Universidade de Brasília
Instituto de Ciências Exatas
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Domain of Attraction for Extremes and Mallows Distance Convergence

by

Somayeh Mousavinasr

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Por

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Dedicated to my parents,
my husband
and my daughters

"As long as you live ,
keep learning how to live"

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Abstract

In this work we explore the connection between domain of attraction for Fréchet distribution and Mallows distance convergence. Under the framework of i.i.d. random variables classical results are derived for Mallows convergence. When the assumption of i.i.d. is dropped sufficient conditions for the desired convergence are proposed. By making use of regeneration approach these results are extended to all three types of extreme distributions. As byproduct, one obtains characterization for domains of attraction for Markov chains with general state space.

Keywords: Mallows distance; Extremes; Domain of attraction; Fréchet distribution; Regenerative Process.

Resumo

Neste trabalho exploramos a conexão entre o domínio de atração da distribuição Fréchet e a convergência na distância de Mallows. Quando as variáveis aleatórias são i.i.d. provamos as convergências clássicas na distância de Mallows. E para estruturas de dependência geral apresentamos as condições suficientes que garantem a convergência desejada. Métodos regenerativos são utilizados possibilitando a análise para todos os tres tipos de distribuições extremais. Como consequência temos a extensão dos resultados clássicos para cadeias de Markov com espaço de estados geral.

Palavras-chaves: Distância Mallows; Extremos; Domínio de atração; distribuição de Fréchet; Processo regenerativo.

Contents

Introduction

Statistics of extremes has been successfully used in a large variety of fields such as floods, heavy rains, extreme temperatures, failure of equipments, breaking strength, air pollution, finance, insurance, risk theory and many others. Extreme Value Theory have faced a huge development in the last decades partially due to the fact that rare events can have catastrophic consequences for human activities through their impact on natural and constructed environments. The pioneer results concerning the possible limiting laws for extremes of a random sample $X_1, ..., X_n$ were obtained by Fisher and Tippett (1928). Rigorous formalization were established by Gnedenko (1943). More specifically, for a given sequence $\{X_n\}_{n\geq 1}$ of independent and identically distributed (i.i.d.) random variables with a common distribution F let the partial maximum be defined by $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$. Assume that there exist norming constants $a_n > 0$ and b_n such that

$$\lim_{n\to\infty} P[a_n^{-1}(X_{(n)} - b_n \leq x)] = \lim_{n\to\infty} F^n(a_n x + b_n) = H(x) \qquad (0.0.1)$$

where H is a non-degenerated distribution. Then the limiting distribution H belongs necessarily to one of the following three classes:

$$\textit{Fréchet} \qquad \Phi_\alpha(x) = \begin{cases} 0 & x < 0 \\ \exp\{-x^{-\alpha}\} & x \geq 0, \end{cases}$$

$$\textit{Weibull} \qquad \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & x < 0 \\ 1 & x \geq 0, \end{cases}$$

$$\textit{Gumbel} \qquad \Lambda(x) = \exp\{-e^{-x}\} \qquad\qquad x \in \mathbb{R}.$$

For properties of these distributions see, for example, Gumbel (1958), de Haan (1970, 1976), Weissman (1978), Galambos (1987), Falk and Marohn (1993) and Worms (1998), among others. The class of distributions F that satisfies (0.0.1) are denominated the domain of attraction of H. Characterization of the domain of attraction is of central interest in the study of extremes and we will address this problem.

Extreme Value Theory (EVT) is the counterpart of the Central Limit Theorem (CLT) type results for partial sums. However, while the CLT is concerned with "small" fluctuations around the mean resulting from an aggregation process, the EVT provides results on the asymptotic behavior of the extreme realizations.

A key assumption for the classical CLT is the finiteness of the second moment. When this assumption is dropped heavy-tailed distributions arise. The most important class of heavy-tailed distributions, namely, the α-stable laws possess infinite variance. Due to their infinitely divisibility property, the stable laws play a central role in the study of asymptotic behavior of normalized partial sums, a similar role normal distribution plays among distributions with finite second moment. With the recognition of the importance of stable laws the interrelation between CLT for stable distributions and EVT emerged. Galambos (1987) devoted the section 4.5 to study this relation by exploring the regularly variation properties of the distribution tails. In fact, if $S_n = \sum_{j=1}^{n} X_j$ and for some constants $A_n > 0$ and B_n we have the convergence in distribution

$$\frac{S_n - B_n}{A_n} \xrightarrow{d} Y_\alpha$$

where Y_α has α-stable distribution with $0 < \alpha < 2$ then the common distribution F of the X_n's has regularly varying tails of index $-\alpha$, $RV_{-\alpha}$ (either left-tail or right-tail or both tails). As for the Fréchet distribution, if the stabilized maximum converges

$$\frac{X_{(n)} - b_n}{a_n} \xrightarrow{d} \Phi_\alpha$$

then for the right-tail we have $1 - F \in RV_{-\alpha}$.

On the other hand, the Mallows (1972) distance measures the discrepancy between two distribution functions and has been successfully used to derive Central Limit Theorem type results for heavy-tailed stable distributions (see, e.g., Johnson and Samworth (2005) or Dorea and Oliveira (2014)). For $r > 0$ the Mallows distance $d_r(F, G)$ between two distribution functions F and G is defined by

$$d_r(F, G) = \inf_{(X,Y)} \left\{ E(|X - Y|^r) \right\}^{1/r}, \;\; X \stackrel{d}{=} F, \; Y \stackrel{d}{=} G$$

where the infimum is taken over all random vectors (X, Y) with marginal distributions F and G ($\stackrel{d}{=}$: equality in distribution). The key connection between convergence in Mallows distance and the convergence in distribution was established by Bickel and Freedman(1981) for distributions with finite r-th moments:

$$d_r(F_n, G) \underset{n}{\rightarrow} 0 \;\Leftrightarrow\; F_n \xrightarrow{d} G \;\text{ and }\; \int |x|^r dF_n(x) \underset{n}{\rightarrow} \int |x|^r dG(x).$$

The above arguments naturally induce us to make use of Mallows distance in order to strengthen the convergence (0.0.1). Other authors have studied different types of convergence : moment convergence in Pickands (1968), convergence in density in Sweeting (1985) and large deviation in Goldie and Smith (1987) and Vivo (2015).

In Chapter 1, we present preliminary concepts and results that are fundamental for understanding the subsequent chapters. It includes some details on extreme distributions, regularly varying functions, stable distributions, Mallows distance and regenerative processes.

In Chapter 2 we will focus on convergence to Fréchet distribution Φ_α. For $\alpha \geq 1$ and for $\{X_n\}$ a sequence of i.i.d. random variables our Theorem 2.3.1 provides sufficient conditions for

$$\lim_{n\to\infty} d_\alpha(F_{M_n}, \phi_\alpha) = 0 \quad \text{and} \quad F_{M_n} \xrightarrow{d} \phi_\alpha$$

where $M_n = \dfrac{X_{(n)} - b_n}{a_n}$ and $M_n \overset{d}{=} F_{M_n}$. The case $0 < \alpha < 1$ is treated in Corollary 2.4.5. In Theorem 2.3.8 we introduce the max-domain of strong normal attraction of Φ_α. The general case is treated in Theorem 2.4.2 where the assumption of independency of the X_n's is dropped but Lindeberg's type conditions are added :

$$\frac{1}{n}\sum_{i=1}^{n} E\{|X_i - Y_i|^\alpha 1_{(|X_i - Y_i| > bn^{\frac{1}{\alpha}})}\} \underset{n}{\to} 0, \quad \forall b > 0$$

where $\{Y_n\}$ is a sequence of i.i.d. random variables with common distribution Φ_α.

