,R} = \|u_k^1\|_{L^p(B_R)}$, $b_R = \|u_0\|_{L^{p'} \cap L^{q'}(\mathbb{R}^N \setminus B_R)}$. Since $b_R \to 0$ as $R \to \infty$, and given R, $a_{k,R} \to 0$ as $k \to \infty$, by above estimate we get

$$\int_{\mathbb{R}^N} \nabla u_k^1(x - y_k^1)\nabla\varphi - f(u_k^1(x - y_k^1))\varphi \, dx \to 0$$

as $k \to \infty$. On the other hand, by (a) of Lemma 2.3.1, it is easy to see that

$$\int_{\mathbb{R}^N} \nabla u_k^1(x - y_k^1)\nabla\varphi - f(u_k^1(x - y_k^1))\varphi \, dx \to$$

$$\int_{\mathbb{R}^N} \nabla w_1 \nabla \varphi - f(w_1))\varphi \, dx.$$

So we get the claim and complete the proof of the lemma. □

Corollary 2.3.7 *(Compactness) I_Ω satisfies the Palais-Smale condition on $\mathcal{N}_\Omega$ at every level $d \in (c, 2c)$.*

Proof. Let (u_k) be a $(PS)_d$-sequence for I_Ω on $\mathcal{N}_\Omega$. If $d \in (c, \bar{c})$ and (u_k) does not have a convergent subsequence then, by the Splitting lemma,

$$\bar{c} > d = I_\Omega(u_0) + \sum_{i=1}^{m} I_{\mathbb{R}^N}(w_j) \geq \begin{cases} mc & \text{if } u_0 = 0 \\ c_\Omega + mc \geq (m+1)c & \text{if } u_0 \neq 0 \end{cases} \tag{2.3.13}$$

then in both cases, $m < 2$ and so $m = 1$. The hypothesis $2c > d \geq (m+1)c$ implies that it is not possible to have $m = 1$ and $u_0 \neq 0$, therefore $u_0 = 0$, which yields $I_\Omega(u_n) \to I_{\mathbb{R}^N}(w_1) = d$ giving a contradiction with the uniqueness of solution of $(\mathcal{P}_{\mathbb{R}^N})$. Hence, I_Ω satisfies the Palais-Smale condition on $\mathcal{N}_\Omega$ at every $d \in (c, 2c)$. □

Remark 2.3.8 *If u is a solution of $(\mathcal{P})$ with $I_\Omega(u) \in [c, 2c)$, then u does not change sign. Since, if u is a solution of $(\mathcal{P})$ then*

$$0 = I'_\Omega(u)u^\pm = J_\Omega(u^\pm),$$

where $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$ and So $u^\pm \in \mathcal{N}_\Omega$, now if $u^+ \neq 0$ and $u^- \neq 0$ then

$$I_\Omega(u) = I_\Omega(u^+) + I_\Omega(u^-) \geq 2c.$$

2.4 Existence of a positive solution

For $R > 0$, $y \in \partial B_2(y_0)$, let us define

$$\varepsilon_R := \int_{\mathbb{R}^N} f(w_0^R) w_y^R.$$

Lemma 2.4.1 *There exists $C > 0$ such that*

$$\varepsilon_R = \int_{\mathbb{R}^N} f(w_0^R) w_y^R \leq C R^{-(N-2)} \tag{2.4.1}$$

for all $y \in \partial B_2(y_0)$ and $R > 0$ sufficiently large.

Proof. It sufficient to take $r = 2^*$ and $s = 1$ in Lemma 2.2.12. □

Note that the previous lemma implies

$$\varepsilon_R \to 0 \quad \text{as } R \to \infty, \quad \text{uniformly for} \quad y \in \partial B_2(y_0).$$

Lemma 2.4.2 *There exists $C > 0$ such that for all $s, t \geq \frac{1}{2}$, $y \in \partial B_2(y_0)$ and R large enough,*

$$\varepsilon_R = \int_{\mathbb{R}^N} f(sw_0^R) tw_y^R \geq CR^{-(N-2)}. \tag{2.4.2}$$

Proof. For $|x| < 1$ and we $R > 1$ we have

$$1 + |x| < 1 + |x - R(y - y_0)| < 1 + |x| + R|(y - y_0)| < 4R. \tag{2.4.3}$$

Now by (f_1), (2.4.3) and the decay estimates (2.2.1) there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} f(sw_0^R) tw_y^R = st \int_{\mathbb{R}^N} \left(\frac{f(sw_0^R)}{sw_0^R} \right) w_0^R w_y^R$$

$$\geq \frac{1}{4} \int_{\mathbb{R}^N} \left(\frac{f(\frac{1}{2}w_0^R)}{\frac{1}{2}w_0^R} \right) w_0^R w_y^R \geq \frac{1}{4} \int_{B_1(Ry_0)} \left(\frac{f(\frac{1}{2}w_0^R)}{\frac{1}{2}w_0^R} \right) w_0^R w_y^R$$

$$\geq \frac{1}{4} \left[\min_{x \in B_1(0)} \frac{f(\frac{1}{2}w(x))}{\frac{1}{2}w(x)} \right] \int_{x \in B_1(0)} w(x) w(x - R(y - y_0))$$

$$\geq C \int_{B_1(0)} (1 + |x|)^{-(N-2)} w(x - R(y - y_0))$$

$$\geq CR^{-(N-2)}.$$

□

If we set $s, t = 1$ in the above lemma we have

$$\varepsilon_R \geq CR^{-(N-2)}. \tag{2.4.4}$$

Lemma 2.4.3 *For every $b > 1$ there is a constant $C > 0$ such that*

$$\left| \int_{\Omega} \left[sf(w_0^R \psi) - f(sw_0^R \psi) \right] w_y^R \psi \right| \leq C \big| s - 1 \big| \; \varepsilon_R,$$

for all $s \in [0, b]$, $y \in \partial B_2(y_0)$ and R large enough.

Proof. Fix $u \in \mathbb{R}$ and consider the function $g(s) := sf(u) - f(su)$. By (2.1.2),

$$g'(s) := f(u) - f'(su)u \le |f(u)| + C(s^{2^*-1}|u|^{2^*})$$

$$\le C|u|^{2^*} \qquad \forall s \in [0,1].$$

Hence, by the Mean Value Theorem,

$$|sf(u) - f(su)| = |g(s) - g(1)| = |g'(t)||s-1|$$

$$\le C|u|^{2^*}|s-1|.$$

This inequality yields

$$\int_\Omega \big|sf(w_0^R\psi) - f(sw_0^R\psi)\big| w_y^R\psi$$

$$\le C\big|s-1\big| \int_\Omega (|w_0^R\psi|^{2^*}) w_y^R\psi,$$

$$= C\big|s-1\big| \int_{\mathbb{R}^N} |w_0^R|^{2^*} w_y^R(\psi)^{2^*+1}.$$

Now apply Lemma 2.2.12 and using that $|\psi| \le 1$ we have

$$\int_{\mathbb{R}^N} \big|sf(w_0^R\psi) - f(sw_0^R\psi)\big| w_y^R\psi \le C\big|s-1\big| O(\varepsilon_R) \le C\big|s-1\big|\ \varepsilon_R$$

for all $s \in [0,b]$, $y \in \partial B_2(y_0)$ as claimed. □

Proposition 2.4.4 *There exists $R_1 > 0$ and, for each $R > R_1$, a number $\eta = \eta_R > 0$, $\eta_R = o_R(1)$ such that*

$$I_\Omega(T_{\lambda,y}^R U_{\lambda,y}^R) \le 2c - \eta,$$

for all $\lambda \in [0,1]$, $y \in \partial B_2(y_0)$.

