Matheus Schmeling Costa

Single-Crossing Choice Correspondences

Brasília

2018

Matheus Schmeling Costa

Single-Crossing Choice Correspondences

Disseração apresentada ao Curso de Mestrado Acadêmico em Economia, Universidade de Brasília, como requisito parcial para a obtenção do título de Mestre em Economia

Universidade de Brasília - UnB

Faculdade de Administração Contabilidade e Economia - FACE

Departamento de Economia - ECO

Programa de Pós-Graduação

Orientador: Prof. Dr. Leandro de Oliveira Nascimento

Coorientador: Prof. Dr. Gil Riella

Brasília

2018

Matheus Schmeling Costa
Single-Crossing Choice Correspondences/ Matheus Schmeling Costa. – Brasília, 2018-
33 p. : il. (algumas color.) ; 30 cm.

Orientador: Prof. Dr. Leandro de Oliveira Nascimento

Dissertação (Mestrado) – Universidade de Brasília - UnB
Faculdade de Administração Contabilidade e Economia - FACE
Departamento de Economia - ECO
Programa de Pós-Graduação, 2018.

1. Single-Crossing. 2. Single-Peak. 3. Choice-Correspondences. 4. Pseudo-Rationality I. Orientador: Prof. Dr. Leandro de Oliveira Nascimento. II. Universidade de Brasília. III. Faculdade de Administração Contabilidade e Economia - FACE. IV. Departamento de Economia. V. Single-Crossing Choice Correspondences.

Matheus Schmeling Costa

Single-Crossing Choice Correspondences

Disseração apresentada ao Curso de Mestrado Acadêmico em Economia, Universidade de Brasília, como requisito parcial para a obtenção do título de Mestre em Economia

Trabalho aprovado. Brasília, 16 de Fevereiro de 2018:

Prof. Dr. Leandro de Oliveira Nascimento
Orientador

Prof. Dr. Gil Riella
Co-orientador

Prof. Dr. José Guilherme de Lara Resende
Convidado

Brasília
2018

Abstract

Classic comparative statics methods relies most commonly on functional form assumptions in order to explicitly derive relations between endogenous and exogenous variables, on the implicit function theorem when no functional form is assumed, or in duality theories when a dual functions exists. Among the several properties an agent's preferences may have there are two known for it's interesting results. First, the single-crossing property, that allows for the ordering of a population of agents with respect to a particular classification of the options available. And secondly, the single-peaked property, that rules out the occurrence of condorcet cycles when the decision system is majority vote by pairs. In this paper we characterize some sufficient and necessary conditions over choice correspondences so that they have a pseudo-rational representation satisfying these properties.

Keywords: Single-Crossing, Single-Peak, Choice Correspondences, Pseudo-Rationality.

Resumo

Métodos clássicos de estática comparativa usualmente se baseiam em hipóteses sobre formas funcionais para derivar explicitamente relações entre variáveis endógenas e exógenas, no teorema da função implícita quando nenhuma forma funcional é assumida, ou em teorias de dualidade quando existem funções duais. Entre as diversas propriedades que as preferências de uma gente podem satisfazer existem duas conhecidas por seus resultados interessantes. Primeiramente, a propriedade de cruzamento-único, que permite a ordenação de uma população de agentes com respeiro a uma classificação particular às alternativas disponíveis. E em segundo lugar, a propriedade de pico-único, que exclui a ocorrência de ciclos de condorcet quando o sistema de decisão é a votação majoritária aos pares. Neste trabalho nós caracterizamos as condições suficientes e necessárias sobre correspondências de escolha para que elas possuam uma representação pseudo-racional que satisfaça estas propriedades.

Palavras-chave: Cruzamento-Único, Pico-Único, Correspondências de Escolha, Pseudo-Racionalidade.

Sumário

1 Introduction

Classic comparative statics methods relies most commonly on functional form assumptions in order to explicitly derive relations between endogenous and exogenous variables, on the implicit function theorem when no functional form is assumed, or in duality theories when a dual functions exists. As pointed by Milgrom e Shannon (1994), these approaches relied on several strong assumptions (convexity of preferred sets or constraint sets, smoothness of indifference curves, interior solutions, among others), which were not necessary to achieve the comparative statics results sought, but rather only to assure that the methods used were applicable. Indeed, Milgrom and Shannon explain that if these assumptions were sufficient, then one could multiply the parameters by -1 and the properties of the functions would remain, although now the relation between the endogenous variables and the parameters could certainly have changed; also the conditions cannot be necessary, since order-preserving changes of the functions would not alter the comparative statics results, but could alter the properties of the functions. In order to address these problems, Milgrom and Shannon developed a theory and methods that depend only on the order structure of the problem, what they called monotone comparative statics.

In view of that, monotone comparative statics has several advantages. Ashworth e Mesquita (2006) list some: first, because the conditions developed by Milgrom and Shannon are not linked to functional forms, the results are more robust to misspecification and allow the researcher to consider more complex models; second, because the results do not depend on the full specification of the model, tests of empirical predictions could be more robust; and third, it is possible to identify critical substantive assumptions to the results, which can be evaluated against reality, as opposed to classical methods which often rely on technical assumptions that have no substantive meaning (e.g., differentiability).

One of the conditions developed by Milgrom and Shannon to obtain monotone comparative statics results is the single-crossing property. This is a condition both on a parametric family of functions $f : X \times T \to \mathbb{R}$ (e.g. the objective function in an optimization problem) and on an order of the choice space X. For instance, in the simple case where the choices and parameters are scalars, we have that f satisfies single-crossing with respect to the natural order on the real line if for all $a > a'$ and $\theta > \theta'$, $f(a, \theta') \geq f(a', \theta')$ implies $f(a, \theta) \geq f(a', \theta)$ and $f(a, \theta') > f(a', \theta')$ implies $f(a, \theta) > f(a', \theta)$. Based on this condition, Milgrom and Shannon show that the optimal choice is weakly increasing in the parameter. This is the classic result of monotone comparative statics.[1]

[1] This result is a simpler version, considering only functions where the choices and parameters are scalars and the optimal choice is unique. Milgrom and Shannon actually show similar results for broader

To get an intuition for the single-crossing property, consider the parameter θ to be aversion to risk and the choice space to be ordered from more to less riskier assets. Then if an individual with less aversion to risk (θ') prefers the less risky asset (a), then an individual more averse to risk (θ) would also prefer the less risky asset. Other examples can be found in Apesteguia, Ballester e Lu (2017).

In fact, Apesteguia, Ballester and Lu mention several applications of the single-crossing property, even outside monotone comparative statics. Thus, given the importance of this condition and also the relevance that Random Utility Models (RUM) have taken in modeling heterogeneity of preferences, Apesteguia, Ballester and Lu characterized a single-crossing RUM (SCRUM) stochastic choice function using the standard monotonicity axiom and a centrality axiom created by them[2,3]. They define a SCRUM as a RUM whose support of preferences satisfy the single-crossing property. When applied to preferences, the single-crossing property states that, given an order $\succ$ on a choice space X, the preferences considered can be ordered such that if $x \succ_s y$ and $s < t$, then $x \succ_t y$. Thus, it is similar to the single-crossing of Milgrom and Shannon.

Apesteguia, Ballester and Lu also briefly describe the classic result of monotone comparative statics in terms of preferences: given a collection of preferences that satisfy the single-crossing property, we can affirm that, for every menu, the best alternatives according to a higher ranked preference are either preferred to or equal to the best alternatives according to a lower ranked preference. The authors then go on and develop what would be the counterpart of the classic monotone comparative statics result in the context of stochastic choices. Furthermore, the authors also characterize single-peaked and single-dipped RUM stochastic choices, given the importance of these concepts as well (for instance, for social choice and political economy).

Inspired by Apesteguia, Ballester and Lu, in this article we characterize a choice correspondence that can be represented by a collection of preferences that satisfy the single-crossing property. We develop our own version of the centrality axiom, applicable to choice correspondences, and show that, together with another axiom called Pseudo-WARP, they characterize correspondences with a single-crossing pseudo-rational representation.