Though our Theorem 2.4.2 does not requires independency or same distribution for the X_n's, the Lindeberg condition imposes a closeness with respect to a sequence of i.i.d. random variables. This suggests that processes $\{X_n\}_{n\geq 0}$ that admit a decomposition into independent blocks could well be studied via Mallows distance. That is the case of Markov chains and more generally the regenerative processes. Namely, processes for which there exist integer-valued random variables $0 < T_0 < T_1 < \dots$ and such that the cycles

$$C_1 = \{X_n, T_0 \leq n < T_1\}, C_2 = \{X_n, T_1 \leq n < T_2\}, \cdots$$

are i.i.d. random vectors. In Chapter 3, our approach for regenerative processes will also allow us to treat all three types of extreme distribution Φ_α (Fréchet) , Ψ_α (Weibull) and Λ (Gumbel). We borrow some of the arguments from Rootzén (1988) by considering the submaxima over the cycles,

$$\xi_j = \max\{X_n : T_{j-1} \leq n < T_j\} \quad j \geq 1.$$

Then approximate $X_{(n)} = \max\{X_1, X_2, \dots, X_n\}$ by $\max\{\xi_0, \dots, \xi_{v_n}\}$ where $v_n = \inf\{k; T_k > n\}$. Our Theorem 3.2.2, for $1 \leq \alpha'$ and under the framework of i.i.d. random variables, exhibits sufficient conditions for the convergence

$$d_{\alpha'}(F_{M_n}, \phi_\alpha) \underset{n}{\rightarrow} 0, \quad d_{\alpha'}(F_{M_n}, \Psi_\alpha) \underset{n}{\rightarrow} 0 \quad \text{and} \quad d_{\alpha'}(F_{M_n}, \Lambda) \underset{n}{\rightarrow} 0.$$

The Corollary 3.3.2 characterizes the max-domain of attraction for Φ_α , Ψ_α and Λ. The Lemma 3.3.4 provides moments convergence and Theorem 3.3.5 summarizes the main results for regenerative processes. And, as byproduct, one obtains characterization for domains of attraction for Markov chains with general state space.

Finally, it is worth pointing out that from the practical point of view it is fairly simple to compute $d_\alpha(F_{M_n}, G)$, being G one of the three extremal distribution. The representation theorem from Dorea and Ferreira (2012) allow us to take $Y^* \stackrel{d}{=} G$, the joint distribution $(M_n, Y^*) \stackrel{d}{=} F_{M_n} \wedge G$ and set

$$d_\alpha^\alpha(F_{M_n}, G) = E\{|M_n - Y^*|^\alpha\}.$$

Chapter 1

Preliminaries

1.1 Introduction

In this chapter we gather the necessary concepts and known results to be used in the subsequent chapters. As basic references we refer the reader to Galambos (1978), Resnick (1987) and Embrechts et al. (1997) for extreme values and regular variation; Breiman (1992), Ibragimov et al. (1971) and Samorodnistky and Taqqu (2000) for stable distributions; Mallows (1972), Bickel and Friedman (1981) and Dorea and Ferreira (2012) for Mallows distance; and Asmussen (1987) and Athreya and Lahiri (2006) for regenerative processes.

Some Notation and Terminology:

i.i.d. : independent and identically distributed

$\xrightarrow{d}$: convergence in distribution

$\stackrel{d}{=}$: equality in distribution

$d_r(F, G)$: Mallows distance of r-th order

$\mathcal{L}_r$: class of distributions with finite r-th moment

$a.s.$: almost surely, with probability 1

Φ_α : Fréchet distribution

Ψ_α : Weibull distribution

Λ : Gumbel distribution

$\mathcal{D}_{\max}(\Phi_\alpha)$: max-domain of attraction of Fréchet

$\mathcal{D}_{\max}(\Psi_\alpha)$: max-domain of attraction of Weibull

$\mathcal{D}_{\max}(\Lambda)$: max-domain of attraction of Gumbel

$S_\alpha(\sigma, \beta, \mu)$: α-stable distribution

G_α : α-stable distribution $S_\alpha(\sigma, \beta, \mu)$

$\mathcal{D}(G_\alpha)$: domain of attraction of G_α

RV_α : regularly varying of index α at ∞

$F \wedge G$: joint distribution $H(x, y) = \min\{F(x), G(y)\}$

$O(1)$: $O_n(1)$ is bounded as $n \to \infty$

$o(1)$: $\lim_{n\to\infty} o_n(1) = 0$

$X_{(n)}$: $\max\{X_1, X_2, \cdots, X_n\}$

M_n : $M_n = \dfrac{X_{(n)} - b_n}{a_n}$

$a_n \sim b_n$: sequences a_n and b_n are such that $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

1.2 Extreme Distribution and Regular Variation

Classical Extreme Value Theory analyses distributional properties of

$$X_{(n)} = \max\left\{X_1, X_2, \cdots, X_n\right\}$$

and

$$X_{(1)} = \min\left\{X_1, X_2, \cdots, X_n\right\}$$

when $X_1, X_2, \cdots$ is a sequence of i.i.d. random variables with a common distribution F. The distribution of $X_{(n)}$ and $X_{(1)}$ are easily computed by

$$P(X_{(n)} \leq x) = P\big(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x\big) = F^n(x)$$

$$P(X_{(1)} \leq x) = 1 - P\big(X_1 > x, X_2 > x, \ldots, X_n > x\big) = 1 - (1 - F(x))^n.$$

If we define right end point as

$$\omega(F) = \sup\{x; F(x) < 1\} \leq \infty$$

and left end point as

$$\nu(F) = \inf\{x; F(x) > 0\} \geq -\infty$$

then

$$X_{(n)} \xrightarrow{a.s} \omega(F) \quad \text{and} \quad X_{(1)} \xrightarrow{a.s} \nu(F).$$

The convergence above shows that a non-degenerate limiting distribution does not exist unless we normalize $X_{(n)}$ or $X(1)$. We will restrict our studies mainly for the maxima since we can always apply the results for minima through the relation

$$\min\big\{X_1, X_2, \cdots, X_n\big\} = -\max\big\{-X_1, -X_2, \cdots, -X_n\big\}.$$

The problem reduces in finding the possible non-degenerate limiting distribution H such that for norming constants $a_n > 0$ and b_n we have

$$P\Big(\frac{X_{(n)} - b_n}{a_n} \leq x\Big) = F^n(a_n x + b_n) \underset{n}{\to} H(x). \qquad (1.2.1)$$

Theorem 1.2.1 *(Extremal Type Theorem) Suppose there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that we have (1.2.1). Then H is one of the following three types:*

$$\textit{Fréchet} \qquad \Phi_\alpha(x) = \begin{cases} 0 & x < 0 \\ \exp\{-x^{-\alpha}\} & x \geq 0 \end{cases}$$

$$\textit{Weibull} \qquad \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$$\textit{Gumbel} \qquad \Lambda(x) = \exp\{-e^{-x}\} \qquad x \in \mathbb{R}$$

where $\alpha > 0$ is a positive parameter.