Proof. Let us denote, for simplicity

$$s := T_{\lambda,y}^R \lambda \quad , \quad t := T_{\lambda,y}^R (1-\lambda),$$

we have that

$$I_\Omega(sw_0^R\psi + tw_y^R\psi) = \frac{1}{2}\int_\Omega |\nabla(sw_0^R\psi + tw_y^R\psi)|^2 - \int_\Omega F(sw_0^R\psi + tw_y^R\psi)$$

$$= \frac{s^2}{2}\int_\Omega |\nabla(w_0^R\psi)|^2 + \frac{t^2}{2}\int_\Omega |\nabla(w_y^R\psi)|^2 + st\int_\Omega \nabla(w_0^R\psi)\nabla(w_y^R\psi)$$

$$-\int_\Omega F(sw_0^R\psi) - \int_\Omega F(tw_y^R\psi) - \int_\Omega F(sw_0^R\psi + tw_y^R\psi) - F(sw_0^R\psi) - F(tw_y^R\psi)$$

$$= \frac{s^2}{2}\int_\Omega |\nabla(w_0^R\psi)|^2 - \int_\Omega F(sw_0^R\psi) \tag{2.4.5}$$

$$+\frac{t^2}{2}\int_\Omega |\nabla(w_y^R\psi)|^2 - \int_\Omega F(tw_y^R\psi) \tag{2.4.6}$$

$$+st\int_\Omega \nabla(w_0^R\psi)\nabla(w_y^R\psi) \tag{2.4.7}$$

$$-\int_\Omega F(sw_0^R\psi + tw_y^R\psi) - F(sw_0^R\psi) - F(tw_y^R\psi) - f(sw_0^R\psi)tw_y^R\psi - f(tw_y^R\psi)sw_0^R\psi \tag{2.4.8}$$

$$-\int_\Omega f(sw_0^R\psi)tw_y^R\psi - \int_\Omega f(tw_y^R\psi)sw_0^R\psi \tag{2.4.9}$$

The sum in line (2.4.5) is equal to $I_{\mathbb{R}^N}(sw_0^R) + o(\varepsilon_R)$ since

$$(2.4.5) = I_{\mathbb{R}^N}(sw_0^R) - I_{\mathbb{R}^N}(sw_0^R) + \frac{s^2}{2}\int_\Omega |\nabla(w_0^R\psi)|^2 - \int_\Omega F(sw_0^R\psi)$$

$$= I_{\mathbb{R}^N}(sw_0^R) + \frac{s^2}{2}\int_{B_{2K}(0)} |\nabla(w_0^R\psi)|^2 - |\nabla w_0^R|^2 - \int_{B_{2K}(0)} F(sw_0^R) - F(sw_0^R\psi)$$

and by (2.2.8) Lemma 2.2.11, (2.4.1), (2.4.4) and s bounded by T_0 we have

$$\frac{s^2}{2}\int_{B_{2K}(0)} |\nabla w_0^R\psi|^2 - |\nabla w_0^R|^2 = o(\varepsilon_R).$$

On other hand, by the Mean Value Theorem, (f_2) and Lemma 2.2.11 we have

$$\int_{B_{2K}(0)} F(sw_0^R) - F(sw_0^R\psi) = \int_{B_{2K}(0)} f(sw_0^R + \theta(x)sw_0^R\psi)(sw_0^R - sw_0^R\psi)$$

$$\leq C\int_{B_{2K}(0)} (|w_0^R|^{2^*-1})w_0^R = C\int_{B_{2K}(0)} |w_0^R|^{2^*} = o(\varepsilon_R).$$

The sum gives that (2.4.5) $= I_{\mathbb{R}^N}(sw_0^R) + o(\varepsilon_R)$ and since w_0^R is a least energy solution of the limit problem $(\mathcal{P}_{\mathbb{R}^N})$, by Lemma 2.2.4 (b) we have that $I_{\mathbb{R}^N}(sw_0^R) \leq c$. Similarly we have the same for the sum in line (2.4.6) and so

$$(2.4.5) + (2.4.6) \leq 2c + o(\varepsilon_R).$$

As to (2.4.8), in Lemma 2.2.10 let $2^* - 2 < \nu < q - 2$ and so $1 + \frac{\nu}{2} > \frac{2^*}{2}$, now by Lemma 2.2.12 we have

$$-\int_{\mathbb{R}^N} F(sw_0^R\psi + tw_y^R\psi) - F(sw_0^R\psi) - F(tw_y^R\psi) - f(sw_0^R\psi)tw_y^R\psi - f(tw_y^R\psi)sw_0^R\psi$$

$$\leq C(st)^{1+\frac{\nu}{2}} \int_{\mathbb{R}^N} (w_y^R\psi w_0^R\psi)^{1+\frac{\nu}{2}} \leq C(st)^{1+\frac{\nu}{2}} \int_{\mathbb{R}^N} (w_y^R w_0^R)^{1+\frac{\nu}{2}} \leq CR^{-(N-2)(1+\frac{\nu}{2})} = o(\varepsilon_R)$$

so we have shown that

$$(2.4.8) \leq o(\varepsilon_R).$$

Now like as line (2.4.5) we have

$$\int_{\Omega} f(sw_0^R\psi)tw_y^R\psi + \int_{\Omega} f(tw_y^R\psi)sw_0^R\psi$$

$$= \int_{\mathbb{R}^N} f(sw_0^R)tw_y^R + \int_{\mathbb{R}^N} f(tw_y^R)sw_0^R + o(\varepsilon_R)$$

and so we can write the sum of the remaining terms as

$$(2.4.7) + (2.4.9) \leq st\int_{\Omega} \nabla w_0^R\psi \nabla w_y^R\psi - \int_{\mathbb{R}^N} f(sw_0^R)tw_y^R - \int_{\mathbb{R}^N} f(tw_y^R)sw_0^R + o(\varepsilon_R)$$

$$= \frac{st}{2}\int_{\mathbb{R}^N} f(w_y^R)w_0^R + \frac{st}{2}\int_{\mathbb{R}^N} f(w_0^R)w_y^R - \int_{\mathbb{R}^N} f(sw_0^R)tw_y^R - \int_{\mathbb{R}^N} f(tw_y^R)sw_0^R + o(\varepsilon_R)$$

$$= \frac{t}{2}\int_{\mathbb{R}^N} [sf(w_0^R) - f(sw_0^R)]w_y^R + \frac{s}{2}\int_{\mathbb{R}^N} [tf(w_y^R) - f(tw_y^R)]w_0^R$$
$$-\frac{1}{2}\int_{\mathbb{R}^N} f(sw_0^R)tw_y^R - \frac{1}{2}\int_{\mathbb{R}^N} f(tw_y^R)sw_0^R + o(\varepsilon_R)$$