The remainder of the article is organized as follows. In Sections 2 and 3 we provide the basic definitions and the main theorem, the characterization of the single-crossing pseudo-rational representation. In Section 4 we characterize a related pseudo-rational choice correspondence that admits a representation by a collection of preferences that

classes of functions, including where there are multiple optimal. But in order to expose those results, further mathematical notation would have to be introduced.

2 The monotonicity axiom states that if an item is added to any menu, the probability, given by this stochastic choice function, of choosing any item contained originally in this menu cannot increase.

3 Given a finite strictly linear ordered set of alternatives $(X, \succ)$ and a stochastic choice function ρ, if $x \succ y \succ z$ and $\rho(y, x, y, z) > 0$, then $\rho(x, x, y) = \rho(x, x, y, z)$ and $\rho(z, y, z) = \rho(z, x, y, z)$.

satisfy the single-peak property. In Section 5 we conclude pointing out future extensions and implications of the theorem which we are currently working on.

2 Setup and Definitions

Let X be a nonempty set of alternatives and Ω_X the collection of nonempty subsets of X. We interpret X as the set of all conceivable alternatives. We call each element $A \in \Omega_X$ a choice problem and (x, Ω_X) a choice space. The idea is that the individual may be required to make a choice from the set A. The primitive of our analysis will be a choice correspondence c on Ω_X. That is, c is a map from Ω_X into Ω_X such that $c(A) \subseteq A$ for every $A \in \Omega_X$.

We could say then that (X, Ω_X, c) characterizes an individual or a population. For instance, in the first case, this tuple indicates which possible choices the individual would make in each different choice problem. Notice that a choice correspondence might indicate that more than one element is selected in a single choice problem. We can then interpret that which of these selected elements is in fact chosen by the decision-maker depends on factors unknown specific to that situation and which are not modeled here. Or we can consider a population as an aggregate decision maker and, for each choice problem available for that population (e.g., the financial assets that are currently available for the people in a certain country), the elements selected by c are the elements that are actually chosen by at least one individual in that population.

The last primitives needed for our analysis are: binary relation $\succcurlyeq^*$ on X and a collection $\mathcal{R}$ of binary relations on X. We say that a binary relation is a linear order if it is complete, transitive and antisymmetric. A binary relation $\succcurlyeq$ is complete if, for every $x, y \in X$, we have either $x \succcurlyeq y$, $y \succcurlyeq x$ or both. $\succcurlyeq$ is transitive if $x \succcurlyeq y$ and $y \succcurlyeq z$ imply $x \succcurlyeq z$. Finally, $\succcurlyeq$ is antisymmetric if $x \succcurlyeq y$ and $y \succcurlyeq x$ imply $x = y$. We also say that a binary relation is a complete preorder if it is complete and transitive. We can then define what it means for a choice correspondence to have a pseudo-rational representation:

Definition 1. *Let (X, Ω_X) be a choice space and c a choice correspondence on (X, Ω_X). We say that c has a pseudo-rational representation if there exists a collection $\mathcal{R}$ of complete preorders on X such that, for every $A \in \Omega_X$,*

$$c(A) = \bigcup_{\succsim \in \mathcal{R}} \max(A, \succsim). \tag{2.1}$$

We also need to define the following property:

Definition 2. *An ordered collection $\mathcal{P} := \{\succcurlyeq_1, \ldots, \succcurlyeq_k\}$ of linear orders on X satisfies the single-crossing property with respect to another linear order $\succcurlyeq^*$ if, for every $x, y \in X$ with $x \succcurlyeq^* y$, if $x \succcurlyeq_i y$ for some $i \in \{1, \ldots, k\}$, then $x \succcurlyeq_j y$ for every $j \geq i$.*

This property can be interpreted intuitively as follows. Imagine a population of individuals choosing among several risky assets, each with its own level of risk aversion. The relation $\succ^*$ can then be considered as a natural ordering of these assets, for instance, $x \succ^* y$ if x is less riskier than y. If each linear order in $\mathcal{P}$ represents a type of individual, based on her level of risk aversion, and $\mathcal{P}$ satisfies single-crossing with respect to $\succ^*$ on X, then we can order $\mathcal{P}$ from less to more risk averse, and we could affirm that, if individuals with a lower rank according to this order (i.e., with less risk aversion) prefer a less risky asset x over a riskier asset y (that is, $x \succ^* y$), then individuals higher ranked (i.e., more risk averse) will also prefer x over y. As indicated in the introduction, this conclusion is basically the classic result of monotonic comparative statics, indicating the importance of the single-property in that context.

Based on these primitives, we can now define the representation of a decision maker that we are going to characterize in this article:

Definition 3. *Let (X, Ω_X) be a choice space, c a choice correspondence on (X, Ω_X), and $\succ^*$ a linear order on X. We say that c has a single-crossing pseudo-rational representation if there exists an ordered collection $\mathcal{P} := \{\succ_1, \dots, \succ_k\}$ of linear orders on X that satisfies the single-crossing property with respect to $\succ^*$, and is a pseudo-rational representation of c.*

3 Representation Theorem

In this chapter we provide a theorem that characterizes correspondences with a single-crossing pseudo-rational representation. Consider the following postulates on c:

Axiom 1 (Pseudo-WARP)**.** *If $x \in c(A) \cap B$, $c(B) \subseteq A$, then $x \in c(B)$.*

Axiom 2 (Centrality)**.** *If $x \succ^* y \succ^* z$ and $y \in c(\{x, y, z\})$, then, for any A with $y \in A$, $x \in c(A \cup \{x\}) \implies x \in c(A \cup \{x, z\})$ and $z \in c(A \cup \{z\}) \implies z \in c(A \cup \{x, z\})$.*

Our Centrality axiom is similar to the one developed by Apesteguia, Ballester e Lu (2017) in the context of stochastic choice functions. Their axiom states that, when an alternative that is ranked in between two other alternatives (say y) according to a (possibly natural) order has a strictly positive probability of being chosen in the presence of these other extreme alternatives (say x, z), then the probability of one extreme alternative being chosen in the presence of only y is the same of being chosen in the presence of y and the other extreme alternative. In other words, the middle ground alternative carries all the information needed to evaluate the extreme alternatives: if this information results in a certain probability when the decision maker faces only one of the extremes and the middle ground, the addition of another extreme does not change the probability.

Our Centrality axiom also has a similar, but somewhat different, interpretation. In both the random choice model and the choice correspondence model, several alternatives from which the decision maker would pick a choice in each choice problem are defined, depending on external factors. But the main difference between the two types of models is that the random choice model seeks to obtain information on these external factors, which are reflected in the different probabilities that each of the possible choices in a choice problem carry. The choice correspondence model does not provide any information as to what could lead to the choice of one or other alternative. Because of that, we cannot affirm that the middle ground carries all the information needed to evaluate the extreme alternatives. We only know that, given any set where the middle ground is present, if one of the extreme alternatives could be chosen, then this remains possible if you add the other extreme alternative - but the model cannot say anything about whether one of them became more or less likely to be chosen, as this is simply not modeled.

We can now state the main theorem of our article:

Theorem 1. *A choice correspondence c satisfies Pseudo-WARP and Centrality if, and only if, there exists an ordered collection $\mathcal{P} := \{\succcurlyeq_1, \ldots, \succcurlyeq_k\}$ of linear orders on X that satisfies the single-crossing property with respect to $\succcurlyeq^*$ and is a pseudo-rational representation of c.*

4 Single-Peakness

As is well-known in the social choice and political economy literature, when preferences satisfy single-peakness, one can assure that condorcet cycles will not occur when the decision system is majority vote by pairs. Furthermore, one can guarantee that the social choice will be that of the median voter. Similar to Apesteguia, Ballester e Lu (2017), we also characterize a choice correspondence that can be represented by a collection of preferences that satisfy single-peakness. We also show that to achieve a representation satisfying single-peakness requires stronger versions of the Centrality axiom used in Theorem 1 and two additional axioms.