As for the minima

$$P\big(\frac{X_{(1)} - b_n}{a_n} \leq x\big) = 1 - (1 - F(a_n x + b_n)^n \underset{n}{\rightarrow} H'(x).$$

The possible types for H' are :

$$\textit{Fréchet} \qquad \Phi'_\alpha(x) = \begin{cases} 1 - \exp\{-(-x)^{-\alpha}\} & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$$\textit{Weibull} \qquad \Psi'_\alpha(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\{-x^{\alpha}\} & x \geq 0 \end{cases}$$

$$\textit{Gumbel} \qquad \Lambda'(x) = 1 - \exp\{-e^{x}\} \qquad x \in \mathbb{R}$$

where $\alpha > 0$ is a positive parameter.

We say that two distributions H_1 and H_2 are of the same type if

$$H_2(x) = H_1(ax + b)$$

for some constants $a > 0$ and $b \in \mathbb{R}$.

Definition 1.2.2 *A non-degenerate random variable Y is called max-stable if for every $n \geq 1$ satisfies*

$$\max\left\{Y_1, Y_2, \cdots, Y_n\right\} \stackrel{d}{=} a_n Y + b_n$$

where $Y_1, \ldots, Y_n$ are independent copies of Y and $\{a_n\}$ and $\{b_n\}$ are sequences of constants with $a_n > 0$.

Max-stable distributions are the only limit laws for normalized maxima.

Theorem 1.2.3 *The class of max-stable distributions coincides with the class of all possible (non-degenerate) limit laws for normalized maxima of i.i.d. random variables. Moreover, for $Y \stackrel{d}{=} H$ we have:*

(i) If $H = \Phi_\alpha$ then $a_n = n^{1/\alpha}$ and $b_n = 0$.

(ii) If $H = \Psi_\alpha$ then $a_n = n^{-1/\alpha}$ and $b_n = 0$.

(iii) If $H = \Lambda$ then $a_n = 1$ and $b_n = \ln n$.

Definition 1.2.4 *We say a distribution F is in the max-domain of attraction of H (write $F \in \mathcal{D}_{\max}(H)$) if for all $n \geq 1$ there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that*

$$F^n(a_n x + b_n) \underset{n}{\to} H(x).$$

Finding necessary and sufficient conditions for $F \in \mathcal{D}_{\max}(H)$ is fundamental for (1.2.1). Before characterizing the max-domain of attraction of extreme value distributions, we will need some basic concepts related to regular variation and collect some fundamental properties of regularly varying functions.

Definition 1.2.5 *A measurable function $U : \mathbb{R}_+ \to \mathbb{R}_+$ is regularly varying at ∞ with index ρ (write $U \in RV_\rho$) if for $x > 0$*

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\rho.$$

The case $\rho = 0$ corresponds to the so called slowly varying function.

Note that any function $U \in RV_\rho$ with $\rho \in \mathbb{R}$ can be written as $U(x) = x^\rho l(x)$, where l is a slowly varying function. It's enough to take $l(x) = x^{-\rho} U(x)$.

Some of the properties of the regularly varying functions are related to H^{-1}. Suppose H is a nondecreasing function on $\mathbb{R}$. With the convention that the infimum of an empty set is $+\infty$ we define the (left continuous) generalized inverse of H as

$$H^{-1}(y) = \inf\{s : H(s) \geq y\}$$

Proposition 1.2.6 *Let $U : \mathbb{R}_+ \to \mathbb{R}_+$ be a regularly varying at ∞ with index ρ. Then*

(i) If U is increasing then $\rho \geq 0$.

(ii) If U is decreasing then $\rho \leq 0$.

(iii) If U is not decreasing and $\rho \in [0, \infty)$ then $U^{-1} \in RV_{\frac{1}{\rho}}$.

Theorem 1.2.7 *(Potter's Bounds). Suppose $U \in RV_\rho$ with $\rho \in \mathbb{R}$. Take $\epsilon > 0$. Then there exists t_0 such that for $x \geq 1$ and $t \geq t_0$*

$$(1 - \epsilon)x^{\rho-\epsilon} < \frac{U(tx)}{U(t)} < (1 + \epsilon)x^{\rho+\epsilon}.$$

The following results give necessary and sufficient conditions for $F \in \mathcal{D}_{\max}(H)$ when H is one of the three extreme value distributions and also characterize a_n and b_n.

Theorem 1.2.8 *$F \in \mathcal{D}_{\max}(\Phi_\alpha)$ if and only if $1 - F \in RV_{-\alpha}$. In this case*

$$F^n(a_n x) \underset{n}{\rightarrow} \Phi_\alpha(x)$$

with $a_n = (1/(1-F))^{-1}(n)$.

Notice that this result implies in particular that every $F \in \mathcal{D}_{\max}(\Phi_\alpha)$ has an infinite right endpoint $\omega(F)$. Furthermore the constants a_n form a regularly varying sequence, more precisely $a_n = n^{1/\alpha} l(n)$ for some slowly varying function l.

Since Ψ_α and Φ_α are closely related, indeed

$$\Psi_\alpha(-x^{-1}) = \Phi_\alpha(x), \qquad x > 0.$$

Therefore one should expect closeness between $\mathcal{D}_{\max}(\Phi_\alpha)$ and $\mathcal{D}_{\max}(\Psi_\alpha)$ will be closely related. The following theorem confirms this.

Theorem 1.2.9 *$F \in \mathcal{D}_{\max}(\Psi_\alpha)$ if and only if $\omega(F) < \infty$ and $1 - F\big(\omega(F) - \frac{1}{x}\big) \in RV_{-\alpha}$. In this case*

$$F^n\big(a_n x + b_n\big) \underset{n}{\rightarrow} \Psi_\alpha(x), \qquad x < 0$$

where $a_n = \omega(F) - (1/(1-F))^{-1}(n)$ and $b_n = \omega(F)$.

Theorem 1.2.10 *$F \in \mathcal{D}_{\max}(\Lambda)$ if and only if there exists a Von Mises function F^* such that for $x \in (z_0, \omega(F))$*

$$1 - F(x) = c(x)(1 - F^*(x)) = c(x)\exp\{-\int_{z_0}^{x} \frac{1}{g(y)} dy\} \tag{1.2.2}$$

with

$$\lim_{x\to\omega(F)} c(x) = c > 0.$$

[A distribution F^* with right end point x_0 is a Von Mises function if there exist $z_0 < x_0$ such that for $x \in (z_0, x_0)$ and $c > 0$

$$1 - F^*(x) = c\exp\{-\int_{z_0}^{x} \frac{1}{g(y)} dy\}$$

where g is absolutely continuous on (z_0, x_0) and $g(u) > 0$, $\lim_{u\to x_0} g'(u) = 0$.]

In this case

$$F^n(a_n x + b_n) \underset{n}{\rightarrow} \Lambda(x)$$

where $b_n = (1/(1-F))^{-1}(n)$ and $a_n = g(b_n)$.

It is possible to analyze the moments of $F(x)$ when it belongs to one of the max-domains of attraction of extreme value distributions.

Proposition 1.2.11 *(i) If $F(x) \in \mathcal{D}_{\max}(\Phi_\alpha)$ then*

$$E[(X^+)^r] = \int_0^{+\infty} x^r dF(x) < \infty, \quad \textit{for all} \;\; r \in (0, \alpha).$$

(ii) If $F(x) \in \mathcal{D}_{\max}(\Lambda)$ then

$$E[(X^+)^r] = \int_0^{+\infty} x^r dF(x) < \infty, \quad \textit{for all} \;\; r \in (0, \infty).$$

1.3 Stable Distributions

Stable distributions are fundamental for the study of the asymptotic behavior of partial sums of random variables. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables. Let $S_n = X_1 + \cdots + X_n$ and consider a nondegenerate distribution G such that

$$\lim_{n\to\infty} P\left(\frac{S_n - B_n}{A_n} \leq x\right) = G(x) \tag{1.3.1}$$

where $A_n > 0$ and $B_n \in \mathbb{R}$. Two important questions arise.