By Lemma 2.4.3 there is a constant $C > 0$ such that

$$\frac{t}{2}\int_{\mathbb{R}^N} [sf(w_0^R) - f(sw_0^R)]w_y^R + \frac{s}{2}\int_{\mathbb{R}^N} [tf(w_y^R) - f(tw_y^R)]w_0^R \leq C(|s-1| + |t-1|)\ \varepsilon_R$$

for all $s, t \in [0, T_0], y \in \partial B_2(y_0)$ and R large enough. Moreover with Lemma 2.4.2, there is a constant $C_0 > 0$ such that

$$-\frac{1}{2}\int_{\mathbb{R}^N} f(sw_0^R)tw_y^R - \frac{1}{2}\int_{\mathbb{R}^N} f(tw_y^R)sw_0^R \geq C_0\,\varepsilon_R$$

for all $s,t \geq \frac{1}{2}$, $y \in \partial B_2(y_0)$ and R large enough. By Lemma 2.2.15, if $\lambda = 1/2$, then $s,t \to 1$ as $R \to \infty$. So taking $R_0 > 0$ sufficiently large and $\delta \in (0,1/2)$ sufficiently small such that for all $\lambda \in [\frac{1}{2}-\delta, \frac{1}{2}+\delta]$, $C(|s-1|+|t-1|) \leq \frac{C_0}{2}$, we have

$$(2.4.7) + (2.4.9) \leq -\frac{C_0}{2}\varepsilon_R + o(\varepsilon_R)$$

for all $y \in \partial B_2(y_0)$ and $R > R_0$. Summing up, we have proved that

$$I_\Omega(sw_0^R + tw_y^R) \leq 2c - \frac{C_0}{2}\varepsilon_R + o(\varepsilon_R)\ , \tag{2.4.10}$$

for all $y \in \partial B_2(y_0)$ and $R > R_0$.
On the other hand, for all $\lambda \in [0, \frac{1}{2}-\delta] \cup [\frac{1}{2}+\delta, 1]$, $y \in \partial B_2(y_0)$ and R sufficiently large, since if $T^R_{\lambda,y} \leq 2$ then $s = T^R_{\lambda,y}\lambda \in [0, 1-2\delta]$ or $t = T^R_{\lambda,y}(1-\lambda) \in [1, 1-2\delta]$ and if $T^R_{\lambda,y} \geq 2$ then $s = T^R_{\lambda,y}\lambda \in [1+2\delta, \infty]$ or $t = T^R_{\lambda,y}(1-\lambda) \in [1+2\delta,\infty]$, in fact one of s or t is in $[0, 1-2\delta] \cup [1+2\delta, \infty]$ and so $(2.4.5) + (2.4.6) \leq 2c - \gamma + O(\varepsilon_R)$. By Lemma 2.2.4(b), there exists $\gamma \in (0, c)$ such that

$$I_{\mathbb{R}^N}(rw_0^R) \leq c - \gamma \qquad \forall r \in [0, 1-2\delta] \cup [1+2\delta, \infty]$$

also with our previous estimates we have (2.4.7)+...+(2.4.9)= $O(\varepsilon_R)$, and so

$$I_\Omega(sw_0^R + tw_y^R) \leq 2c - \gamma + O(\varepsilon_R). \tag{2.4.11}$$

Inequalities (2.4.10) and (2.4.11), together, yield the statement of the proposition. □

Lemma 2.4.5 *For any $\delta > 0$, there exists $R_2 > 0$ such that*

$$I_\Omega(T^R_{\lambda,y}U^R_{\lambda,y}) < c + \delta,$$

for $\lambda = 0$ and every $y \in \partial B_2(y_0)$ and $R \geq R_2$.

Proof. $T^R_{\lambda,y}$ is bounded uniformly in λ, y and R. As w_y^R is a ground state of problem $(\mathcal{P}_{\mathbb{R}^N})$, like we saw for (2.4.5) we have

$$I_\Omega(T^R_{0,y}U^R_{0,y})$$

$$\leq I_{\mathbb{R}^N}(T^R_{0,y} w^R_y) + o(\varepsilon_R)$$

$$\leq \max_{s>0} I_{\mathbb{R}^N}(s w^R_y) + o(\varepsilon_R) \leq c + o(\varepsilon_R).$$

This proves the lemma. □

Let $\beta : \mathcal{D}^{1,2}(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ be a barycenter map, i.e. a continuous map such that, for every $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and every isometry A of $\mathbb{R}^N$,

$$\beta(u(\cdot - y)) = \beta(u) + y \quad \text{and} \quad \beta(u \circ A^{-1}) = A(\beta(u)). \tag{2.4.12}$$

Note that $\beta(u) = 0$ if u is radial. Barycenter maps have been constructed in [4,9].

Lemma 2.4.6 *There exists $\delta > 0$ such that*

$$\beta(u) \neq 0, \qquad \forall u \in \mathcal{N}_\Omega \cap I^{c+\delta}_\Omega$$

where $I^c_\Omega = \{u \in H^1_0(\Omega), \quad I_\Omega(u) \leq c\}$.

Proof. Arguing by contradiction, assume that for each $k \in \mathbb{N}$ there exists $v_k \in \mathcal{N}_\Omega$ such that $I_\Omega(v_k) < c_\Omega + \frac{1}{k}$ and $\beta(v_k) = 0$. By Ekeland's variational principle [29], there exists a $(PS)_d$-sequence (u_k) for I_Ω on $\mathcal{N}_\Omega$ at the level $d = c_\Omega$ such that $\|u_k - v_k\| \to 0$ [24, Theorem 8.5]. As c_Ω is not attained, Lemma 2.3.5 (splitting) implies that there exists a sequence (y_k) in $\mathbb{R}^N$ such that $|y_k| \to \infty$ and $\|u_k - w(\cdot - y_k)\| \to 0$, where w is the (positive or negative) radial ground state of $(\mathcal{P}_{\mathbb{R}^N})$. Setting $\tilde{v}_k(x) := v_k(x + y_k)$, and using properties (2.4.12) and the continuity of the barycenter, we conclude that

$$-y_k = \beta(v_k) - y_k = \beta(\tilde{v}_k) \to \beta(w) = 0$$

this is a contradiction. □

Proof of Theorem 2.1.2. We will show that I has a critical value in $(c, 2c)$. By Lemma 2.4.6, we may fix $\delta \in (0, \frac{c}{4})$ such that

$$\beta(u) \neq 0, \qquad \forall\, u \in \mathcal{N}_\Omega \cap I^{c+\delta}_\Omega.$$

Proposition 2.4.4 and Lemma 2.4.5 allow us to choose $\eta \in (0, \frac{c}{4})$ and $R > 0$ such that

$$I_\Omega(T^R_{\lambda,y} U^R_{\lambda,y}) \leq \begin{cases} 2c - \eta & \text{for all } \lambda \in [0,1] \text{ and all } y \in \partial B_2(y_0) \\ c + \delta & \text{for } \lambda = 0 \text{ and all } y \in \partial B_2(y_0). \end{cases}$$

Define $\alpha : B_2(y_0) \to \mathcal{N}_\Omega \cap I_\Omega^{2c-\eta}$ by

$$\alpha((1-\lambda)y_0 + \lambda y) := T_{\lambda,y}^R U_{\lambda,y}^R \quad \text{with } \lambda \in [0,1], \quad y \in \partial B_2(y_0).$$

Arguing by contradiction, assume that I_Ω does not have a critical value in $(c, 2c)$. As, by Corollary 2.3.7, I_Ω satisfies the Palais-Smale condition on $\mathcal{N}_\Omega$ at every level in $(c, 2c)$, there exists $\varepsilon > 0$ such that

$$\|\nabla_{\mathcal{N}_\Omega} I_\Omega(u)\| \geq \varepsilon, \qquad \forall u \in \mathcal{N}_\Omega \cap I_\Omega^{-1}[c+\delta, 2c-\eta].$$