First we define single-peakness with respect to a particular order. In this context:

Definition 4. *A collection $\mathcal{P} := \{\succcurlyeq_1, \ldots, \succcurlyeq_k\}$ of linear orders on X satisfies the single-peak property with respect to $\succcurlyeq^*$ on X if $\forall \succcurlyeq_i \in \mathcal{P}$, whenever $y \succcurlyeq^* x \succcurlyeq^* max(X, \succcurlyeq_i)$ or $max(X, \succcurlyeq_i) \succcurlyeq^* x \succcurlyeq^* y$, then we must have $x \succcurlyeq_i y$.*

In words, given a peak according to a certain order, the alternatives on each side of the peak, according to the natural order, must be ordered by closeness to the peak: the closer they are, the better they are. We could interpret this in terms of political positions, for instance. Consider several policy proposals, ordered from the most liberal to the least liberal. If the choice correspondence representing the choices of the individuals in this society satisfy the single-peak property, then we can say that each individual has a preference such that, given its most preferred alternative, say a middle ground between the most and least liberal policies, it will prefer alternatives in each side that are closer to this middle ground over more extreme alternatives. On the other hand, individuals with more extreme tastes prefer policies closer to their tastes (but not the other extreme policies).

We must now introduce a formal definition of neighbors according to the main order, $\succcurlyeq^*$, and the axioms that we need to characterize a pseudo-rational choice correspondence that admits a representation by a collection of orders satisfying single-peakness:

Definition 5. *Let A be a choice problem and $x \in X$. We say that y is x's neighbor in A, or $y \in N(A, x)$, if $y \in A \setminus \{x\}$, and for no $z \in A \setminus \{x\}$ it is true that $y \succ^* z \succ^* x$ or $x \succ^* z \succ^* y$.*

Axiom 3 (Strong Centrality)**.** *If $x \succ y \succ z$ and $y \in A$, then $x \in c(A \cup \{x\}) \implies x \in c(A \cup \{x, z\})$ and $z \in c(A \cup \{z\}) \implies z \in c(A \cup \{x, z\})$.*

Axiom 4 (Neighbors)**.** *If $x \in c(A)$, then $N(A, x) \cap c(A \setminus \{x\}) \neq \emptyset$, provided that $A \neq \{x\}$.*

Axiom 5 (Archimedian Centrality)**.** *If* $y_2 \succcurlyeq^* y_1 \succ^* x \succ^* z_1 \succcurlyeq^* z_2$ *and* $x \in c(\{y_2, x, z_2\})$*, then either* $x \in c(\{y_2, x, z_1\})$ *or* $x \in c(\{y_1, x, z_2\})$*.*

The necessity of the Neighbors axiom is clear when we consider the example above of the policies ordered from more to less liberal. As we stated, individuals that have single-peak preferences must prefer alternatives close to their preferred alternatives. Therefore, when deprived of their preferred alternative in a choice problem, their second option must be the one closer to their preferred alternative. Its intuitive meaning is thus clear. Now, Strong Centrality is very similar to the weaker version above, but has a meaning more related to the single-peak property. If x is chosen when y is present, then one could interpret that at least one individual prefers x over y. For the preferences of this individual to obey single-peakness, then, she must also prefer y over z, since z is worse according to the main order. Hence she must choose again x when faced with the same choice problem, but with the addition of z. Finally, Archimedian Centrality axiom indicates that when a middle-ground option x is chosen in the presence of more extreme ones like y_2 and z_2, it will remain being chosen in the presence of at least one of its closer but still more extreme neighbors, like $y_1$or z_1. In other words, if x is perceived as a good choice among y_2 and z_2, but not so when y_1 is available, it indicates that at least one agent in this hypothetical population would like to choose something in between x and y_2, so we should expect that when faced with an option between y_2, x and z_1 she would keep perceiving x as a good choice.

Theorem 2. *A choice correspondence satisfies Pseudo-WARP, Strong Centrality, Neighbors and Archimedian Centrality if, and only if, it has a single-peak pseudo-rational representation.*

It is imediate from theorem 2 that if a choice correspondence has a single-peak pseudo-rational representation then it must also have a single-crossing pseudo-rational representation. It's possible, though, to go further. In the proof of theorem 2 we allow the collection of preferences $\mathcal{P}$ to contain every single preference relation that satisfies single-peak and does not violate the pseudo-rational representation of c, but by carefully choosing and ordering a subset of this collection of preferences we are able to arrive in a collection $\mathcal{P}'$ that also satisfies single-crossing without loosing the pseudo-rational representation of c and, evidently, the single-peak property of it's preference relations. This result implies that any choice correspondence that admits a single-peak pseudo-rational representation also admits a single-peak single-crossing pseudo-rational representation, as stated in the following theorem.

Theorem 3. *A choice correspondence satisfies Pseudo-WARP, Strong Centrality, Neighbors and Archimedian Centrality if, and only if, it has a single-peak and single-crossing pseudo-rational representation.*

5 Conclusion

Following Apesteguia, Ballester e Lu (2017), we characterized pseudo-rational choice correspondences that have a representation by collections of preferences satisfying the single-crossing and single-peak properties. Whereas Apesteguia, Ballester and Lu provided a characterization of a stochastic choice model, our characterization does not model probabilities among the possible choices in each choice problem. Therefore, it may be useful when the researcher does not have this data. As seen above, the single-crossing property is most important in the context of comparative statics, because it allows one to obtain results that are robust to misspecification of functional forms, what is called monotone comparative statics. The single-peak property is mostly known for its role in the median voter theorem, according to which, if all citizens have preferences that are single-peaked with respect to a natural order of the policies being chosen, then the choice by voting on pairs of options will be the preferred option of the voter with the median preferences.

There are still several extensions that can be done. Most immediately, we are working on developing a more elegant characterization of the single-peaked pseudo-rational choice correspondence. We are also working on obtaining representation theorems in a world of menus. For instance, if the individuals have preferences over menus of policies, like in the case of a social security reform in which the congress must approve a menu of policies among which the voter will choose one to validate it's retirement, we want to characterize when a choice correspondence over this menus will have a single-crossing and single-peak representation. With respect to the single-peakness, we believe this could be useful in modeling preferences over political parties, since they can, in principle, [also] be interpreted as menus of policies. Also, we are working on an application of the single-crossing and single-peak properties to the context of categories, where the individuals divide the world in categories and, each time they have to make a decision, they pick the most preferred option in a certain category - the choice of the category being determined by factors outside the model.

At last, we have already seen in the introduction that it is possible to obtain monotone comparative statics results using preferences only, in a world with finite alternatives, which is the world we are considering here. As described by Apesteguia, Ballester e Lu (2017), given a collection of preferences over risky assets that satisfy the single-crossing property, we can affirm that, for every asset, the best alternative for an individual with more risk aversion is either less riskier than or equal to the best alternative for an individual with less risk aversion. But, as seen in the introduction, the literature on monotone comparative statics is mostly based on objective functions (e.g., utility or profit functions) that satisfy a single-crossing property, instead of preferences. Therefore, further extensions that would be interesting, and we are working on, involve characterization theorems for arbitrary choice spaces and theorems that allow us to obtain representations of the single-crossing preferences through a family of objective functions that satisfy single-crossing as defined by the literature on monotone comparative statics.

Referências

AIZERMAN, M.; MALISHEVSKI, A. General theory of best variants choice: Some aspects. *IEEE Transactions on Automatic Control*, v. 26, n. 5, p. 1030–1040, Oct 1981. ISSN 0018-9286. Citado na página 25.

APESTEGUIA, J.; BALLESTER, M. A.; LU, J. Single-crossing random utility models. *Econometrica*, Blackwell Publishing Ltd, v. 85, n. 2, p. 661–674, 2017. ISSN 1468-0262. Disponível em: <http://dx.doi.org/10.3982/ECTA14230>. Citado 4 vezes nas páginas 12, 17, 19 e 21.

ASHWORTH, S.; MESQUITA, E. B. de. Monotone comparative statics for models of politics. *American Journal of Political Science*, [Midwest Political Science Association, Wiley], v. 50, n. 1, p. 214–231, 2006. ISSN 00925853, 15405907. Disponível em: <http://www.jstor.org/stable/3694267>. Citado na página 11.