First: What is the form of the class of all limit distributions G?

Second: What are the necessary and sufficient conditions on the common distribution function of $X_1, X_2, ...$ for (1.3.1) hold?

These two questions lead to the stable laws and the domains of attraction of the stable laws.

Definition 1.3.1 *A random variable X is said to have a stable law if for every integer $k > 0$, and $X_1, ..., X_k$ independent with the same distribution as X, there are constants $a_k > 0, b_k$ such that*

$$X_1 + X_2 + \cdots + X_k \stackrel{d}{=} a_k X + b_k \ . \tag{1.3.2}$$

X is called strictly stable if (1.3.2) *hold with $b_k = 0$, for every k.*

Proposition 1.3.2 *X is the limit in distribution of normed sums if and only if X has a stable law.*

Definition 1.3.3 *(Equivalent to Definition 1.3.1) A random variable X is said to have a stable law if there are parameters $0 < \alpha \leq 2$, $\sigma > 0$, $-1 \leq \beta \leq 1$ and μ real such that its characteristic function has the following form:*

$$E\{\exp(itX)\} = \begin{cases} \exp\{-\sigma^\alpha |t|^\alpha (1 - i\beta(sign\ t)\tan\frac{\pi\alpha}{2}) + i\mu t\} & if \ \ \alpha \neq 1, \\ \exp\{-\sigma^\alpha |t|^\alpha (1 + i\beta(sign\ t)\ln|t|) + i\mu t\} & if \ \ \alpha = 1. \end{cases}$$

The parameters α, σ, β and μ are called respectively index of stability, scale parameter, skewness parameter and shift parameter.

This characterization motivates to denote stable distribution by

$$S_\alpha(\sigma, \beta, \mu).$$

Just a few of α-stable distribution are known in a close form. We present them in the following example.

Example 1.3.4 *(i) The Gaussian distribution* $S_2(\sigma, 0, \mu) = \mathcal{N}(\mu, 2\sigma^2)$

(ii) The Cauchy distribution $S_1(\sigma, 0, \mu)$

(iii) The Lévy distribution $S_{\frac{1}{2}}(\sigma, 1, \mu)$

Proposition 1.3.5 *Let* $X \overset{d}{=} S_\alpha(\sigma, \beta, \mu)$ *with* $0 < \alpha < 2$. *Then*

(i) X *with* $\alpha \neq 1$ $(\alpha = 1)$ *is strictly stable if and only if* $\mu = 0$ $(\beta = 0)$.

(ii) $X + a \overset{d}{=} S_\alpha(\sigma, \beta, \mu + a)$, $a \in \mathbb{R}$ *constant*

(iii) $X \overset{d}{=} S_\alpha(\sigma, \beta, 0) \Longleftrightarrow -X \overset{d}{=} S_\alpha(\sigma, -\beta, 0)$

(iv) X *is symmetric if and only if* $\beta = \mu = 0$. *It is symmetric about* μ *if and only if* $\beta = 0$

Proposition 1.3.6 *Let* $X \overset{d}{=} S_\alpha(\sigma, \beta, \mu)$ *with* $0 < \alpha < 2$. *Then*

$$\lim_{x \to \infty} x^\alpha P(X > x) = C_\alpha \frac{1 + \beta}{2} \sigma^\alpha$$

and

$$\lim_{x \to \infty} x^\alpha P(X < -x) = C_\alpha \frac{1 - \beta}{2} \sigma^\alpha$$

where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha}\sin x\ dx\right)^{-1} = \begin{cases} \dfrac{1-\alpha}{\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})} & \text{if } \alpha \neq 1, \\ \dfrac{2}{\pi} & \text{if } \ \alpha = 1 \end{cases}$$

Remark 1.3.7 *(a) Note that when α is restricted to the range $(0,1)$ and β is fixed at 1, the α-stable distribution has support (μ,∞).*

(b) If $X \stackrel{d}{=} S_\alpha(\sigma,1,0)$ then by Proposition 1.3.6 we have

$$\lim_{x\to\infty} x^\alpha P(X < -x) = 0.$$

That is $P(X < -x)$ tends to 0 faster than $x^{-\alpha}$.

(c) If $X \stackrel{d}{=} S_\alpha(\sigma,-1,0)$ then by Proposition 1.3.6 we have

$$\lim_{x\to\infty} x^\alpha P(X > x) = 0.$$

That is $P(X > x)$ tends to 0 faster than $x^{-\alpha}$.

Proposition 1.3.8 *Let $X \stackrel{d}{=} S_\alpha(\sigma,\beta,\mu)$ with $0<\alpha<2$. Then*

$$\begin{aligned} E|X|^k &< \infty \qquad \text{for any} \quad 0<k<\alpha\ , \\ E|X|^k &= \infty \qquad \text{for any} \quad k \geq \alpha\ . \end{aligned}$$

Definition 1.3.9 *A distribution F is said to be in the domain of attraction of a stable law G_α with exponent $0<\alpha\leq 2$ if there are sequences of constants $\{A_n\}$ and $\{B_n\}$ with $A_n > 0$ such that*

$$(S_n - B_n)/A_n \stackrel{d}{\to} Z$$

where $Z \stackrel{d}{=} G_\alpha$, $S_n = \sum_{j=1}^n X_j$ and $X_1, , X_2, \ldots$ are i.i.d. random variables with a common distribution F. Denote this by $F \in \mathcal{D}(G_\alpha)$.

The following theorem give necessary and sufficient conditions for $F \in \mathcal{D}(G_\alpha)$.

Theorem 1.3.10 *F is in the domain of attraction of a stable law with exponent $0 < \alpha < 2$ if and only if there are constants $M^+ \geq 0$ and $M^- \geq 0$ with $M^+ + M^- > 0$ and such that*

$$\lim_{y \to \infty} \frac{F(-y)}{1 - F(y)} = \frac{M^-}{M^+}$$

and for every $\xi > 0$

$$M^+ > 0 \Rightarrow \lim_{y \to \infty} \frac{1 - F(\xi y)}{1 - F(y)} = \frac{1}{\xi^\alpha},$$

$$M^- > 0 \Rightarrow \lim_{y \to \infty} \frac{F(-\xi y)}{F(-y)} = \frac{1}{\xi^\alpha}.$$

It's possible to analyze the moments of F when it belongs to domains of attraction of α-stable distribution.

Theorem 1.3.11 *If F belong to the domain of attraction of a stable law, with index α then*

$$\int_{-\infty}^{+\infty} |x|^r dF(x) < \infty, \quad \textit{for any} \quad 0 \leq r < \alpha$$

and

$$\int_{-\infty}^{+\infty} |x|^r dF(x) = \infty, \quad \textit{for any} \quad r > \alpha.$$

1.4 Mallows Distance

The Mallows distance (1972) between two distributions functions F and G generalizes the "Wasserstein distance" appeared for the first time in 1970 (case $r = 1$). Thus, in the literature, the name distance of Wasserstein has also been used instead of Mallows.