Hence, the negative gradient flow of I on $\mathcal{N}_\Omega$ yields a continuous function

$$\rho : \mathcal{N}_\Omega \cap I_\Omega^{2c-\eta} \to \mathcal{N}_\Omega \cap I_\Omega^{c+\delta}$$

such that $\rho(u) = u$ for all $u \in \mathcal{N}_\Omega \cap I_\Omega^{c+\delta}$ (see [2] or [48], Lemma 5.15). Now we define $\Gamma(x) := (\beta \circ \rho \circ \alpha \circ \tau)(x)$, where $\tau(x) = x + y_0$ is a normal transfer. By Lemma 2.4.6 $\Gamma(x) \neq 0$ and so the function $\tilde{h} : B_2(0) \to \partial B_2(0)$ given by

$$\tilde{h} := 2\frac{\Gamma(x)}{|\Gamma(x)|}$$

is well defined and continuous. Moreover, if $y \in \partial B_2(y_0)$, then

$$\alpha(y) = T_{0,y}^R U_{0,y}^R = T_{0,y}^R w_y^R \in \mathcal{N}_\Omega \cap I_\Omega^{c+\delta}$$

and hence

$$(\beta \circ \rho \circ \alpha)(y) = \beta(T_{0,y}^R w_y^R) = y.$$

Therefore, $h(x) = \frac{\Gamma(x)}{2}\tilde{h}(x) - y_0 = x$ for every $x \in \partial B_2(0)$ and since by Brouwer Fixed Point Theorem such a map does not exist, I_Ω must have a critical point $u \in \mathcal{N}_\Omega$ with $I_\Omega(u) \in (c, 2c)$. By Remark 2.3.8 u does not change sign, now if $u \geq 0$ with the maximum principle, we get $u > 0$ is a solution of $(\mathcal{P})$. On other hand if $u \leq 0$, then by oddness of f, $f(u) \leq 0$ and so $-u$ is a positive solution. This proves that problem $(\mathcal{P})$ has a positive solution.
Now we can write $(\mathcal{P})$ as

$$-\Delta u = au$$

where $a = \frac{f(u)}{u}$, if we show $a \in L_{loc}^{\frac{N}{2}}(\mathbb{R}^N)$ then by Brezis-Kato theorem [17] $u \in L_{loc}^p(\mathbb{R}^N)$ for all $1 \leq p < \infty$ and so $u \in W_{loc}^{2,p}(\mathbb{R}^N)$ and by Sobolev embedding $u \in C_{loc}^{0,1-\frac{N}{p}}(\mathbb{R}^N)$, now let $p > N$ we have u is Holder continuous and by continuity of f we have $f(u)$ is hölder continuous and so by elliptic regularity theorems, $u \in C^2(\mathbb{R}^N)$ and so u is classic solution. In order to complete the proof we show $a \in L^{\frac{N}{2}}_{loc}(\mathbb{R}^N)$. By (f_2) we have

$$|a(x)| = \frac{f(u)}{u} \leq C|u|^{2^*-2}$$

and so

$$\int_\Gamma |a(x)|^{\frac{N}{2}} \leq C \int_\Gamma |u|^{\frac{(2^*-2)N}{2}} = C \int_\Gamma |u|^{2^*} < \infty$$

for any open set $\Gamma \subset\subset \mathbb{R}^N$. Hence the theorem is proved. $\square$

Chapter 3

Appendix

In this section we show and prove some useful properties of Orlicz space.

Lemma 3.0.1 *(a) If $v \in L^p + L^q$, the following inequalities hold:*

$$\max\{\|v\|_{L^q(\mathbb{R}^N\setminus\Gamma_v)} - 1, \frac{1}{1+|\Gamma_v|^{pq/q-p}}\|v\|_{L^p(\Gamma_v)}\}$$

$$\leq \|v\|_{L^p+L^q}$$

$$\leq \max\{\|v\|_{L^q(\mathbb{R}^N\setminus\Gamma_v)}, \|v\|_{L^p(\Gamma_v)}\};$$

(b) Let $\{v_k\} \in L^p + L^q$ and set $\Gamma_k = \{x \in \Omega : |v_k(x)| > 1\}$. Then $\{v_k\}$ is bounded in $L^p + L^q$ if and only if the sequences $\{|\Gamma_k|\}$ and $\{\|v_k\|_{L^q(\mathbb{R}^N\setminus\Gamma_k)} + \|v_k\|_{L^p(\Gamma_k)}\}$ are bounded;

(c) f is a bounded map from $L^p + L^q$ into $L^{p'} \cap L^{q'}$.

Proof. (a) First we prove the inequality

$$\|v\|_{L^p+L^q} \leq \max\{\|v\|_{L^q(\mathbb{R}^N\setminus\Gamma_v)}, \|v\|_{L^p(\Gamma_v)}\}.$$

Let $\phi \in L^{p'} \cap L^{q'}$ we have

$$\left|\int_{\mathbb{R}^N} v(x)\phi(x)dx\right| = \left|\int_{\mathbb{R}^N\setminus\Gamma_v} v(x)\phi(x)dx + \int_{\Gamma_v} v(x)\phi(x)dx\right|$$

$$\leq \|v\|_{L^q(\mathbb{R}^N\setminus\Gamma_v)}\|\phi\|_{L^{q'}(\mathbb{R}^N\setminus\Gamma_v)} + \|v\|_{L^p(\Gamma_v)}\|\phi\|_{L^{p'}(\Gamma_v)}$$

$$\leq \max(\|v\|_{L^q(\mathbb{R}^N\setminus\Gamma_v)}, \|v\|_{L^p(\Gamma_v)})(\|\phi\|_{L^{q'}(\mathbb{R}^N\setminus\Gamma_v)} + \|\phi\|_{L^{p'}(\Gamma_v)}).$$

By equivalent norm obtain

$$\|v\|_{L^p+L^q} = \|v\|_{L^{p'}\cap L^{q'}} \leq \max\{\|v\|_{L^q(\mathbb{R}^N\setminus\Gamma_v)}, \|v\|_{L^p(\Gamma_v)}\}.$$

Next we prove the inequality

$$\max\{\|v\|_{L^q(\mathbb{R}^N\setminus\Gamma_v)} - 1, \frac{1}{1+|\Gamma_v|^{pq/q-p}}\|v\|_{L^p(\Gamma_v)}\} \leq \|v\|_{L^p+L^q}.$$