MILGROM, P.; SHANNON, C. Monotone comparative statics. *Econometrica*, [Wiley, Econometric Society], v. 62, n. 1, p. 157–180, 1994. ISSN 00129682, 14680262. Disponível em: <http://www.jstor.org/stable/2951479>. Citado na página 11.

A Proofs

A.1 Proof of Theorem 1

[Necessity] Suppose $\mathcal{P} := \{\succcurlyeq_1, \dots, \succcurlyeq_k\}$ is a collection of linear orders on X that satisfies the single-crossing property with respect to $\succcurlyeq^*$ and is a pseudo-rational representation of c. As already shown by Aizerman e Malishevski (1981), this implies that c satisfies Pseudo-WARP. Now consider $x, y, z \in X$ such that $x \succ^* y \succ^* z$ and $y \in c(\{x, y, z\})$, and an arbitrary $A \subseteq X$, such that $y \in A$. If $x \in c(A \cup \{x\})$, by the representation we must have $x \succ_j w$, $\forall w \in A$ (in particular $x \succ_j y$), for some $\succcurlyeq_j \in \mathcal{P}$. Also, because of $y \in c(\{x, y, z\})$ and the single-crossing property, we must have $y \succ_i x$ and $y \succ_i z$ for some $i < j$. But then, again because of the single-crossing property, we must have $y \succ_j z$, implying by transitivity of $\succcurlyeq_j$ that $x \succ_j z$ and, by the representation, that $x \in c(A \cup \{x, z\})$.

For the second part, if $z \in c(A \cup \{z\})$, then we must have $z \succ_i w$, $\forall w \in A$ (in particular $z \succ_i y$), for some $\succcurlyeq_i \in \mathcal{P}$. Also, because of $y \in c(\{x, y, z\})$ and the single-crossing property, we must have $y \succ_j z$ and $y \succ_j x$ for some $j > i$. The, by the single-crossing property, we must have $y \succ_i x$, implying by transitivity of $\succcurlyeq_i$ that $z \succ_i x$ and, by the representation, that $z \in c(A \cup \{x, z\})$.

[Sufficiency] Suppose c is a choice correspondence that satisfies Pseudo-WARP and Centrality. Define a binary relation $\succcurlyeq$ by $x \succcurlyeq y$ if, and only if, $x \succcurlyeq^* y$ and $x \in c(\{x, y\})$ or $\{x\} = c(\{x, y\})$. We begin with the following claim:

Claim 1. *The relation $\succcurlyeq$ is a linear order such that c satisfies Centrality with respect to $\succcurlyeq$.*

Proof of Claim. Suppose c satisfies centrality with respect to $\succcurlyeq^*$, $x \succcurlyeq y \succcurlyeq z$ and $y \in c(\{x, y, z\})$. Then, by Pseudo-WARP, $y \in c(\{x, y\}), y \in c(\{y, z\})$ wich implies $x \succcurlyeq^* y$. If $y, z = c(\{y, z\})$, then we would also have $z \succ^* y$. If $\{y\} = c(\{y, z\})$, then $\forall A \subseteq X$ with $y \in A, z \notin c(A \cup \{z\})$. We must then have that c satisfies centrality with respect to $\succcurlyeq$. ‖

From now on, whenever we refer to Centrality, we mean Centrality with respect to $\succcurlyeq$.

Define a binary relation $\succcurlyeq_1 \subseteq X \times X$ by $x \succcurlyeq_1 y$ if, and only if, $x \succcurlyeq y$ and $\{x\} = c(\{x, y\})$ or $y \succ x$ and $x \in c(\{x, y\})$. We note that $\succcurlyeq_1$ is complete and anti-symmetric. We need the following claim:

Claim 2. *The relation $\succcurlyeq_1$ is a linear order.*

Demonstração. We only need to show that $\succcurlyeq_1$ is transitive. Suppose, thus, that $x \succcurlyeq_1 y$ and $y \succcurlyeq_1 z$. If $x = y$ or $y = z$, there is nothing to prove, so suppose that $x \succ_1 y$ and $y \succ_1 z$. If $y \notin c(\{x, y\})$ and $z \notin c(\{y, z\})$, then Pseudo-WARP implies that $c(\{x, y, z\}) = \{x\}$. Now Pseudo-WARP implies that $c(\{x, z\}) = c(\{x, y, z\})$ and, consequently, $x \succcurlyeq_1 z$. If $y \in c(\{x, y\})$ and $z \in c(\{y, z\})$, then we must have $y \succ x$, $z \succ y$, $x \in c(\{x, y\})$ and $y \in c(\{y, z\})$. By the transitivity of $\succcurlyeq$, we learn that $z \succ x$. If $x \notin c(\{x, y, z\})$, Pseudo-WARP implies that $c(\{x, y, z\}) = \{y, z\}$. But then Centrality plus $x \in c(\{x, y\})$ implies that $x \in c(\{x, y, z\})$, which is a contradiction. We learn that $x \in c(\{x, y, z\})$. Now Pseudo-WARP implies that $x \in c(\{x, z\})$ and, since $z \succ x$, we learn that $x \succcurlyeq_1 z$. Now suppose that $y \notin c(\{x, y\})$, but $z \in c(\{y, z\})$. By Pseudo-WARP, $y \notin c(\{x, y\})$ implies that $y \notin c(\{x, y, z\})$. By the definition of $\succcurlyeq_1$, $z \in c(\{y, z\})$ implies that $z \succ y$ and $y \in c(\{y, z\})$. Now, by Pseudo-WARP, we must have $x \in c(\{x, y, z\})$, which, again by Pseudo-WARP, gives us that $x \in c(\{x, z\})$. We cannot have $x \succ z$ and $z \in c(\{x, y, z\})$, otherwise Centrality would imply that $y \in c(\{x, y, z\})$ which is not true. We conclude that $x \succcurlyeq_1 z$ if $x \succ z$. Since $x \in c(\{x, z\})$, we also get that $x \succcurlyeq_1 z$ if $z \succ x$. We are left with the case $y \in c(\{x, y\})$ and $z \notin c(\{y, z\})$. In this case, we must have $y \succ x$ and $x \in c(\{x, y\})$. Also, Pseudo-WARP implies that $z \notin c(\{x, y, z\})$. Now Pseudo-WARP implies that $c(\{x, y, z\}) = \{x, y\}$ and another application of Pseudo-WARP implies that $x \in c(\{x, z\})$. If $z \succ x$, we immediately get $x \succcurlyeq_1 z$. If $x \succ z$, then Centrality would imply that $z \in c(\{x, y, z\})$ if $z \in c(\{x, z\})$. Since $z \notin c(\{x, y, z\})$, we learn that $z \notin c(\{x, z\})$ and, consequently, $x \succcurlyeq_1 z$. We conclude that $\succcurlyeq_1$ is a linear order. ||

We can now prove the following claim:

Claim 3. *For every A, if $x \in \max(A, \succcurlyeq_1)$, then $x \in c(A)$.*

Demonstração. This is true by definition when $|A| = 2$. We proceed by induction. Suppose the claim is true whenever $|A| \leq n$ and fix A such that $|A| = n + 1$ and $x \in \max(A, \succcurlyeq_1)$. Since $\succcurlyeq_1$ is a linear order, this implies that $x \succ_1 y$ for every $y \in A \setminus \{x\}$. If there is $y \in A$ such that $x \succ y$, then the definition of $\succcurlyeq_1$ implies that $y \notin c(\{x, y\})$ and, by Pseudo-WARP, $c(A) \subseteq A \setminus \{y\}$. Now Pseudo-WARP and the induction hypothesis imply that $x \in c(A \setminus \{y\}) \subseteq c(A)$. The only remaining case is when $y \succ x$ for every $y \in A \setminus \{x\}$. Let $z \in A$ be such that $z \succ w$ for every $w \in A \setminus \{z\}$. If $c(A) = \{z\}$, then Pseudo-WARP implies that $c(\{x, z\}) \subseteq c(A)$, which is a contradiction. We conclude that $c(A) \neq \{z\}$. If there exists $y \in A \setminus \{x, z\}$ such that $y \in c(A)$, then Centrality and the induction hypothesis imply that $x \in c(A)$. If there does not exist such a y, then we must also have $x \in c(A)$, because $c(A) \neq \{z\}$. ||

Now fix $i \in \mathbb{N} \setminus \{1\}$ and let's make the following induction hypotheses:

Induction Hypotheses. The collection $\{\succsim_1, \dots, \succsim_n\}$ is an ordered set of linear orders on X that satisfies single-crossing with respect to $\succsim$ and such that

$$\bigcup_{i=1}^{n} \max(A, \succsim_i) \subseteq c(A)$$

for every choice problem A.