Definition 1.4.1 *For $r > 0$, the Mallows r-distance between distributions F and G is given by*

$$d_r(F,G) = \inf_{(X,Y)} \left\{E(|X-Y|^r)\right\}^{1/r}, \quad X \overset{d}{=} F, Y \overset{d}{=} G \tag{1.4.1}$$

where the infimum is taken over all random vectors (X, Y) with marginal distributions F and G, respectively.

For $r \geq 1$ the Mallows distance represents a metric on the space of distribution functions

$$\mathcal{L}_r = \left\{F : \int_{\mathbb{R}} |x|^r dF(x) < +\infty\right\}.$$

There is a close connection between convergence in Mallows distance and the convergence in distribution.

Theorem 1.4.2 *(Bickel and Freedman (1981)). For $r \geq 1$ and for distributions $G \in \mathcal{L}_r$ and $\{F_n\}_{n\geq 1} \subset \mathcal{L}_r$ we have*

$$d_r(F_n, G) \underset{n}{\to} 0 \iff F_n \overset{d}{\to} G \text{ and } \int |x|^r dF_n(x) \underset{n}{\to} \int |x|^r dG(x).$$

Theorem 1.4.3 *(Dorea and Ferreira (2012)). Let $r \geq 1$, $X^* \overset{d}{=} F$, $Y^* \overset{d}{=} G$ and $(X^*, Y^*) \overset{d}{=} H$, where $H(x,y) = F(x) \wedge G(y) = \min\{F(x), G(y)\}$. Then the following representation holds*

$$\begin{aligned} d_r^r(F,G) &= E\{|F^{-1}(U) - G^{-1}(U)|^r\} = \int_0^1 |F^{-1}(u) - G^{-1}(u)|^r du \\ &= E\{|X^* - Y^*|^r\} = \int_{R^2} |x-y|^r dH(x,y) \end{aligned} \tag{1.4.2}$$

where U is uniformly distributed on the interval $(0,1)$ and $0 < u < 1$.

By making use of Mallows distance several Central Limit type results for stable distributions were successfully derived.

Theorem 1.4.4 *(Barbosa and Dorea (2009). Fix $0 < \alpha < 2$. Let $\{X_n\}_{n\geq 1}$ be a sequence of independent random variables, with zero-mean if $\alpha > 1$. Let G_α be a strictly α-stable distribution and assume that there exists a random variable $Y \stackrel{d}{=} G_\alpha$ such that for $Y_1, Y_2, \ldots$ independent copies of Y we have for all $b > 0$*

$$\frac{1}{n}\sum_{i=1}^{n} E\big\{|X_i - Y_i|^\alpha 1_{(|X_i - Y_i| > bn^{\frac{2-\alpha}{2\alpha}})}\big\} \underset{n}{\to} 0 \tag{1.4.3}$$

then

$$d_\alpha(F_{S_n}, G_\alpha) \underset{n}{\to} 0$$

where $F_{S_n} \stackrel{d}{=} \dfrac{X_1 + \cdots + X_n - c_n}{n^{1/\alpha}}$, being $\{c_n\}_{n\geq 1}$ a sequence of constants.

As a corollary of Theorem 1.4.4 for i.i.d. sequence we have:

Corollary 1.4.5 *Under the hypotheses of Theorem 1.4.4 if, in addition, the random variables $X_1, X_2, \ldots$ are i.i.d. and*

$$d_\alpha(F, G_\alpha) < \infty. \tag{1.4.4}$$

then there exists a sequence of constants $\{c_n\}_{n\geq 1}$ such that

$$d_\alpha(F_{S_n}, G_\alpha) \underset{n}{\to} 0 \quad ,$$

where $F_{S_n} \stackrel{d}{=} \dfrac{X_1 + \cdots + X_n - c_n}{n^{1/\alpha}}$.

Also, for $1 \leq \alpha < 2$ and under the same notation as above Corollary there is an equivalence between convergence in Mallows distance and convergence in distribution (cf. Dorea and Ferreira (2012))

Theorem 1.4.6 *Let $1 \leq \alpha < 2$. The condition*

$$d_\alpha(F, G_\alpha) < \infty$$

guarantees the equivalence

$$d_\alpha(F_{S_n}, G_\alpha) \underset{n}{\to} 0 \quad \Longleftrightarrow \quad F_{S_n} \xrightarrow{d} G \ .$$

1.5 Regenerative Process

The classical concept for a stochastic process $\{X_n\}_{n\geq 0}$ to be regenerative means, in intuitive terms, that the process can be splitted into i.i.d. cycles. That is, we assume that a collection of time points exists, so that between any two consecutive time points in this sequence, (i.e. during a cycle), the process $\{X_n\}_{n\geq 0}$ has the same probabilistic behavior. For references on this section see Athreya and Lahiri (2006) or Embrechts et al. (1997).

Definition 1.5.1 *A stochastic process $\{X_n\}_{n\geq 0}$ with values in a measurable space $(E, \mathcal{E})$ is regenerative if there exist integer-valued random variables $0 < T_0 < T_1 \dots$ which split the sequence up into independent "cycles" or "excursions", $C_0, C_1, \dots$. If*

$$C_0 = \{X_n, 0 \leq n < T_0\} \ , \ C_1 = \{X_n, T_0 \leq n < T_1\} \ , \ \dots$$

then $C_1, C_2, \dots$ are i.i.d. random vectors. Clearly $\{T_k\}_{k=0}^{\infty}$ is a renewal process i.e.

$$Y_0 = T_0, Y_1 = T_1 - T_0, Y_2 = T_2 - T_1, \dots$$

are i.i.d. random variables. A regenerative stochastic process $\{X_n\}_{n\geq 0}$ is called zero-delayed when the first cycle, C_0 has the same distribution as all the other cycles.

The notation P_0 for the probability and E_0 for the expectation will be used for the zero-delayed case. Also, if the process has initial distribution λ then we shall write P_λ and E_λ, respectively. Analogously, P_x and E_x will stand for the case λ gives probability 1 to the point $\{x\}$.

Proposition 1.5.2 *Let $\{X_n\}_{n\geq 0}$ be a regenerative process with renewal times $\{T_n\}_{n\geq 0}$ then*

(i) If $\phi : E \to F$ is any measurable mapping, then $\{\phi(X_n)\}_{n\geq 0}$ is regenerative process with the same renewal times.

(ii) Let $v_n = \inf\{k; T_k > n\}$ then the Law of Large Numbers applies $\dfrac{v_n}{n} \underset{n}{\rightrightarrows} \dfrac{1}{\mu_Y}$, where $\mu_Y = E(Y_1) = E(T_1 - T_0)$, expected length of a cycle.

Renewal theory plays a key role in the analyze of the asymptotic structure of many kinds of stochastic processes, and especially in the development asymptotic properties of general Markov chains. The underlying ground consists in the fact that limit theorems proved for sums of independent random vectors may be extended to regenerative processes. Any Markov chain $\{X_n\}_{n\geq 0}$ with a countable state space S that is irreducible and recurrent is regenerative with $\{T_i\}_{i\geq 1}$ being the times of successive returns to a given state $\{x\}$. Harris chains on a general state space that possess an atom, are special cases of regenerative processes. And this illustrates the range of applications of the regenerative methods.