Since $p' > q'$ we have $\|\phi\|_{L^{q'}(\Gamma_v)} \leq |\Gamma_v|^{\frac{q-p}{pq}}\|\phi\|_{L^{p'}(\Gamma_v)}$, then for $\phi \in L^{p'} \cap L^{q'}$ we have

$$\begin{aligned}
\|v\|_{L^p+L^q} &= \sup_{\phi\neq 0} \frac{\int v(x)\phi(x)dx}{\|\phi\|_{L^{p'}} + \|\phi\|_{L^{q'}}} \\
&\geq \sup_{\phi\neq 0, \phi(\mathbb{R}^N\setminus\Gamma_v)\equiv 0} \frac{\int v(x)\phi(x)dx}{\|\phi\|_{L^{p'}} + \|\phi\|_{L^{q'}}} \\
&= \sup_{\phi\neq 0, \phi\in L^{p'}(\Gamma_v)} \frac{\int v(x)\phi(x)dx}{\|\phi\|_{L^{p'}(\Gamma_v)} + \|\phi\|_{L^{q'}(\Gamma_v)}} \\
&\geq \sup_{\phi\neq 0, \phi\in L^{p'}(\Gamma_v)} \frac{\int v(x)\phi(x)dx}{\|\phi\|_{L^{p'}(\Gamma_v)} + |\Gamma_v|^{\frac{q-p}{pq}}\|\phi\|_{L^{p'}(\Gamma_v)}} \\
&= \frac{1}{1+|\Gamma_v|^{\frac{q-p}{pq}}} \sup_{\phi\neq 0, \phi\in L^{p'}(\Gamma_v)} \frac{\int v(x)\phi(x)dx}{\|\phi\|_{L^{p'}(\Gamma_v)}} \\
&= \frac{1}{1+|\Gamma_v|^{\frac{q-p}{pq}}}\|v\|_{L^p(\Gamma_v)}
\end{aligned}$$

and

$$\frac{1}{1+|\Gamma_v|^{\frac{q-p}{pq}}}\|v\|_{L^p(\Gamma_v)} \leq \|v\|_{L^p+L^q}.$$

So, in order to complete the proof of inequality it remains to show that

$$\|v\|_{L^q(\mathbb{R}^N\setminus\Gamma_v)} - 1 \leq \|v\|_{L^p+L^q}.$$

Let $\varepsilon > 0$, by (2.2.3) there exists $v_1 \in L^p$ such that $v - v_1 \in L^q$ and

$$\begin{aligned}
\|v\|_{L^p+L^q} &\geq \|v_1\|_{L^P} + \|v - v_1\|_{L^q} - \varepsilon \\
&\geq (\int_{(\mathbb{R}^N\setminus\Gamma_v)\cap\Gamma_{v_1}} |v_1|^p + \int_{(\mathbb{R}^N\setminus\Gamma_v)\setminus\Gamma_{v_1}} |v_1|^p)^{1/p} \\
&+(\int_{(\mathbb{R}^N\setminus\Gamma_v)\cap\Gamma_{v_1}} |v-v_1|^q + \int_{(\mathbb{R}^N\setminus\Gamma_v)\setminus\Gamma_{v_1}} |v-v_1|^q)^{1/q} - \varepsilon. \qquad (3.0.1)
\end{aligned}$$

Now set

$$\tilde{v}(x) = \begin{cases} v(x) & \text{for} \quad x \in (\mathbb{R}^N \setminus \Gamma_v) \cap \Gamma_{v_1}, \\ v_1(x) & \text{for} \quad x \in (\mathbb{R}^N \setminus \Gamma_v) \setminus \Gamma_{v_1}. \end{cases}$$

Since $\mathbb{R}^N \setminus \Gamma_v = \{x \in \mathbb{R}^N; v(x) \leq 1\}$ and $\Gamma_{v_1} = \{x \in \mathbb{R}^N; v_1(x) > 1\}$ we obtain

$$|\tilde{v}(x)| = \begin{cases} |v(x)| \leq 1 < |v_1(x)| & \text{for} \quad x \in (\mathbb{R}^N \setminus \Gamma_v) \cap \Gamma_{v_1}, \\ |v_1(x)| \leq 1 & \text{for} \quad x \in (\mathbb{R}^N \setminus \Gamma_v) \setminus \Gamma_{v_1}. \end{cases}$$

Now, since $p < q$, we have

$$\int_{(\mathbb{R}^N \setminus \Gamma_v) \cap \Gamma_{v_1}} |v_1|^p + \int_{(\mathbb{R}^N \setminus \Gamma_v) \setminus \Gamma_{v_1}} |v_1|^p$$

$$\geq \int_{(\mathbb{R}^N \setminus \Gamma_v) \cap \Gamma_{v_1}} |\tilde{v}|^q + \int_{(\mathbb{R}^N \setminus \Gamma_v) \setminus \Gamma_{v_1}} |\tilde{v}|^q = \|\tilde{v}\|^q_{L^q(\mathbb{R}^N \setminus \Gamma_v)}. \tag{3.0.2}$$

Moreover,

$$\int_{(\mathbb{R}^N \setminus \Gamma_v) \setminus \Gamma_{v_1}} |v - v_1|^q = \int_{(\mathbb{R}^N \setminus \Gamma_v) \setminus \Gamma_{v_1}} |v - \tilde{v}|^q$$

and since $v - \tilde{v} = 0$ in $(\mathbb{R}^N \setminus \Gamma_v) \cap \Gamma_{v_1}$, we get

$$\int_{(\mathbb{R}^N \setminus \Gamma_v) \setminus \Gamma_{v_1}} |v - v_1|^q$$

$$= \int_{(\mathbb{R}^N \setminus \Gamma_v) \setminus \Gamma_{v_1}} |v - \tilde{v}|^q + \int_{(\mathbb{R}^N \setminus \Gamma_v) \cap \Gamma_{v_1}} |v - \tilde{v}|^q = \|v - \tilde{v}\|^q_{L^q(\mathbb{R}^N \setminus \Gamma_v)}. \tag{3.0.3}$$

By (3.0.1), (3.0.2) and (3.0.3) we easily deduce that

$$\|v\|_{L^p + L^q} \geq \|\tilde{v}\|^{q/p}_{L^q(\mathbb{R}^N \setminus \Gamma_v)} + \|v - \tilde{v}\|_{L^q(\mathbb{R}^N \setminus \Gamma_v)} - \varepsilon. \tag{3.0.4}$$

So, if $\|\tilde{v}\|_{L^q(\mathbb{R}^N \setminus \Gamma_v)} - 1 < 0$, by (3.0.4) obtain

$$\|v\|_{L^p + L^q} \geq \|v - \tilde{v}\|_{L^q(\mathbb{R}^N \setminus \Gamma_v)} + \|\tilde{v}\|_{L^q(\mathbb{R}^N \setminus \Gamma_v)} - 1 - \varepsilon \geq \|v\|_{L^q(\mathbb{R}^N \setminus \Gamma_v)} - 1 - \varepsilon, \tag{3.0.5}$$

and if $\|\tilde{v}\|_{L^q(\mathbb{R}^N \setminus \Gamma_v)} - 1 \geq 0$, by (3.0.4) obtain

$$\|v\|_{L^p + L^q} \geq \|v - \tilde{v}\|_{L^q(\mathbb{R}^N \setminus \Gamma_v)} + \|\tilde{v}\|_{L^q(\mathbb{R}^N \setminus \Gamma_v)} - \varepsilon \geq \|v\|_{L^q(\mathbb{R}^N \setminus \Gamma_v)} - \varepsilon. \tag{3.0.6}$$

Finally, since $\varepsilon > 0$ is arbitrary, (3.0.5) and (3.0.5) imply the claim.