Enumerate the elements of X according to $\succsim_n$. That is, $x_1 \succsim_n \cdots \succsim_n x_k$. Let $i \in \mathbb{N}$ be the first natural number such that $x_{i+1} \succ x_i$ and $x_{i+1} \in c(\{x_i, \dots, x_k)$. Define $\succsim_{n+1} := (\succsim_n \setminus \{(x_i, x_{i+1})\}) \cup \{(x_{i+1}, x_i)\}$. It is easy to see that $\succsim_{i+1}$ is a linear order such that $\max(A, \succsim_{i+1}) \subseteq c(A)$ for every $A \in 2^X \setminus \{\emptyset\}$ and $(\succsim_1, \dots, \succsim_n)$ satisfies single-crossing. This observation gives us an inductive procedure to build an ordered set $\{\succsim_1, \dots, \succsim_n\}$ of linear orders that satisfies single-crossing and such that $\max(A, \succsim_i) \subseteq c(A)$ for every $A \in 2^X \setminus \{\emptyset\}$ and every $i \in \{1, \dots, n\}$.

Let $\{\succsim_1, \dots, \succsim_n\}$ be the collection of linear orders constructed by the inductive procedure above. We need the following claim:

Claim 4. *If $\{\succsim_1, \dots, \succsim_n\}$ is the collection of linear orders constructed by the inductive procedure above, then $\succsim_n = \succsim$.*

Proof of Claim. Suppose the claim is not true. Let x be the $\succsim_n -$*minimal* element for which there exists $y \in X$ with $x \succ_n y$, but $y \succ x$. In fact, by the minimality of x, there exists such y such that x is the $\succsim_n$-successor of y.[1] If there exists $z \in L(y, \succ_n)$ with $y \succ z \succ x$ and $z \in c(\{x, y, z\})$, then $y \in c(L(x, \succsim_n))$ whenever $y \in c(L(x, \succsim_n) \setminus \{x\})$, by Centrality. But $L(x, \succsim_n) \setminus \{x\} = L(y, \succsim_n)$ and we know that $y \in c(L(y, \succsim_n))$. Otherwise, let $A := \{z \in L(y, \succ_n) : z \notin c(L(x, \succsim_n))\}$. By Pseudo-WARP, $c(L(x, \succsim_n)) = c(L(x, \succsim_n) \setminus A)$. Define also $B := \{z \in L(y, \succ_n) : y \succ x \succ z\}$. Successive appliactions of Centrality give us that $y \in c(L(x, \succsim_n) \setminus A)$ if $y \in c(L(x, \succsim_n) \setminus (A \cup B))$. But $L(x, \succsim_n) \setminus (A \cup B) = \{x, y\}$ and we know that $y \in c(\{x, y\})$. We have already seen this implies that $y \in c(L(x, \succsim_n) \setminus A) = c(L(x, \succsim_n))$. But then we should have an $\succsim_{n+1}$ with $y \succ_{n+1} x$, which is a contradiction. ‖

We now need the following claim:

Claim 5. *For any distinct $x, y, z \in X$ with $x \succ y \succ z$ and $y \in c(\{x, y, z\})$, and any $i \in \{1, \dots, n\}$, it cannot be true that $z \succ_i x \succ_i y$.*

Proof of Claim. Suppose the claim is not true and let i^* be the minimal $i \in \{1, \dots, n\}$ such that there exist $x, y, z \in X$ with $x \succ y \succ z$, $y \in c(\{x, y, z\})$ and $z \succ_{i^*} x \succ_{i^*} y$. By the minimality of i^* and the construction of the collection $\{\succsim_1, \dots, \succsim_n\}$, we must have that

[1] By $\hat{w}$ being the successor of w with respect to $\succsim_{i^*-1}$ we mean that $\hat{w} \succ_{i^*-1} w$ and for no $\tilde{w} \in X$ it is true that $\hat{w} \succ_{i^*-1} \tilde{w} \succ_{i^*-1} w$.

1. $z \succ_{i^*-1} y \succ_{i^*-1} x$;
2. there exist no $w, \hat{w} \in X$ with $w \succcurlyeq_{i^*-1} y$, $w \succ \hat{w}$, $\hat{w}$ being the successor of w with respect to $\succcurlyeq_{i^*-1}$ and $w \in c(L(\hat{w}, \succcurlyeq_{i^*-1}))$;[2,3]
3. $x \in c(L(y, \succcurlyeq_{i^*-1}))$.

We claim that there must exist $w \in X$ such that $w \succ z$, $z \succ_{i^*-1} w \succcurlyeq_{i^*-1} y$ and $w \in c(L(z, \succcurlyeq_{i^*-1}))$. To see that, suppose that there does not exist w such that $w \succ z$, $z \succ_{i^*-1} w \succ_{i^*-1} y$ and $w \in c(L(z, \succcurlyeq_{i^*-1}))$. Let $A := \{w \in X : z \succ_{i^*-1} w \succ_{i^*-1} y \text{ and } w \notin c(L(z, \succcurlyeq_{i^*-1}))\}$. By Pseudo-WARP, $c(L(z, \succcurlyeq_{i^*-1})) = c(L(z, \succcurlyeq_{i^*-1}) \setminus A)$. Now let $B := \{w \in X : z \succ w \text{ and } z \succ_{i^*-1} w \succ_{i^*-1} y\}$. Since $z \in c(L(z, \succcurlyeq_{i^*-1}))$, $z \in c(\{y, w, z\})$ for every $w \in B$. Now successive applications of Centrality give us that $y \in c(L(z, \succcurlyeq_{i^*-1}) \setminus A)$ if $y \in c(L(z, \succcurlyeq_{i^*-1}) \setminus (A \cup B)) = c(L(y, \succcurlyeq_{i^*-1}) \cup \{z\})$. Since $x \in c(L(y, \succcurlyeq_{i^*-1}))$ and $y \in c(\{x, y, z\})$, Centrality implies that $x \in c(L(y, \succcurlyeq_{i^*-1}) \cup \{z\})$. Now let $C := \{\hat{w} \in X : x \succ_{i^*-1} \hat{w} \text{ and } \hat{w} \notin c(L(y, \succcurlyeq_{i^*-1}) \cup \{z\})\}$ and $D := \{\hat{w} \in X : x \succ_{i^*-1} \hat{w} \text{ and } \hat{w} \in c(L(y, \succcurlyeq_{i^*-1}) \cup \{z\})\}$. By Pseudo-WARP, $c(L(y, \succcurlyeq_{i^*-1}) \cup \{z\}) = c((L(y, \succcurlyeq_{i^*-1}) \cup \{z\}) \setminus C)$. Now, fix $\hat{w} \in D$. By Pseudo-WARP, this implies that $\hat{w} \in c(\{x, \hat{w}, z\})$, so, the minimality of i^* implies we cannot have that $x \succ \hat{w} \succ z$. That is, either $\hat{w} \succ x \succ y$ or $y \succ z \succ \hat{w}$. Now successive applications of Centrality give us that $y \in c((L(y, \succcurlyeq_{i^*-1}) \cup \{z\}) \setminus C)$ if $y \in c((L(y, \succcurlyeq_{i^*-1}) \cup \{z\}) \setminus (C \cup D))$. But notice that $(L(y, \succcurlyeq_{i^*-1}) \cup \{z\}) \setminus (C \cup D) = \{x, y, z\}$, so we know that $y \in c((L(y, \succcurlyeq_{i^*-1}) \cup \{z\}) \setminus (C \cup D))$. As we have argued above, this implies that $y \in c((L(y, \succcurlyeq_{i^*-1}) \cup \{z\}) \setminus C) = c(L(y, \succcurlyeq_{i^*-1}) \cup \{z\})$. In turn, this implies that $y \in c(L(z, \succcurlyeq_{i^*-1}) \setminus A) = c(L(z, \succcurlyeq_{i^*-1}))$ and, consequently, it is always true that there exists $w \in X$ such that $w \succ z$, $z \succ_{i^*-1} w \succcurlyeq_{i^*-1} y$ and $w \in c(L(z, \succcurlyeq_{i^*-1}))$. Fix, then, a $w \in X$ with $w \succ z$, $z \succ_{i^*-1} w \succcurlyeq_{i^*-1} y$ and $w \in c(L(z, \succcurlyeq_{i^*-1}))$. Let $\hat{w}$ be the successor of w with respect to $\succcurlyeq_{i^*-1}$. We cannot have $\hat{w} \succ w$, otherwise we would have a contradiction to the minimality of i^*. But then we arrive at a contradiction to 2 above. This proves the claim. ‖