Now, let $\{X_n\}_{n\geq 0}$ be a Markov chain on a measurable space $(E, \mathcal{E})$ with transition probability function $P(.,.)$. That is, for all $A \in \mathcal{E}$, we have

$$P(X_{n+1} \in A|\sigma(X_0, X_1, \cdots, X_n)) = P(X_{n+1} \in A|\sigma(X_n)) = P(X_n, A) \quad \text{a.s.}$$

for any given initial distribution of X_0. We have used the notation $\sigma(X_n)$ for the sub-σ-algebra of $\mathcal{E}$ generated by X_n and $\sigma(X_0, X_1, \cdots, X_n)$ the one generated by $(X_0, X_1, \cdots, X_n)$.

For any $A \in \mathcal{E}$ we define the first entrance time to A as

$$\tau_A = \tau_A^1 = \begin{cases} \inf\{n : n \geq 1, X_n \in A\} \\ \infty \quad \text{if} \quad X_n \notin A \quad \forall\, n \geq 1. \end{cases}$$

Note that τ_A or τ_A^1 is a stopping time with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 1}$ where $\mathcal{F}_n = \sigma(X_0, X_1, \cdots, X_n)$. We can also define the successive return times to A by

$$\tau_A^j = \inf\{n : n \geq \tau_A^{j-1}, X_n \in A\}, \quad j \geq 2.$$

Definition 1.5.3 *Let ψ be a nonzero σ-finite measure on $(E, \mathcal{E})$.*

(i) The Markov chain $\{X_n\}_{n\geq 0}$ (or equivalently, its transition function $P(.,.)$) is said to be ψ-irreducible (or irreducible in the sense of Harris with reference measure ψ) if for any $A \in \mathcal{E}$ and all $x \in E$ we have

$$\psi(A) > 0 \Rightarrow \quad P_x(\tau_A < \infty) > 0.$$

(ii) A Markov chain$\{X_n\}_{n\geq 0}$ that is Harris irreducible with respect to ψ is said to be Harris recurrent if for all $x \in E$ we have

$$A \in \mathcal{E}\ ,\ \psi(A) > 0 \ \Rightarrow\ P_x(\tau_A < \infty) = 1.$$

(iii) The set $A \in \mathcal{E}$ is an atom if there exists a probability measure ν such that $P(x, B) = \nu(B)$, $x \in A$ and $B \in \mathcal{E}$.The set A is an accessible atom for a ψ-irreducible Markov chain if $\psi(A) > 0$ and for all $x \in E$ and $y \in E$ we have $P(x, .) = P(y, .)$.

Remark 1.5.4 *If a chain has an accessible atom then the times at which the chain enters the atom are regeneration times.*

When the chain is Harris recurrent then, for any initial distribution, the probability of returning infinitely often to the atom A is equal to one. By the strong Markov property it follows that, for any initial distribution λ, the sample paths of the chain can be divided into i.i.d. blocks of random length corresponding to consecutive visits to A. The cycles can be defined by

$$C_1 = \{X_{\tau_A^1}, X_{\tau_A^1+1}, \ldots, X_{\tau_A^2}\}, \ldots, C_n = \{X_{\tau_A^n}, X_{\tau_A^n+1}, \ldots, X_{\tau_A^{n+1}}\}, \ldots$$

Characterization of max-domain of attraction for regenerative process will be treated in our Chapter 3.

Chapter 2

Mallows Distance Convergence to Fréchet Distribution

2.1 Introduction

Mallows distance has been successfully used to derive Central Limit Theorem type results for heavy-tailed stable distributions (see, e.g., Johnson and Samworth (2005) or Dorea and Oliveira (2014)). On the other hand, the regularly varying behavior of tail distributions establishes the connection between stable laws and the Fréchet distributions. The connection is treated in section 2.2. This leads us to study the role of Mallows distance in characterizing the domain of attraction of Φ_α and to provide conditions under which

$$d_\alpha(F_{M_n}, \Phi_\alpha) \underset{n}{\to} 0 \quad \Rightarrow \quad F_{M_n} \xrightarrow{d} \Phi_\alpha, \tag{2.1.1}$$

where for given random variables $X_1, X_2, \dots$ we define

$$M_n = \frac{\max\{X_1, \dots, X_n\}}{n^{1/\alpha}} \quad \text{and} \quad M_n \overset{d}{=} F_{M_n}. \tag{2.1.2}$$

In section 2.3 we study the case that $\{X_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables. For $\alpha \geq 1$ our Theorem 2.3.1 provides sufficient conditions for (2.1.1). The case $0 < \alpha < 1$ is treated in Corollary 2.4.5. We will give some examples that clarify the connection between stable laws and Fréchet distribution and the role of Mallows distance. Theorem 2.3.8 provides sufficient conditions for equivalence between Mallows convergence and convergence in distribution.

In Section 2.4 we study the case when i.i.d. hypothesis is dropped. Theorem 2.4.2 proves that Lindeberg's type conditions suffices for (2.1.1). As a side result moment convergences are also derived.

2.2 Partial Sums Versus Fréchet Distribution

With the recognition of the importance of stable laws the interrelation between CLT for stable distributions and asymptotics for EVT emerged. This relation is due to the behavior of distribution tails. In fact, if $S_n = \sum_{j=1}^{n} X_j$ and for some constants $A_n > 0$ and B_n we have the convergence in distribution

$$\frac{S_n - B_n}{A_n} \xrightarrow{d} Y_\alpha \tag{2.2.1}$$

where Y_α has α-stable distribution with $0 < \alpha < 2$ then the common distribution F of the X_n's has regularly varying tails of index $-\alpha$, $RV_{-\alpha}$ (either left-tail or right-tail or both tails). As for the Fréchet distribution, if the stabilized maximum converges

$$\frac{X_{(n)} - b_n}{a_n} \xrightarrow{d} \Phi_\alpha \tag{2.2.2}$$

then for the right-tail we have $1 - F \in RV_{-\alpha}$. In fact, we have the following proposition that, in a sense, extends Theorem 4.5.1 from Galambos (1978).

Proposition 2.2.1 *Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with common distribution function F. Assume F belongs to the domain of attraction of a stable law with exponent $\alpha < 2$ and skewness parameter $\beta \neq -1$. Then $F \in \mathcal{D}_{\max}(\Phi_\alpha)$ and as for the constants $A_n > 0$, $a_n > 0$ in (2.2.1) and (2.2.2) , respectively, the following relationship holds*

$$a_n/A_n \sim K, \quad 0 < K < \infty.$$

Proof. Since F is in domain of attraction of a stable law with exponent $\alpha < 2$ and the skewness parameter $\beta \neq -1$ we have, by Proposition 1.3.6 and Theorem 1.3.10, that $1 - F \in RV_{-\alpha}$. Theorem 1.2.8 concludes the first part of the proof.