(b) The "if" part clearly follows from the second inequality in (a). Now we prove the "only if" part, so we assume that $|v_k|$ is a bounded sequence in $L^p + L^q$. Then, by the first inequality in (a), there exists $c > 0$ such that, for any positive integer k,

$$\frac{1}{1 + |\Gamma_{v_k}|^{\frac{q-p}{pq}}} \|v_k\|_{L^p(\Gamma_{v_k})} \leq C.$$

So, since $|v_k(x)| > 1$ for $x \in \Gamma_{v_k}$, we have

$$|\Gamma_{v_k}| \leq \int_{\Gamma_{v_k}} |v_k|^p \leq C^p(1 + |\Gamma_{v_k}|^{\frac{q-p}{pq}})^p, \tag{3.0.7}$$

and so $\{|\Gamma_{v_k}|\}$ is bounded, since $(\frac{q-p}{pq})p = 1 - \frac{p}{q} < 1$. Then, from the first inequality in (a) we find that

$$\{\|v_k\|_{L^q(\mathbb{R}^N \setminus \Gamma_k)} + \|v_k\|_{L^p(\Gamma_k)}\}$$

is bounded.

(c) Let $v \in L^p + L^q$, since $q = p + \gamma$ for $\gamma > 0$ and by (f_2), for any $\phi \in L^p(\mathbb{R}^N)$ we have

$$\left|\int_{\mathbb{R}^N} f(v(x))\phi(x)dx\right| = C \int_{\mathbb{R}^N \setminus \Gamma_v} |v|^{q-1}|\phi|dx + C \int_{\Gamma_v} |v|^{p-1}|\phi|dx$$

$$= C \int_{\mathbb{R}^N \setminus \Gamma_v} |v|^{p-1}|\phi||v|^\gamma dx + C \int_{\Gamma_v} |v|^{p-1}|\phi|dx$$

$$\leq C(\int_{\mathbb{R}^N \setminus \Gamma_v} |v|^p|v|^\gamma dx)^{1/p'}(\int_{\mathbb{R}^N \setminus \Gamma_v} |\phi|^p|v|^\gamma dx)^{1/p} + C\|v\|^{p-1}_{L^p(\Gamma_v)}\|\phi\|_{L^p(\mathbb{R}^N)}$$

$$\leq C(\int_{\mathbb{R}^N \setminus \Gamma_v} |v|^q dx)^{1/p'}(\int_{\mathbb{R}^N \setminus \Gamma_v} |\phi|^p dx)^{1/p} + C\|v\|^{p-1}_{L^p(\Gamma_v)}\|\phi\|_{L^p(\mathbb{R}^N)}$$

$$C\|v\|^{q/p'}_{L^p(\mathbb{R}^N \setminus \Gamma_v)}\|\phi\|_{L^p(\mathbb{R}^N)} + C\|v\|^{p-1}_{L^p(\Gamma_v)}\|\phi\|_{L^p(\mathbb{R}^N)}, \tag{3.0.8}$$

and by part (a)

$$f(v) \in L^{p'}. \tag{3.0.9}$$

Now let $\phi \in L^q(\mathbb{R}^N)$, by boundedness of (Γ_v) we have

$$\left|\int_{\mathbb{R}^N} f(v(x))\phi(x)dx\right| = C \int_{\mathbb{R}^N \setminus \Gamma_v} |v|^{q-1}|\phi|dx + C \int_{\Gamma_v} |v|^{p-1}|\phi|dx$$

$$\leq C\|v\|^{q-1}_{L^q(\mathbb{R}^N \setminus \Gamma_v)}\|\phi\|_{L^q(\mathbb{R}^N)} + C|\Gamma_v|^{(q-p)/qp}(\int_{\Gamma_v} |v|^p)^{1/p'}\|\phi\|_{L^q(\mathbb{R}^N)}$$

$$\leq C\|v\|^{q-1}_{L^q(\mathbb{R}^N\setminus\Gamma_v)}\|\phi\|_{L^q(\mathbb{R}^N)} + C|\Gamma_v|^{(q-p)/qp}\|v\|^{p/p'}_{L^p(\Gamma_v)}\|\phi\|_{L^q(\mathbb{R}^N)}, \tag{3.0.10}$$

and by part (a)

$$f(v) \in L^{p'}. \tag{3.0.11}$$

Finally (3.0.9) and (3.0.11) imply that f maps $L^p + L^q$ into $L^{p'} \cap L^{q'}$.
Now we show that f is bounded. Let $\{v_k\}$ be bounded sequence in $L^p + L^q$ by part (b), the sequences

$$\{|\Gamma_k|\} \quad \text{and} \quad \{\|v_k\|_{L^q(\mathbb{R}^N\setminus\Gamma_k)} + \|v_k\|_{L^p(\Gamma_k)}\}$$

are bounded. Here (3.0.8) and (3.0.10) imply that $f(v_k)$ is a bounded in $L^{p'} \cap L^{q'}$.

□

Lemma 3.0.2 *(a) If u and v are in a bounded subset of $L^p + L^q$, then $f'(u)v$ is in a bounded subset of $L^{p'} + L^{q'}$;*

(b) f' is a bounded continuous map from $L^p + L^q$ into $L^{p/p-2} + L^{q/q-2}$.

Proof. (a) By (f_2) we have

$$\left|\int_\Omega f'(u)v\phi dx\right| \leq C\int_{\Gamma_u}|u|^{p-2}|v||\phi|dx + C\int_{\Omega\setminus\Gamma_u}|u|^{q-2}|v||\phi|dx$$

$$= C\int_{\Gamma_u, u>v}|u|^{p-2}|v||\phi|dx + C\int_{\Gamma_u, u\leq v}|u|^{p-2}|v||\phi|dx$$

$$+C\int_{\Omega\setminus\Gamma_u, u>v}|u|^{q-2}|v||\phi|dx + C\int_{\Omega\setminus\Gamma_u, u\leq v}|u|^{q-2}|v||\phi|dx$$

$$\leq C\int_{\Gamma_u, u>v}|u|^{p-1}|\phi|dx + C\int_{\Gamma_u, u\leq v}|v|^{p-1}|\phi|dx$$

$$+C\int_{\Omega\setminus\Gamma_u, u>v}|u|^{q-1}|\phi|dx + C\int_{\Omega\setminus\Gamma_u, u\leq v}|v|^{q-1}|\phi|dx$$

$$\leq C\int_{\Gamma_u}|u|^{p-1}|\phi|dx + C\int_{\Gamma_v}|v|^{p-1}|\phi|dx$$

$$+C\int_{\Omega\setminus\Gamma_u}|u|^{q-1}|\phi|dx + C\int_{\Omega\setminus\Gamma_v}|v|^{q-1}|\phi|dx. \tag{3.0.12}$$

Now let $q = p + \alpha$, $\alpha > 0$ and $\phi \in L^p(\Omega)$ we obtain

$$C\int_{\Omega\setminus\Gamma_u}|u|^{q-1}|\phi|dx + C\int_{\Gamma_u}|u|^{p-1}|\phi|dx$$

$$= C\int_{\Omega\setminus\Gamma_u}|u|^{p-1}|u|^{\alpha}|\phi|dx + C\int_{\Gamma_u}|u|^{p-1}|\phi|dx$$

$$\leq C\left(\int_{\Omega\setminus\Gamma_u}|u|^p|u|^{\alpha}dx\right)^{\frac{1}{p'}}\left(\int_{\Omega\setminus\Gamma_u}|\phi|^p|u|^{\alpha}dx\right)^{\frac{1}{p}} + C\|u\|^{p-1}_{L^p(\Gamma_u)}\|\phi\|_{L^p(\Omega)}$$

$$\leq C\|u\|^{q/p'}_{L^q(\Omega\setminus\Gamma_u)}\|\phi\|_{L^p(\Omega)} + C\|u\|^{p-1}_{L^p(\Gamma_u)}\|\phi\|_{L^p(\Omega)}. \tag{3.0.13}$$

So by (3.0.12) and (3.0.13), $f'(v)u \in L^{p'}$. On other hand for $\phi \in L^q(\Omega)$ we obtain

$$C\int_{\Omega\setminus\Gamma_u}|u|^{q-1}|\phi|dx + C\int_{\Gamma_u}|u|^{p-1}|\phi|dx$$

$$\leq C\|u\|^{q-1}_{L^q(\Omega\setminus\Gamma_u)}\|\phi\|_{L^q(\Omega)} + C|\Gamma_u|^{pq/q-p}\|u\|^{p/p'}_{L^p(\Gamma_u)}\|\phi\|_{L^q(\Omega)}. \tag{3.0.14}$$

Since $|\Gamma_u|$ is bounded, by (3.0.12) and (3.0.14), $f'(v)u \in L^{p'}$. By (3.0.12), (3.0.13), (3.0.14), and (b) of Lemma 3.0.1 we get the claim.

