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Notas e Discussões

Buoyant force in a nonuniform gravitational field

(Força de empuxo em um campo gravitacional não-uniforme)

F.M.S. Lima[1], F.F. Monteiro[2]

[1] *Instituto de Física, Universidade de Brasília, Brasília, DF, Brazil*
[2] *Faculdade UnB Planaltina, Universidade de Brasília, Brasília, DF, Brazil*
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When an arbitrarily-shaped body is fully immersed in a liquid in equilibrium, it gets from the liquid a non-null hydrostatic force known as *buoyant force*. It is an easy task to apply the divergence theorem to show that this force agrees to that predicted by the well-known Archimedes' principle, namely an upward force whose magnitude equals the weight of the displaced liquid. Whenever this topic is treated in physics and engineering textbooks, a uniform gravitational field is assumed, which is a good approximation near the surface of the Earth. Would this approximation be essential for that law to be valid? In this note, starting from a surface integral of the pressure forces exerted by the fluid, we obtain a volume integral for the buoyant force valid for *nonuniform* gravitational fields. By comparing this force to the weight of the displaced fluid we show that the above question admits a *negative* answer *as long as these forces are measured in the same place*. The subtle possibility, missed in literature, of these forces to be distinct when measured in different places is pointed out.
Keywords: hydrostatics, Archimedes' principle, divergence theorem.

Quando um corpo com uma forma qualquer encontra-se completamente submerso em um líquido em equilíbrio, ele recebe do líquido uma força hidrostática não-nula conhecida como *força de empuxo*. É uma tarefa fácil aplicar o teorema da divergência para mostrar que esta força está em acordo com aquela prevista pelo famoso princípio de Arquimedes, ou seja, uma força vertical, pra cima, cuja intensidade é igual ao peso do líquido deslocado. Sempre que este tópico é abordado em livros-texto de física e engenharia, considera-se que o campo gravitacional é uniforme, o que é uma boa aproximação nas proximidades da superfície da Terra. Seria esta aproximação essencial para que a lei de Arquimedes seja válida? Nesta nota, partindo da integral de superfície das forças de pressão exercidas pelo fluido, nós obtemos uma integral de volume para a força de empuxo válida para campos gravitacionais *não-uniformes*. Ao comparar esta força com o peso do fluido deslocado, nós mostramos que a pergunta acima tem resposta *negativa*, desde que *estas forças sejam sejam medidas no mesmo local*. A possibilidade sutil, não observada na literatura, dessas forças serem distintas quando medidas em lugares diferentes é apontada aqui.
Palavras-chave: hidrostática, princípio de Arquimedes, teorema da divergência.

1. Introduction

In modern texts on hydrostatics, the original propositions introduced by Archimedes describing the force exerted by a liquid on a body immersed in it are reduced to a single statement known as Archimedes' law of buoyancy, or simply Archimedes' principle (AP), which asserts that "the buoyant force (BF) exerted by a fluid on a body immersed in it points upward and has a magnitude equal to the weight of the displaced fluid" [1]. Note that Archimedes did not call his discoveries in hydrostatics by *laws*, nor did present them as a consequence of experiments. He, instead, treated them as *mathematical theorems*, as those proposed by Euclides for geometry [2]. This law has been thoroughly tested experimentally since the times of Stevinus and Galileo.[2] On the theoretical hand, the simple case of a symmetric solid body (*e.g.*, a right-circular cylinder or a rectangular block) immersed in a *liquid* is used in textbooks for deriving AP from the Stevinus law (*i.e.*, from the linear increase of pressure with depth) [3,4]. Symmetry arguments are then taken into account to show that the net

[1]E-mail: fabio@fis.unb.br.

[2]An accurate experiment for testing AP, devised by Gravesande (1688-1742), deserves citation. The experiment uses a bucket and a metallic cylinder that fits snugly inside the bucket. By suspending the bucket and the cylinder from a balance and bringing it into equilibrium, he then immersed the cylinder in a water container. The balance equilibrium is only restored when one fills the bucket (to the rim) with water.

force exerted by the liquid is equal to the difference of the pressure forces exerted on the top and bottom surfaces, resulting in an upward force with a magnitude that agrees to AP, which also explains the origin of the BF. Though this proof is valid only for *symmetric* bodies with flat, horizontal top and bottom immersed in a liquid, it can be extended to *arbitrarily-shaped* bodies by making use of the divergence theorem [5, 6]. This derivation, in turn, can be adapted to the more general case of *inhomogeneous* fluids, as shown by the first author in a very recent work [7]. There, the BF is defined as the net force exerted by a fluid on the portion S of a body that effectively touches the fluid and then a gradient version of the divergence theorem is applied to the surface integral of the pressure forces, which leads to a volume integral that can be easily compared to the weight of the displaced fluid. This has validated the use of AP for any fluid in equilibrium. In fact, it also elucidates the origin of some known *exceptions* to AP, including the so-called 'bottom case' (see Sec. 3 of Ref. [7]).