Now, we rewrite the $1 - F \in RV_{-\alpha}$ as

$$1 - F(x) = \frac{L(x)}{x^\alpha} \tag{2.2.3}$$

where L is a slowly varying function. Since $n[1 - F(a_n)] \underset{n}{\to} 1$ (cf Galambos (1978)page 273) we have by (2.2.3)

$$\frac{nL(a_n)}{a_n^\alpha} \underset{n}{\to} 1. \tag{2.2.4}$$

On the other hand, when F is in the domain of attraction of a stable law with exponent $\alpha < 2$ then A_n satisfies

$$n(1 - F(A_n x)) \underset{n}{\to} sx^{-\alpha} \quad , \; x > 0 \; , \; 0 < s < +\infty$$

(cf Ibragimov and Linnik (1971)). With $x = 1$ and (2.2.3), we can write

$$\frac{nL(A_n)}{A_n^\alpha} \underset{n}{\to} s \qquad 0 < s < +\infty.$$

This along with (2.2.4) yields

$$\begin{aligned}(a_n/A_n)^\alpha \sim \frac{nL(a_n)}{\frac{1}{s}nL(A_n)} = s\frac{L(a_n)}{L(A_n)} &= s\frac{L\big(A_n(a_n/A_n)\big)}{L(A_n)} \\ &= s\frac{L(a_n)}{L\big(a_n(A_n/a_n)\big)}.\end{aligned}$$

The right hand side always tends to s because either (a_n/A_n) or (A_n/a_n) is bounded. $\square$

Remark 2.2.2 *(i) If $F(0) = 0$ and $F \in D_{max}(\Phi_\alpha)$ then there exists a α-stable law G_α with skewness parameter $\beta = 1$ such that $F \in D(G_\alpha)$.*

(ii) If $F \in D_{max}(\Phi_\alpha) \cap D_{min}(\Phi'_{\alpha'})$ with $0 < \alpha \leq \alpha' < 2$, then by Theorems 1.2.8 and 1.3.10 there exists a α-stable law G_α such that $F \in D(G_\alpha)$. Note that in this case we have

$$\beta \neq 1, -1 \quad \text{if} \;\; \alpha = \alpha'$$

$$\text{and} \;\; \beta = 1 \quad \text{if} \;\; \alpha < \alpha'$$

Next for the i.i.d. case, we explore the connection between maxima and the partial sums.

2.3 The I.I.D. Case

Throughout this section we will assume that $\{X_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables with a common distribution F. The following theorem provide sufficient conditions for convergence to Φ_α in Mallows distance and in distribution.

Theorem 2.3.1 *Let $\alpha \geq 1$. Assume that $d_\alpha(F, \Phi_\alpha) < \infty$ then for M_n defined by* (2.1.2) *we have*

$$\lim_{n\to\infty} d_\alpha(F_{M_n}, \Phi_\alpha) = 0 \quad \textit{and} \quad F_{M_n} \xrightarrow{d} \Phi_\alpha. \tag{2.3.1}$$

Before proving the theorem the following preliminary results will be needed.

Lemma 2.3.2 *For sequences of real numbers $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ we have*

$$\begin{aligned} &\big|\max\{x_1, \ldots, x_n\} - \max\{y_1, \ldots, y_n\}\big| \\ &\quad \leq \max\big\{|x_1 - y_1|, \ldots, |x_n - y_n|\big\} \end{aligned} \tag{2.3.2}$$

Proof. Suppose that $\max\{x_1, \ldots, x_n\} = x_i$ and $\max\{y_1, \ldots, y_n\} = y_j$. Then we have

$$\begin{aligned} x_i - y_j \leq x_i - y_i &\leq |x_i - y_i| \\ &\leq \max\big\{|x_1 - y_1|, \ldots, |x_n - y_n|\big\} \end{aligned}$$

and

$$\begin{aligned} y_j - x_i \leq y_j - x_j &\leq |x_j - y_j| \\ &\leq \max\big\{|x_1 - y_1|, \ldots, |x_n - y_n|\big\}. \end{aligned}$$

It follows

$$\begin{aligned} \big|\max\{x_1, \ldots, x_n\} - \max\{y_1, \ldots, y_n\}\big| &= |x_i - y_j| \\ &\leq \max\big\{|x_1 - y_1|, \ldots, |x_n - y_n|\big\} \end{aligned}$$

This completes the proof. $\square$

Lemma 2.3.3 *Let $\xi_1, \xi_2, \dots$ be i.i.d. random variables. Assume that for some $\gamma > 0$ we have $E\{|\xi_n|^\gamma\} < \infty$. Then*

$$\frac{1}{n} E\{|\xi_{(n)}|^\gamma\} \underset{n}{\to} 0 \quad , \quad \xi_{(n)} = \max\{\xi_1, \dots, \xi_n\}. \tag{2.3.3}$$

Proof. Let $G \overset{d}{=} \xi_n$ and $\omega(G) = \sup\{x : G(x) < 1\}$.

(i) If $\omega(G) < \infty$ then we can choose $\epsilon > 0$ small enough such that $\int_{\omega(G)-\epsilon}^{\omega(G)} |x|^\gamma dG(x)$ is small. Now,

$$\begin{aligned}
\frac{1}{n} E\{|\xi_{(n)}|^\gamma\} &= \frac{1}{n} \int_{-\infty}^{\omega(G)} |x|^\gamma dG^n(x) \\
&= \int_{-\infty}^{\omega(G)} |x|^\gamma G^{n-1}(x) dG(x) \\
&= \int_{-\infty}^{\omega(G)-\epsilon} |x|^\gamma G^{n-1}(x) dG(x) + \\
&\quad \int_{\omega(G)-\epsilon}^{\omega(G)} |x|^\gamma G^{n-1}(x) dG(x) \\
&\leq G^{n-1}(\omega(G)-\epsilon) \int_{-\infty}^{\omega(G)-\epsilon} |x|^\gamma dG(x) + \\
&\quad \int_{\omega(G)-\epsilon}^{\omega(G)} |x|^\gamma dG(x).
\end{aligned}$$

Since $E\{|\xi_n|^\gamma\} < \infty$ and $G^{n-1}(\omega(G)-\epsilon) \underset{n}{\to} 0$, (2.3.3) follows.

(ii) If $\omega(G) = \infty$ then we can choose d large enough so that $\int_d^\infty |x|^\gamma dG(x)$ is small and $0 < G(d) < 1$. Now,

$$\begin{aligned}
\frac{1}{n} E\{|\xi_{(n)}|^\gamma\} &= \frac{1}{n} \int_{-\infty}^{+\infty} |x|^\gamma dG^n(x) = \int_{-\infty}^{+\infty} |x|^\gamma G^{n-1}(x) dG(x) \\
&= \int_{-\infty}^{d} |x|^\gamma G^{n-1}(x) dG(x) + \int_{d}^{+\infty} |x|^\gamma G^{n-1}(x) dG(x) \\
&\leq G^{n-1}(d) \int_{-\infty}^{d} |x|^\gamma dG(x) + \int_{d}^{+\infty} |x|^\gamma dG(x).
\end{aligned}$$

Since $E\{|\xi_n|^\gamma\} < \infty$ and $G^{n-1}(d) \underset{n}{\rightarrow} 0$, result follows. $\square$

Proof of Theorem 2.3.1. Let $Y_1^*, Y_2^*, \dots$ be i.i.d. random variables with common distribution Φ_α. Since Φ_α is max-stable we have

$$\frac{\max\{Y_1^*, Y_2^*, \dots, Y_n^*\}}{n^{1/\alpha}} \stackrel{d}{=} \Phi_\alpha. \tag{2.3.4}$$

By (1.4.1) and (2.1.2) we have

$$\begin{aligned} d_\alpha^\alpha(F_{M_n}, \Phi_\alpha) &\leq E\big\{\big|M_n - \frac{\max\{Y_1^*, \dots, Y_n^*\}}{n^{1/\alpha}}\big|^\alpha\big\} \\ &= \frac{1}{n}E\big\{\big|\max\{X_1, \dots, X_n\} - \max\{Y_1^*, \dots, Y_n^*\}\big|^\alpha\big\} \\ &\leq \frac{1}{n}E\big\{\big|\max\{|X_1 - Y_1^*|, \dots, |X_n - Y_n^*|\}\big|^\alpha\big\}. \end{aligned}$$

In the last inequality we have used (2.3.2).