(b) Let $u \in L^p + L^q$, $q = p + \alpha$ with $\alpha > 0$ and $\phi \in L^{p/2}(\Omega)$, by (f_2) we have

$$\left|\int_{\Omega} f'(u)\phi dx\right| \leq C\int_{\Omega\setminus\Gamma_u}|u|^{q-2}|\phi|dx + C\int_{\Gamma_u}|u|^{p-2}|\phi|dx$$

$$\leq C\int_{\Omega\setminus\Gamma_u}|u|^{p-2}|u|^{\alpha}|\phi|dx + C\int_{\Gamma_u}|u|^{p-2}|\phi|dx$$

$$\leq C\left(\int_{\Omega\setminus\Gamma_u}|u|^p|u|^{\alpha}dx\right)^{\frac{p-2}{p}}\left(\int_{\Omega\setminus\Gamma_u}|\phi|^{p/2}|u|^{\alpha}dx\right)^{\frac{2}{p}} + C\|u\|^{p-2}_{L^p(\Gamma_u)}\|\phi\|_{L^{p/2}(\Omega)}$$

$$\leq C\|u\|^{q(p-2)/p}_{L^q(\Omega\setminus\Gamma_u)}\|\phi\|_{L^{p/2}(\Omega)} + C\|u\|^{p-2}_{L^p(\Gamma_u)}\|\phi\|_{L^{p/2}(\Omega)}. \tag{3.0.15}$$

So, $f'(u) \in L^{p/p-2}$. On other hand for $\phi \in L^{q/2}(\Omega)$ we have

$$\left|\int_{\Omega} f'(u)\phi dx\right| \leq C\int_{\Omega\setminus\Gamma_u}|u|^{q-2}|\phi|dx + C\int_{\Gamma_u}|u|^{p-2}|\phi|dx$$

$$\leq C\int_{\Omega\setminus\Gamma_u}|u|^{p-2}|u|^{\alpha}|\phi|dx + C\int_{\Gamma_u}|u|^{p-2}|\phi|dx$$

$$\leq C\left(\int_{\Omega\backslash\Gamma_u}|u|^q dx\right)^{\frac{q-2}{q}}\left(\int_{\Omega\backslash\Gamma_u}|\phi|^{q/2}dx\right)^{\frac{2}{q}}+C|\Gamma_u|^{\frac{2(q-p)}{pq}}\left(\int_{\Omega\backslash\Gamma_u}|u|^p dx\right)^{\frac{p-2}{p}}\left(\int_{\Omega\backslash\Gamma_u}|\phi|^{q/2}dx\right)^{\frac{2}{q}}$$

$$\leq C\|u\|^{q-2}_{L^q(\Omega\backslash\Gamma_u)}\|\phi\|_{L^{q/2}(\Omega)}+C|\Gamma_u|^{\frac{2(q-p)}{pq}}\|u\|^{p-2}_{L^p(\Gamma_u)}\|\phi\|_{L^{q/2}(\Omega)}. \tag{3.0.16}$$

Since $|\Gamma_u|$ is bounded, $f'(u)\in L^{q/q-2}$ and by (3.0.15), (3.0.16) and (b) of Lemma 3.0.1 we get the claim.

□

Lemma 3.0.3 *The map* $(u,v)\to uv$ *from* $(L^p+L^q)^2$ *in* $L^{\frac{p}{2}}+L^{\frac{q}{2}}$ *is a bounded map.*

Proof. We set

$$A_1=\Gamma_u\cap\Gamma_v$$

$$A_2=\{x\in\Gamma_u : v(x)\leq 1, |u(x)v(x)|>1\}$$

$$A_3=\{x\in\Gamma_v : u(x)\leq 1, |u(x)v(x)|>1\}.$$

Then we have $\Gamma_{uv}=A_1\cup A_2\cup A_3$ and by (b) of Lemma 3.0.1, $|\Gamma_{uv}|\leq 2(|\Gamma_u|+|\Gamma_v|)<\infty$. Now we set $I_j=\int_{A_j}|uv|^{p/2}dx$ for $j=1,2,3$. By hölder inequality we obtain

$$I_1=\int_{A_1}|uv|^{p/2}dx\leq\|u\|^{p/2}_{L^p(\Gamma_u)}+\|v\|^{p/2}_{L^p(\Gamma_v)}$$

$$I_2=\int_{A_2}|uv|^{p/2}dx\leq\int_{\Gamma_u\cap(\Omega\backslash\Gamma_v)}|uv|^{p/2}dx$$

$$\leq\left(\int_{\Gamma_u}|u|^p dx\right)^{1/2}\left(\int_{\Omega\backslash\Gamma_v}|v|^p dx\right)^{1/2}$$

$$\leq\|u\|^{p/2}_{L^p(\Gamma_u)}\left(\int_{\Omega\backslash\Gamma_v}|v|^{p\frac{q}{p}}dx\right)^{\frac{p}{2q}}|A_2|^{(1-\frac{p}{q})\frac{1}{2}}$$

$$\leq\|u\|^{p/2}_{L^p(\Gamma_u)}\|v\|^{p/2}_{L^q(\Omega\backslash\Gamma_v)}|\Gamma_u|^{q-p/2q}$$

and likewise

$$I_3\leq\|u\|^{p/2}_{L^p(\Gamma_u)}\|v\|^{p/2}_{L^q(\Omega\backslash\Gamma_u)}|\Gamma_v|^{q-p/2q}.$$

So

$$\|uv\|_{L^{p/2}(\Gamma_{uv})}\leq C(\|u\|_{L^p(\Gamma_u)}+\|v\|_{L^p(\Gamma_v)}+\|u\|_{L^p(\Gamma_u)}\|v\|_{L^q(\Omega\backslash\Gamma_v)}|\Gamma_u|^{q-p/pq}$$

$$+\|v\|_{L^p(\Gamma_v)}\|u\|_{L^q(\Omega\backslash\Gamma_u)}|\Gamma_v|^{q-p/pq}).$$