Here in this note, by following the divergence theorem approach of Ref. [7], as shortly described above, we show, for the more general case of an arbitrarily-shaped body immersed in any fluid (homogeneous or not), that the assumption of an uniform gravitational field is not essential for the validity of the Archimedes' law of buoyancy.

2. Buoyant force in a *uniform* gravitational field

Let us recall the precise statement of the *Archimedes' principle*, as found in modern texts [1, 4],

> When a body is fully or partially submerged in a fluid, a buoyant force $\mathbf{B}$ from the surrounding fluid acts on the body. The force is directed *upward* and has a magnitude equal to the weight $m_f\, g$ of the fluid displaced by the body.

Here, m_f is the mass of the fluid that is displaced by the body and g is the local acceleration of gravity. Whenever this topic is discussed in textbooks, an *uniform* gravitational field is assumed. This is equivalent to say that $\boldsymbol{g}$ is a constant, which means that

$$\mathbf{B} = -\, m_f\, \mathbf{g} = m_f\, g\, \hat{\mathbf{k}}\,, \tag{1}$$

is also a constant vector pointing everywhere along the (vertical) z-axis direction, as identified in Fig. 1. In his original propositions, Archimedes does not mention any force field, which is a modern concept in physics [8]. He only mentions the *weight* of a body, treating it as a physical quantity proportional to the amount of matter. Therefore, for Archimedes a given body would got a constant weight, independently of its position in space (its height, in particular). For us, moderns, a body of fixed mass has a constant weight only when the gravitational field is *uniform*. In fact, an uniform gravitational field is a good approximation for points near the Earth surface.[3] Rigorously speaking, this field could only be generated by a *flat* Earth, as can be shown by taking into account an analogy with the electric field of a planar distribution of charge (edge effects being neglected), as shown in the Appendix. Could Archimedes have used a flat Earth model to derive his propositions? Though this would make his results mathematically exact, it would be a regression because the idea of a *spherical* Earth,[4] as suggested by Pythagoras (6th century b.C.), was very common among Greek intellectuals since 330 b.C., when Aristotle maintained it on the basis of physical theory and observational evidence [9]. Since Archimedes makes explicit references to a *spherical* Earth on both the text and figures of his *On floating bodies* [2], the answer to our preliminary question is definitely *negative*. For instance, in Proposition 2 of book I, Archimedes states that "The surface of any fluid at rest is the surface of a *sphere* whose centre is the same as that of the *Earth*."

According to Newton's law of gravitation, a spherical Earth (in fact, any spherical distribution of mass) creates a gravitational field that points radially and decays with the square of the distance to the center, hence a *nonuniform* field. Let us then check if the assumption of an uniform gravitational field is essential for the validity of the Archimedes' law of buoyancy.

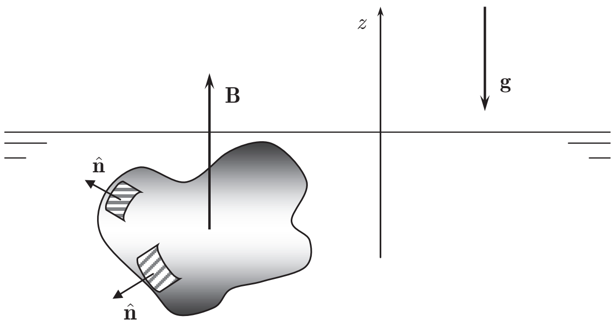

Figura 1 - An arbitrarily-shaped body fully submerged in a fluid. Note that the unit vector $\hat{\mathbf{n}}$ is directed along the outward normal to the (external) surface Σ of the body, changing its direction when one goes from a point to another over Σ.

[3] For instance, the gravitational field at the top of Mount Everest (8,850 m) is only 0.3% smaller than that at sea level.

[4] The Earth's radius was first estimated by Eratosthenes in 240 b.C. His result is surprisingly accurate (less than 2% distinct from modern measurements).