We can now prove the following result:

Claim 6. *For any $A \in 2^X \setminus \{\emptyset\}$, if $y \in c(A)$, then there exists $\succcurlyeq_i \in \{\succcurlyeq_1, \ldots, \succcurlyeq_n\}$ with $y \in \max(A, \succcurlyeq_i)$.*

Proof of Claim. Let $B := \{x \in A : x \succ_n y\} = \{x \in A : x \succ y\}$ and $C := \{z \in A : y \succ_n z\} = \{z \in A : y \succ z\}$. By Pseudo-WARP, $y \in c(\{x, y, z\})$ for every $x \in B$ and $z \in C$. Let i^* be the first $i \in \{1, \ldots, n\}$ such that $y \succ_{i^*} z$ for every $z \in C$. By Claim 5, we must have $y \succ_{i^*} x$ for every $x \in B$. Consequently, $\{y\} = \max(A, \succcurlyeq_{i^*})$. ‖

[2] By $\hat{w}$ being the successor of w with respect to $\succcurlyeq_{i^*-1}$ we mean that $\hat{w} \succ_{i^*-1} w$ and for no $\tilde{w} \in X$ it is true that $\hat{w} \succ_{i^*-1} \tilde{w} \succ_{i^*-1} w$.

[3] *Notation.* For any $\hat{w} \in X$ and any preorder $\succcurlyeq$, by $L(\hat{w}, \succcurlyeq)$ we mean the set $\{\tilde{w} \in X : \hat{w} \succcurlyeq \tilde{w}\}$ and by $U(\hat{w}, \succcurlyeq)$ we mean the set $\{\tilde{w} \in X : \tilde{w} \succcurlyeq \hat{w}\}$.

This shows that c has the desired representation, except that, for now, we only know that $\{\succcurlyeq_1, \ldots, \succcurlyeq_n\}$ satisfies single-crossing with respect to $\succcurlyeq$. We conclude the proof with the following claim:

Claim 7. *The collection $\{\succcurlyeq_1, \ldots, \succcurlyeq_n\}$ satisfies single-crossing with respect to $\succcurlyeq^*$.*

Demonstração. Suppose $x, y \in X$ are such that $x \succ^* y$ and $x \succ_i y$ for some $i \in \{1, \ldots, n-1\}$. If $x \succ y$, then $x \succ_j y$ for every $j \geq i$ by the fact that $\{\succcurlyeq_1, \ldots, \succcurlyeq_n\}$ satisfies single-crossing with respect to $\succcurlyeq$. If $y \succ x$, then we must have that $\{y\} = c(\{x, y\})$, by the definition of $\succcurlyeq$. But then we would have $y \succ_j x$ for every $j \in \{1, \ldots, n\}$, which is a contradiction. We conclude that $\{\succcurlyeq_1, \ldots, \succcurlyeq_n\}$ satisfies single-crossing with respect to $\succcurlyeq^*$. ‖

This concludes the proof of the theorem.

A.2 Proof of Theorem 2

[Necessity] Suppose $\mathcal{P} := \{\succcurlyeq_1, \ldots, \succcurlyeq_k\}$ is a collection of linear orders on X that satisfies the single-peak property with respect to $\succcurlyeq^*$ and is a pseudo-rational representation of c.

Claim 8. *c satisfies Pseudo-Warp.*

Proof of Claim. Suppose $x \in c(A) \cap B$ and $c(B) \subseteq A$. We must have that there is a linear order $\succcurlyeq_i \in \mathcal{P}$, such that $\{x\} = max(A, \succcurlyeq_i)$. As $c(B) \subseteq A$, $max(B, \succcurlyeq_i) \subseteq A$, which implies $\{x\} = max(B, \succcurlyeq_i)$ and $x \in c(B)$. □

Claim 9. *c satisfies Strong-Centrality.*

Proof of Claim. Given $x \succcurlyeq^* y \succcurlyeq^* z$ and an arbitrary set A such that $y \in A$, if $x \in c(A \cup \{x\})$, then we must have that, for some $\succcurlyeq_i$, $x \succcurlyeq_i y$. But then, by single-peakness, we cannot have $z \succcurlyeq_i x$, since: if z is the peak of $\succcurlyeq_i$, we should have $y \succcurlyeq_i x$, and if z is not the peak, either the peak if below z in $\succcurlyeq^*$and we would also have $y \succcurlyeq_i x$, or the peak is above z and we would have either $y \succcurlyeq_i x$ or $x \succcurlyeq_i y \succcurlyeq_i z$. Therefore, we must have $x \in c(A \cup \{x, z\})$, and Strong Centrality is satisfied. □

Claim 10. *c satisfies Neighbors.*

Proof of Claim. Also, given an arbitrary A such that $|c(A)| > 1$. If $x \in c(A)$, then we must have that $x = \max(A, \succcurlyeq_i)$ for some $\succcurlyeq_i$. Suppose the second best in A according to $\succcurlyeq_i$is not a neighbor of x. If this second best in below x in $\succcurlyeq^*$, then it is inverted with the below-neighbor of x, which can happen only if the peak is also below this second best in

$\succcurlyeq^*$, contradicting $x = \max(A, \succcurlyeq_i)$, due to single-peakness. If this second best is above x in $\succcurlyeq^*$, then we would have that it is not inverted with the neighbor of x in A according to $\succcurlyeq_i$, which can happen only if the peak is above x, but that would contradict $x = \max(A, \succcurlyeq_i)$ again. $\square$

Claim 11. *c satisfies Archimedian Centrality.*

Proof of Claim. Suppose $x \in c(\{x, y_2, z_2\})$ and $y_2 \succcurlyeq^* y_1 \succ^* x \succ^* z_1 \succcurlyeq^* z_2$. This implies there is $\succcurlyeq_i \in \mathcal{P}$ with $x \in max(\{x, y_2, z_2\}, \succcurlyeq_i)$, $x \succ_i y_2$ and $x \succ_i z_2$. Let $\hat{x} := max(X, \succcurlyeq_i)$. The single-peakedness of $\succcurlyeq_i$ implies that either $\hat{x} \succcurlyeq_i x \succ_i z_1$ or $\hat{x} \succcurlyeq_i x \succ_i y_1$. But then, we must have that either $x \in max(\{x, y_2, z_1\})$ or $x \in c(\{x, y_1, z_2\})$, which proves the claim. $\square$

[Sufficiency] Suppose c satisfies Strong Centrality, Pseudo-WARP, Neighbors and Archimedian Centrality. Take $w_1 \in c(X)$ and let $N := |X|$. For each $n \in \{1, 2, \ldots, N-1\}$, choose $w_{n+1} \in c(X \setminus \{w_1, \ldots, w_n\}) \cap N(X \setminus \{w_1, \ldots, w_{n-1}\}, w_n)$. Notice that Neighbors guarantees there will always be a w_{n+1} to be chosen this way. Let $\succcurlyeq_i := \{(w_n, w_{n+1}) \in X \times X : n \in \{1, 2, \ldots, N-1\}\}$ and $\mathcal{P}$ be the collection of every linear order $\succcurlyeq_i$ that can be defined this way. Because of it's construction $\succcurlyeq_i$ must satisfy single-peak with respect to $\succcurlyeq^*$ and Pseudo-WARP implies that $\forall \succcurlyeq_i \in \mathcal{P}, \forall B \in \Omega_X, max(B, \succcurlyeq_i) \subseteq c(B)$. We must now show that for each $A \in \Omega_X$ and each $x \in c(A)$ there is a linear order $\succcurlyeq_i \in \mathcal{P}$ such that $x \in max(A, \succcurlyeq_i)$.