Hence, by Lemma 3.0.1, we get that $\|uv\|_{L^{q/2}(\Omega\setminus\Gamma_{uv})}$ is bounded. In the same way we set

$$B_1 = (\Omega \setminus \Gamma_u) \cap (\Omega \setminus \Gamma_v)$$

$$B_2 = \{x \in \Gamma_v : u(x) < 1, |u(x)v(x)| \leq 1\}$$

$$B_3 = \{x \in \Gamma_u : v(x) < 1, |u(x)v(x)| \leq 1\}.$$

Then we have $\Omega \setminus \Gamma_{uv} = B_1 \cup B_2 \cup B_3$. Now we set $\tilde{I}_j = \int_{B_j} |uv|^{q/2}dx$ for $j = 1, 2, 3$. By holder inequality we obtain

$$\tilde{I}_1 = \int_{B_1} |uv|^{q/2}dx \leq \|u\|^{q/2}_{L^q(\Omega\setminus\Gamma_u)} + \|v\|^{q/2}_{L^q(\Omega\setminus\Gamma_v)}$$

$$\tilde{I}_2 = \int_{B_2} |uv|^{q/2}dx = \int_{B_2} |uv|^{p/2}|uv|^{q-p/2}dx \leq \int_{\Gamma_v\cap(\Omega\setminus\Gamma_u)} |uv|^{p/2}dx$$

$$\leq \left(\int_{\Gamma_v} |v|^p dx\right)^{1/2} \left(\int_{\Omega\setminus\Gamma_u} |u|^p dx\right)^{1/2}$$

$$\leq \|v\|^{p/2}_{L^p(\Gamma_v)} \left(\int_{\Omega\setminus\Gamma_u} |u|^{p\frac{q}{p}} dx\right)^{\frac{p}{2q}} |B_2|^{(1-\frac{p}{q})\frac{1}{2}}$$

$$\leq \|v\|^{p/2}_{L^p(\Gamma_v)}\|u\|^{p/2}_{L^q(\Omega\setminus\Gamma_u)}|\Gamma_v|^{q-p/2q}$$

and likewise

$$\tilde{I}_3 \leq \|u\|^{p/2}_{L^p(\Gamma_u)}\|v\|^{p/2}_{L^q(\Omega\setminus\Gamma_v)}|\Gamma_u|^{q-p/2q}.$$

So, like as above and by Lemma 3.0.1, we get that $\|uv\|_{L^{q/2}(\Omega\setminus\Gamma_{uv})}$ is bounded. Moreover, for any $\phi \in L^{p/p-2} \cap L^{q/q-2}(\Omega)$ we have

$$\int_\Omega uv\phi dx = \int_{\Omega\setminus\Gamma_{uv}} uv\phi dx + \int_{\Gamma_{uv}} uv\phi dx$$

$$\leq \|uv\|_{L^{p/2}(\Gamma_{uv})}\|\phi\|_{L^{p/p-2}(\Gamma_{uv})} + |uv\|_{L^{q/2}(\Omega\setminus\Gamma_{uv})}\|\phi\|_{L^{q/q-2}(\Omega\setminus\Gamma_{uv})}$$

$$\leq \max(\|uv\|_{L^{p/2}(\Gamma_{uv})}, \|uv\|_{L^{q/2}(\Omega\setminus\Gamma_{uv})})(\|\phi\|_{L^{p/p-2}(\Gamma_{uv})} + \|\phi\|_{L^{q/q-2}(\Omega\setminus\Gamma_{uv})}).$$

Hence, by a duality argument, we get that $uv \in L^{p/2} + L^{q/2}(\Omega)$ and by boundedness of $\|uv\|_{L^{q/2}(\Omega\setminus\Gamma_{uv})}$ and $\|uv\|_{L^{p/2}(\Gamma_{uv})}$ we get the claim. □

Lemma 3.0.4 *The functional* $\mathcal{F} : L^p + L^q \to \mathbb{R}$ *defined by*

$$\mathcal{F}(u) := \int_\Omega F(u)dx,$$

is of class C^2 and we have

$$\mathcal{F}'(u_0)v = \int_\Omega f(u_0)vdx, \tag{3.0.17}$$

$$\mathcal{F}''(u_0)vw := \int_\Omega f'(u_0)vwdx. \tag{3.0.18}$$

Proof. Step 1: Existence of the first derivative of $\mathcal{F}$ at u_0.
To verify that $\mathcal{F}$ is differentiable at u_0 and that (3.0.17) holds, by (b) of Lemma 3.0.1 and by the fact that

$$\int_\Omega F(u_0+v) - F(u_0) - f(u_0)vdx = \int_\Omega f'(u_0+\theta_x v)v^2dx$$

where $0 < \theta_x < 1$, it is enough to show that

$$\lim_{v\to 0} \frac{\int_\Omega f'(u_0+\theta_x v)v^2dx}{\|v\|_{L^p+L^q}} = 0.$$

Since $v \to 0$ in $L^p + L^q$, by (a) of Lemma 3.0.2 we have $f'(u_0+\theta_x v)v$ is bounded in $L^{p'} \cap L^{q'}$, and we get the claim.
Step 2: Existence of the second derivative of $\mathcal{F}$ at u_0.
We will show that

$$\sup_{\|v\|_{L^p+L^q}=1} \int_\Omega [f(u_0+w) - f(u_0) - f'(u_0)w]vdx \to 0$$

as $\|w\|_{L^p+L^q} \to 0$. We can write

$$\int_\Omega [f(u_0+w) - f(u_0) - f'(u_0)w]vdx = \int_\Omega f'(u_0+\theta_x w) - f'(u_0))wvdx$$

where $0 < \theta_x < 1$. Since $\|v\|_{L^p+L^q} = 1$ and $\|\theta w\|_{L^p+L^q} \leq \| w\|_{L^p+L^q} \to 0$, by Lemma 3.0.2, we have that $f'(u_0+\theta_x w) - f'(u_0))v$ is bounded in $L^{p'} \cap L^{q'}$, and so, we get the result.
Step : $\mathcal{F}$ is of class C^2.
We will show that

$$\sup_{\|w\|_{L^p+L^q}=\|v\|_{L^p+L^q}=1} \int_\Omega (f'(u_0+u) - f'(u_0))vwdx \to 0$$

as $\|u\|_{L^p+L^q} \to 0$. We can write

$$\int_\Omega (f'(u_0+u) - f'(u_0))vwdx = \langle (f'(u_0+u) - f'(u_0)), vw\rangle$$

where by Lemma 3.0.3 uv is bounded in $L^{p/2} + L^{q/2}$, and, by (b) of Lemma 3.0.2 we have that $f'(u_0+u) - f'(u_0) \to 0$ in $L^{(p/2)'} + L^{(q/2)'}$ as $u \to 0$ in $L^p + L^q$, and this complete the proof. □

Lemma 3.0.5 *If the sequence $\{u_k\}$ converges to u in L^p+L^q, then the sequence $\int_\Omega f(u_k)u_k dx$ converges to $\int_\Omega f(u_k)u_k dx$.*

Proof. By the Mean-Value Theorem exists $0 < \theta < 1$ such that

$$\int_\Omega |f(u_k)u_k - f(u)u|dx$$

$$\leq \int_\Omega |f(u_k)||u_k - u|dx + \int_\Omega |f(u_k) - f(u)||u|dx$$
$$\leq \int_\Omega |f(u_k)||u_k - u|dx + \int_\Omega |f'(u_k + \theta u)|u_k - u||u|dx$$

since by Lemma 2.2.1 (c) $\{f(u_k)\}$ is bounded and by Lemma 3.0.2 (b) $\{f'(u_k + \theta u)|u|\}$ is bounded, we get the claim. □

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