3. Buoyant force in a *nonuniform* gravitational field

When a *nonuniform* gravitational field is taken into account, we of course cannot use $m_f\, g$ for representing the weight of the displaced fluid because its magnitude is not a constant anymore. However, by dividing the region previously occupied by the displaced fluid in a large number N of small elements, each with a volume $\Delta V_f = V_f/N$, and then taking the limit as $N \to \infty$, one finds that the weight of any differential element of displaced fluid is $d\mathbf{W_f} = dm_f\, \mathbf{g} = \rho\, \mathbf{g}\, dV_f$. The (total) weight of the displaced fluid is then given by

$$\mathbf{W_f} = \int_{V_f} \rho\, \mathbf{g}\, dV_f\,, \tag{2}$$

where V_f is the volume of the displaced fluid, $\rho = \rho(\mathbf{r})$ is the density of the fluid, and $\mathbf{g} = \mathbf{g}(\mathbf{r})$ is the spatially variable gravitational field on the region occupied by the body, $\mathbf{r}$ being the position vector that locates the differential element of volume. Now, all we have to do is to determine the BF acting on the body and compare it to the weight $\mathbf{W_f}$, above. For this, we follow Ref. [7], starting by defining the BF as *the net pressure force that a fluid exerts on the part S of the (external) surface of a body that is effectively in contact to the fluid.* This corresponds to the following general formula for BF evaluations, valid for arbitrarily-shaped bodies totally[5] or partially immersed in a fluid in equilibrium:[6]

$$\mathbf{B} \equiv -\int_S p\; \hat{\mathbf{n}}\, dS\,, \tag{3}$$

where $p = p(\mathbf{r})$ is the fluid pressure and $\hat{\mathbf{n}} = \hat{\mathbf{n}}(\mathbf{r})$ is the *outward* normal unit vector at a point P of the surface S, as indicated in Fig. 1. The minus signal comes from the direction of the pressure force exerted by the fluid at a point P of S, which is that of the *inward* normal to S by P, thus opposite to $\hat{\mathbf{n}}$. This integral can be easily evaluated for a body with a *symmetric* surface immersed in a liquid, but it appears to be intractable analytically in the more general case of an arbitrarily-shaped body, due to the dependence of the direction of $\hat{\mathbf{n}}$ on the position over S, which depends on the (arbitrary) shape of S, as indicated in Fig. 1. However, this task can be easily worked out for a body fully submerged if one takes into account the following gradient version of the divergence theorem [10].

3.1. Gradient theorem

Let R be a bounded region in space whose boundary S is a closed, piecewise smooth surface which is positively oriented by a unit normal vector $\hat{\mathbf{n}}$ directed outward from R. If $f = f(\mathbf{r})$ is a scalar function with continuous partial derivatives in all points of an open region that contains R (including S), then[7]

$$\oint_S f\, \hat{\mathbf{n}}\, dS = \int_R \boldsymbol{\nabla} f\; dV\,. \tag{4}$$

At the only appendix of Ref. [7], the usual form of the divergence theorem is taken into account for proving the above theorem. The advantage of using this theorem is that it allows for a prompt conversion of the surface integral in Eq. (3) into a volume integral of $\boldsymbol{\nabla} p$, which, in virtue of the *hydrostatic equation* [5], can be written as

$$\boldsymbol{\nabla} p = \rho(\mathbf{r})\; \mathbf{g}(\mathbf{r})\,. \tag{5}$$

On putting $f(\mathbf{r}) = -p(\mathbf{r})$ in the integrals of the gradient theorem, Eq. (4), one finds

$$-\oint_S p\, \hat{\mathbf{n}}\, dS = -\int_{V_f} \boldsymbol{\nabla} p\;\; dV_f\,. \tag{6}$$

The surface integral above is, according to our definition, the BF itself whenever the surface S is closed, *i.e.* when the body is *fully submerged* in a fluid. In this case, by substituting the pressure gradient in Eq. (5) on the right-hand side of Eq. (6), one finds

$$\mathbf{B} = -\int_{V_f} \boldsymbol{\nabla} p\;\; dV_f = -\int_{V_f} \rho\, \mathbf{g}\;\; dV_f\,. \tag{7}$$

As the density of our arbitrary fluid can change with position, the pressure gradient will be integrable over V_f whenever the product $\rho(\mathbf{r})\; \mathbf{g}(\mathbf{r})$ is a continuous function of position in all points of V_f, in conformity to the hypothesis of the gradient theorem, a condition usually satisfied by a fluid in equilibrium. Within this condition, the comparison of the volume integrals found in Eqs. (2) and (7) promptly yields $\mathbf{B} = -\mathbf{W_f}$, thus confirming the validity of AP. Of course, we have to modify the AP statement slightly in order to adapt it to cover nonuniform gravitational fields, which can be done by correcting the direction of the BF from "directed upward" to "directed oppositely to the weight of the displaced fluid" and removing the expression "$m_f\, g$," in agreement to the discussion that precedes Eq. (2). Note that the former is the only modification needed for validating the Archimedes original propositions, more specifically his Propositions 5–7 [2].