Fix some choice problem A and some $x \in c(A)$. If $x \in c(X)$, define $x := w_1$ and we are done. Suppose then $x \notin c(X)$. If we have that $\forall y \in A, x \succcurlyeq^* y$ and $\exists w \in c(X)$ with $w \succcurlyeq^* x$, then define $w := w_1$ and again by the construction of $\succcurlyeq_i$ we must have that $x \in max(A, \succcurlyeq_i)$. The same argument is valid when we have that $\forall y \in A, y \succcurlyeq^* x$ and $\exists w \in c(X)$ with $x \succcurlyeq^* w$.

We need now the following claim.

Claim 12. *Let $A, B \in \Omega_X$. If $x \in c(A) \cap B$ and for every $y \in c(B)$ there exists $z \in A$ with $x \succ z \succcurlyeq y$ or $y \succcurlyeq z \succ x$, then $x \in c(B)$.*

Proof of Claim. As c satisfies Pseudo-WARP and Strong Centrality and for every $y \in c(B)$ there exists $z \in A$ with $x \succ z \succcurlyeq y$ or $y \succcurlyeq z \succ x$, then successive applications of Strong Centrality give us that $x \in c(A \cup c(B))$. Now, as $x \in c(A \cup c(B)) \cap B$ and $c(B) \subseteq A \cup c(B)$, then Pseudo-WARP implies $x \in c(B)$. $\square$

By claim 12 we must have that if $x \in c(A)$ and $\forall y \in A, x \succcurlyeq^* y$ but now for no $w \in c(X)$ it is true that $w \succcurlyeq^* x$, then there must be $w \in c(X)$ with $x \succcurlyeq^* w \succ^* y$, otherwise claim 12 would imply $x \in c(X)$. And again, the same argument is valid when $\forall y \in A, y \succsim^* x$ and $c(X) \cap L(x, \succsim^*) = \emptyset$. In these cases, as $x \in c(A)$, taking $y \in N(A, x)$, claim 12 also implies that $x \in c(U(x, \succsim^*) \cup L(y, \succsim *))$, when $x \succ^* y$, and $x \in c(U(y, \succsim^*) \cup L(x, \succsim *))$, when $y \succ^* x$.

We now proceed to the following claim:

Claim 13. *If $x \in c(A), y \in N(A, x), w \in c(X)$, with $x \succ^* w \succ^* y$ and $x \in c(U(x, \succsim^*) \cup L(y, \succsim *))$ or $y \succ^* w \succ^* x$ and $x \in c(U(y, \succsim^*) \cup L(x, \succsim *))$, then we must have that $\exists \succsim_i \in \mathcal{P}$ with $x \in max(A, \succsim_i)$.*

Proof of Claim. Suppose we have the first case, that is, $x \in c(A), y \in N(A, x), w \in c(X)$, with $x \succ^* w \succ^* y$ and $x \in c(U(x, \succsim^*) \cup L(y, \succsim *))$. Take $w_1 := w$. Then there must be a set $\{w_1, \ldots, w_n\} \subseteq X \setminus (U(x, \succsim^*) \cup L(y, \succsim *))$ with $x \in N(X \setminus \{w_1, \ldots, w_{n-1}\}, w_n) \cap c(X \setminus \{w_1, \ldots, w_n\})$, so that we can define $x := w_{n+1}$ and, as $A \subseteq X \setminus \{w_1, \ldots, w_n\}$, $x \in max(A, \succsim_i)$, where $\succsim_i \in \mathcal{P}$ is a linear order derived from the set $\{w_1, \ldots, w_{n+1}\}$ as above. □

With claim 13 and the preceding arguments, we prove that $\exists \succsim_i \in \mathcal{P}$ such that $x \in max(A, \succsim_i)$ whenever $x \in (max(A, \succsim^*) \cup min(A, \succsim^*))$. We must now prove that the result is also valid when $x \in c(A) \setminus (max(A, \succsim^*) \cup min(A, \succsim^*))$. In this case, we must have that $\exists y, z \in A$ with $\{y, z\} = N(A, x)$. Let $\{\hat{y}, \hat{z}\} := N(X, x)$ such that $y \succsim^* \hat{y} \succ^* x \succ^* \hat{z} \succsim^* z$. Then Archimedian Centrality gives us that either $x \in c(\{y, x, \hat{z}\})$ or $x \in c(\{\hat{y}, x, z\})$. In each case successive applications of Strong Centrality imply that $x \in c(U(y, \succsim^*) \cup L(x, \succsim^*))$, in the first case, or $x \in c(U(x, \succsim^*) \cup L(z, \succsim^*))$, in the second. If we have that $x \in c(U(y, \succsim^*) \cup L(x, \succsim^*))$, then Pseudo-WARP implies that $\exists w \in c(X) \setminus (U(y, \succsim^*) \cup L(x, \succsim^*))$, otherwise we would have $x \in c(X)$. Now, claim 13 implies that there is $\succsim_i \in \mathcal{P}$ such that $x \in max(A, \succsim_i)$. The case where $x \in c(U(x, \succsim^*) \cup L(z, \succsim^*))$ is similar and with it we conclude the proof of the theorem.

A.3 Proof of Theorem 3

[Sufficiency] Suppose c satisfies Strong Centrality, Pseudo-WARP, Neighbors and Archimedian Centrality. We have already proved that in this setting c has a single-peak representation $\mathcal{P}$, we will show now that it is possible to make changes to $\mathcal{P}$ so that it also satisfies single-crossing without losing the representation property. For that let $\mathcal{P}$ be defined as in the proof of theorem 2, in the sense that it contains every preference relation that satisfies single peak and does not violate the pseudo-rational representation of c. Let $\mathcal{P}' = \{\succsim_1, \ldots, \succsim_m\} \subseteq \mathcal{P}$ be a $\supseteq$ -maximal subset of $\mathcal{P}$ that satisfies single-crossing, which means that, if $y \succ^* x$ and $y \succ_i x$ then $y \succ_j x, \forall \succ_j \in \mathcal{P}'$ with $j > i$. We must show that for an arbitrary choice problem $A \in \Omega_X$ and $x \in c(A)$, there is $\succsim_i \in \mathcal{P}'$ such that $x \in max(A, \succcurlyeq_i)$. Suppose then, by contradiction, that for no $\succcurlyeq \in \mathcal{P}'$ it is true that $x \in max(A, \succcurlyeq)$. We proceed with the following claim:

Claim 14. *If* $x \in (max(A, \succcurlyeq^*) \cup min(A, \succcurlyeq^*)) \cap c(A)$*, then* $\exists \succcurlyeq_l \in \mathcal{P}'$ *such that* $x \in max(A, \succcurlyeq_l)$.