Note that the *potential energy minimization* technique cannot be used for deriving the AP in these cases since it works only for incompressible fluids [11]. Of course, the same result would be found by assuming that the body is fully submerged in a *single* fluid

[5] Of course, when the body is fully submerged in a fluid, the surface S of contact coincides with the entire external surface of the body.

[6] We are assuming a piecewise smooth surface S and that $p(\mathbf{r})\; \hat{\mathbf{n}}(\mathbf{r})$ is integrable over S, a condition fulfilled in most practical situations.

[7] We are using here a general version of the theorem which is valid for piecewise smooth surfaces. This version covers bodies in which S is not smooth in *all* points, as *e.g.* rectangular blocks and cylinders.

whose density is not a continuous function of position, but only a *piecewise* continuous function, with a leap discontinuity at the interface between the fluids.

As the weight, in general, changes with the position in a *nonuniform* gravitational field, we point out that Propositions 5–7 of Archimedes *On floating bodies* [2], in which he explicitly mentions the "weight of the fluid displaced," will be valid only if this weight is measured at the same place where fluid has been displaced, otherwise it can be different from the buoyant force, which would erroneously suggest that AP is invalid. This subtle point is overlooked in textbooks.

4. Conclusions

In this note, we show that the BF predicted by AP can be derived from the hydrostatic equation for a body of arbitrary shape, submerged in any fluid, even in a *nonuniform* gravitational field, under certain continuity conditions usually satisfied in applications. For this, we have made use of a theorem for the gradient of scalar fields to convert the surface integral of the pressure forces exerted by the fluid into a volume integral of the gradient of the fluid pressure [7]. The hydrostatic equation is then applied to substitute this gradient by the product of the fluid density and the gravitational field, yielding a volume integral for the net pressure force that is equal to the opposite of the weight of the displaced fluid, in agreement to AP. *A priori*, this result can be applied in geologic studies, *e.g.* accurate modeling of the *isostasy* phenomenon [12], and astrophysics, *e.g.* in the study of objects attracted by the gravitational field of giant stars and black holes, or in any physical system in which the gravitational field changes significantly with position. As our vector calculus approach is not so advanced, it could well be explored in undergraduate courses.

Appendix

Gravitational field of a planar, uniform distribution of mass

The Gauss's law of electrostatics, namely $\oint_S \mathbf{E} \cdot \hat{\mathbf{n}}\, dS = Q_{in}/\epsilon_0$, ϵ_0 being the permittivity of free space, allows for a simple derivation of the electric field created by a planar plate uniformly charged with a surface charge density σ. By taking S as the surface of a right-circular cylinder whose central axis is perpendicular to the charged plane, it is easy to show that (see, *e.g.*, Ref. [1, p. 617])

$$|\mathbf{E}| = \frac{\sigma}{2\,\epsilon_0} = 2\,\pi\,k\,\sigma\,, \tag{8}$$

where $k = 1/(4\,\pi\,\epsilon_0)$ is the Coulomb constant. Being the planar distribution of charge horizontal, we have $\mathbf{E} = 2\,\pi\,k\,\sigma\,\hat{\mathbf{k}}$ in every point *above* the xy plane, hence an *uniform* electric field.

From the similarity between the Coulomb's law for the electrostatic force and the Newton's law for the gravitational force (always attractive), it is easy to deduce that

$$\oint_S \mathbf{g} \cdot \hat{\mathbf{n}}\, dS = -\,4\,\pi\,G\,M_{in}\,, \tag{9}$$

where G is the gravitational constant and M_{in} is the mass enclosed by the surface S. By analogy with the electric field, the gravitational field created by a planar, uniform distribution of mass is then

$$|\mathbf{g}| = 2\,\pi\,G\,\sigma \tag{10}$$

where σ is now the areal mass density. Therefore, $\mathbf{g} = -2\,\pi\,G\,\sigma\,\hat{\mathbf{k}}$ in every point *above* our horizontal plate, which shows that the gravitational field generated there is *uniform*.

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