Demonstração. Suppose $x \in max(A, \succcurlyeq^*) \cap c(A)$ and let $\hat{w}_1 := max(c(X), \succcurlyeq^*)$ and for each $i < |X|$, $\hat{w}_{i+1} := max(c(X \setminus \{\hat{w}_1, \dots, \hat{w}_i\}) \cap N(X \setminus \{\hat{w}_1, \dots, \hat{w}_{i-1}\}, \hat{w}_i), \succcurlyeq^*)$. Let now $\succcurlyeq_l := \{(w_i, w_{i+1}) \in X^2 : i \in \{1, \dots, |X|-1\}\}$. As $\succcurlyeq_l$ satisfies single-peak and in no way violates the pseudo-rational representation of c, we must have that $\succcurlyeq_l \in \mathcal{P}$. Note then that, if for some $\succcurlyeq_i \in \mathcal{P}$ and $x, y \in X$ it is true that both $x \succcurlyeq_i y$ and $x \succcurlyeq^* y$, then either $\hat{w}_1 \succcurlyeq^* x \succcurlyeq^* y$ or $x \succcurlyeq^* \hat{w}_1 \succcurlyeq^* y$ and in both cases we must have $x \succcurlyeq_l y$. As we chose $\succcurlyeq_i$ arbitrarily, this implies that $\succcurlyeq_l$ satisfies single-crossing with respect to $\succcurlyeq^*$ towards any other preference in $\mathcal{P}$ and, by the $\supseteq$-maximality of $\mathcal{P}'$, $\succcurlyeq_l \in \mathcal{P}'$. Finally, as $x \in c(A)$, by the theorem 2, there must be $\succcurlyeq_j \in \mathcal{P}$ such that $x \succ_j y, \forall y \in A \setminus \{x\}$, and then we ought to have $x \in max(A, \succcurlyeq_l)$. The proof when $x \in min(A, \succcurlyeq^*) \cap c(A)$ is symmetrical. □

Let then $y, z \in N(A, x)$ be such that $y \succcurlyeq^* x \succcurlyeq^* z$. Note we must have that $\forall \succcurlyeq \in \mathcal{P}'$, either $z \succcurlyeq x$ or $y \succcurlyeq x$. Let also $j := max\{n \in \mathbb{N} : z \succ_n x, \text{ and } \succcurlyeq_n \in \mathcal{P}'\}$, which implies that, as $\mathcal{P}'$ satisfies single crossing, $\forall k < j, z \succ_k x$ and $\forall k > j, y \succ_k x$, and let $z^* := max(X, \succcurlyeq_j)$ and $y^* := max(X, \succcurlyeq_{j+1})$, which implies $y^* \succ^* x \succ^* z^*$. We will show that $\exists \succcurlyeq_i \in \mathcal{P}$, such that $x \in max(A, \succcurlyeq_i)$ and $\{\succcurlyeq_1, \dots, \succcurlyeq_j, \succcurlyeq_i, \succcurlyeq_{j+1}, \dots, \succcurlyeq_m\}$ satisfies single crossing. Archimedian Centrality implies that either $x \in c(U(x, \succcurlyeq^*) \cup L(z, \succcurlyeq^*))$ or $x \in c(U(y, \succcurlyeq^*) \cup L(x, \succcurlyeq^*))$. Suppose then $x \in c(U(x, \succcurlyeq^*) \cup L(z, \succcurlyeq^*)) \setminus c(X)$ and take $w_1 \in N(c(X), x)$ such that $y^* \succ^* x \succ^* w_1 \succcurlyeq^* z^*$.[4] Notice now that, if $w_1 \succ^* b$, then, by Strong Centrality, $y^* \succ_{j+1} w_1 \succ_{j+1} b$, and, in the same way, if $b \succ^* w_1$, then $z^* \succcurlyeq_j w_1 \succ_j b$. We now need the following claim to proceed.

Claim 15. *(@)Suppose* $w_1, \dots, w_n \subseteq M(x, z, \succ^*)$[5] *is an ordered set in which* $w_{k+1} \in c(X \setminus \{w_1, \dots, w_k\}) \cap N(X \setminus \{w_1, \dots, w_{k-1}\}, w_k)$ *for all* $k < n$, $w_{k+1} \succ^* w_k$ *implies* $w_k \succ_j w_{k+1}$, $w_k \succ^* w_{k+1}$ *implies* $w_k \succ_{j+1} w_{k+1}$ *and whenever there are* $a, b \in X$ *fulfilling these conditions, we take* $w_k := max(\{a, b\}, \succcurlyeq^*)$. *We must then have that either* $x \in c(X \setminus \{w_1, \dots, w_n\}) \cap N(X \setminus \{w_1, \dots, w_{n-1}\}, w_n)$ *and* $w_n \succ_j x$ *or there is* $w_{n+1} \in M(x, y, \succ^*) \cap c(X \setminus \{w_1, \dots, w_n\}) \cap N(X \setminus \{w_1, \dots, w_{n-1}\}, w_n)$ *such that if* $w_{n+1} \succ^* w_n$ *then* $w_n \succ_j w_{n+1}$ *and ifw*$_n \succ^* w_{n+1}$ *then* $w_n \succ_{j+1} w_{n+1}$.

Demonstração. As $w_n \in M(x, z, \succ^*)$, if $x \notin c(X \setminus \{w_1, \dots, w_n\}) \cap N(X \setminus \{w_1, \dots, w_{n-1}\}, w_n)$, we either have $x \succ^* w_n \succ^* z^*$ or $x \succ^* z^* \succ^* w_n \succ^* z$ and, since $\succ_j$ satisfies single-peak and $z \succ_j x$, which in both cases implies $w_n \succ_j x$. Suppose then $x \notin c(X \setminus \{w_1, \dots, w_n\}) \cap N(X \setminus$

[4] The case where $x \in c(X)$ is rather trivial.

[5] Where $M(x, z, \succ^*) := \{a \in X : x \succ^* a \succ^* z\}$.

$\{w_1, \ldots, w_{n-1}\}, w_n)$, as $x \in c(U(x, \succsim^*) \cup L(z, \succsim^*))$ we must have that $M(x, z, \succ^*) \neq \emptyset$. Take then $w_{n+1} \in max(M(x, y, \succ^*) \cap c(X \setminus \{w_1, \ldots, w_n\}) \cap N(X \setminus \{w_1, \ldots, w_{n-1}\}, w_n), \succsim^*)$. If we have that $y^* \succ^* w_n \succ^* w_{n+1}$, then, by single-peakness of $\succ_{j+1}$, $w_n \succ_{j+1} w_{n+1}$. If, otherwise, $w_{n+1} \succ^* w_n$, we must have that either $w_{n+1} \succ^* w_n \succsim^* z^*$ or $w_{n+1} \succ^* w_1 \succsim^* z^* \succ^* w_n$. In the first case we get that $w_n \succ_j w_{n+1}$ by single-peakness of $\succ_j$. For the second case, as $w_{n+1} \succ^* w_{n-1} \succ^* w_n$, we must have $w_n \succ_j w_{n+1}$, otherwise we should have taken w_{n+1} instead of $w_n$on the previous step. □

With claim @ we prove we can form a sequence of alternatives $\{w_1, \ldots, w_n, x\}$ that satisfies single peak, if we consider $w_k \succ_i w_{k+1}, \forall k \leq n$, in no way violates either the pseudo-rational representation or the single-crossing property towards $\succ_j$ or $\succ_{j+1}$ and for which $x = max(A, \succ_i)$. To conclude the proof we proceed by choosing the following alternatives in a way that $w_{k+1} = max(N(X \setminus \{w_1, \ldots, w_n, x, \ldots, w_{k-1}\}, w_k), \succ_j)$. As $max(B, \succ_j) \subseteq c(B)$, this does not violate the pseudo-rational representation. If now we have that $w_k \succ^* w_{k+1}$, we must have $y^* \succ_{j+1} w_k \succ_{j+1} w_{k+1}$, and if $w_{k+1} \succ^* w_k$ then $z^* \succ_j w_k \succ_j w_{k+1}$ so that $\succsim_i := \{(w_k, w_{k+1}) \in X^2 : k \in \{1, \ldots, |X| - 1\}\}$ and $\{\succsim_1, \ldots, \succsim_j, \succsim_i, \succsim_{j+1}, \ldots, \succsim_m\}$ satisfies single-crossing, contradicting the $\supseteq$ −maximality of $\mathcal{P}'$.