We may take Y_n^* satisfying $(X_n, Y_n^*) \stackrel{d}{=} F \wedge \Phi_\alpha$, $n = 1, 2, \dots$. Since $d_\alpha(F, \Phi_\alpha) < \infty$, we have by representation Theorem 1.4.2

$$d_\alpha^\alpha(F, \Phi_\alpha) = E\{|X_n - Y_n^*|^\alpha\} < \infty , \quad n = 1, 2, \dots .$$

Let $\xi_n = X_n - Y_n^*, n = 1, 2, \dots$, so we have $E\{|\xi_n|^\alpha\} < \infty$. Using Lemma 2.3.3 we have

$$d_\alpha^\alpha(F_{M_n}, \Phi_\alpha) = \frac{1}{n}E\{|\xi_{(n)}|^\alpha\} \underset{n}{\rightarrow} 0.$$

Now, take $Y^* \stackrel{d}{=} \Phi_\alpha$ and $(M_n, Y^*) \stackrel{d}{=} F_{M_n} \wedge \Phi_\alpha$ then by representation Theorem 1.4.2

$$d_\alpha^\alpha(F_{M_n}, \Phi_\alpha) = E\{|M_n - Y^*|^\alpha\} \underset{n}{\rightarrow} 0.$$

Now from the α mean convergence follows that $M_n \xrightarrow{d} Y^*$ or, equivalently, $F_{M_n} \xrightarrow{d} \Phi_\alpha$. $\square$

Remark 2.3.4 *(a) The proof of Theorem 2.3.1 shows that under the assumption $d_\alpha(F, \Phi_\alpha) < \infty$ we have*

$$d_\alpha(F_{M_n}, \Phi_\alpha) \not\to_n 0 \quad \Rightarrow \quad F_{M_n} \xrightarrow{d} \Phi_\alpha. \tag{2.3.5}$$

(b) Let $G_\alpha = S_\alpha(\sigma, \beta, \mu)$ with $\beta \neq -1$. By Proposition 2.2.1 if $F \in \mathcal{D}(G_\alpha)$ then $F \in \mathcal{D}_{max}(\Phi_\alpha)$ and $F_{M_n} \xrightarrow{d} \Phi_\alpha$. From Johnson and Samworth (2005) we have:

$$\text{if} \;\; 1 \leq \alpha < 2 \;\; \text{and} \;\; d_\alpha(F, G_\alpha) < \infty \;\; \text{then} \;\; F \in \mathcal{D}(G_\alpha).$$

It does not follows that $d_\alpha(F, \Phi_\alpha) < \infty$ as the left tail $\int_{-\infty}^0 |x|^\alpha 1_{(x\leq 0)} dF(x)$ needs to be finite. By Proposition 1.3.6 the finiteness can be assumed if $\beta = 1$. Thus if $d_\alpha(F, G_\alpha) < \infty$ with $G_\alpha = S_\alpha(\sigma, 1, \mu)$ then $d_\alpha(F, \Phi_\alpha) < \infty$ and (2.3.1) hold.

The following examples illustrates the above remarks.

Example 2.3.5 *Let $F = G_1 = S_1(1, 0, 0)$, the standard Cauchy distribution. Then $F \in \mathcal{D}(G_1)$ and $d_1(F, G_1) = 0$. By Theorem 1.3.10 we have $1 - F \in RV_{-1}$ and by Theorem 1.2.8 we have $F \in \mathcal{D}_{max}(\Phi_1)$. Since*

$$\int_{-\infty}^0 |x| d\Phi_1(x) = 0 \;\; \text{and} \;\; \int_{-\infty}^0 |x| dF(x) = \infty$$

we can not have $d_1(F, \Phi_1) < \infty$.

Example 2.3.6 *Assume $F = G_{\frac{1}{2}} = S_{\frac{1}{2}}(1, 1, 0)$, the Lévy distribution function*

$$F(x) = 2\left(1 - H(\sqrt{1/x})\right), \quad x > 0$$

where H is the distribution function of $\mathcal{N}(0,1)$ (cf Samorodnitsky and Taqqu (2000)). Clearly $F \in \mathcal{D}(G_{\frac{1}{2}})$ and $d_{\frac{1}{2}}(F, G_{\frac{1}{2}}) = 0$. Since $1 - F \in RV_{-\frac{1}{2}}$ we also have $F \in \mathcal{D}_{max}(\Phi_{\frac{1}{2}})$. We can write

$$H(\sqrt{1/x}) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}}\big(1 + x^{-1}O(1)\big)$$

where $\lim_{x\to\infty} |O(1)| = k < \infty$. It follows that for some constant c we have

$$1 - F(x) = cx^{-\frac{1}{2}}\big(1 + x^{-1}O(1)\big). \tag{2.3.6}$$

Our Theorem 2.3.8 will show that in this case we will have

$$d_{\frac{1}{2}}(F, \Phi_{\frac{1}{2}}) < \infty.$$

Example 2.3.7 *We borrow from Dorea and Ferreira (2012) the following example that shows*

$$F \in \mathcal{D}_{max}(\Phi_\alpha) \quad \not\Rightarrow \quad d_\alpha(F, \Phi_\alpha) < \infty.$$

Define

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{1}{2} & 0 \le x < 1 \\ 1 - \dfrac{1}{2}x^{-1}\dfrac{1}{1+\log x} & x \ge 1. \end{cases}$$

Then $1 - F \in RV_{-1}$ and $F \in D_{max}(\Phi_1)$. Write for $x > 1$

$$\begin{aligned} 1 - \Phi_1(x) &= 1 - e^{-x^{-1}} = x^{-1}[1 + b_{\Phi_1}(x)] \\ 1 - F(x) &= x^{-1}[1 + b_F(x)] \end{aligned}$$

where

$$\begin{aligned} b_{\Phi_1}(x) &= x - 1 - xe^{-x^{-1}} = x^{-1}O(1) \\ b_F(x) &= \frac{1}{2}\frac{1}{1+\log x} - 1. \end{aligned}$$

Let $u^+ > 1/2$ *such that* $\Phi_1^{-1}(u^+) \wedge F^{-1}(u^+) > 1$. *Then for* $u \geq u^+$ *we have*

$$\begin{aligned} F^{-1}(u) &= \inf\{x : 1 - F(x) < 1 - u\} \\ &= \frac{1}{1-u}\big(1 + b_F(F^{-1}(u))\big) \end{aligned}$$

and

$$\begin{aligned} \Phi_1^{-1}(u) &= \frac{1}{1-u}\big(1 + b_{\Phi_1}({\Phi_1}^{-1}(u))\big) \\ &= \frac{1}{1-u}(1 + \frac{O(1)}{{\Phi_1}^{-1}(u)}). \end{aligned}$$

By (1.4.2) with $U \stackrel{d}{=} U[0,1]$,

$$\begin{aligned} d_1(F,\Phi_1) &= E\big\{\big|F^{-1}(U) - \Phi_1^{-1}(U)\big|\big\} \\ &= E\big\{\big|\frac{1}{1-U} - F^{-1}(U) - \frac{1}{1-U} + \Phi_1^{-1}(U)\big|\big\} \\ &\geq E\big\{\big|\big|\frac{1}{1-U} - F^{-1}(U)\big| - \big|\frac{1}{1-U} - \Phi_1^{-1}(U)\big|\big|\big\}. \\ &\geq \Big|E\big\{\big|\frac{1}{1-U} - F^{-1}(U)\big|\big\} - E\big\{\big|\frac{1}{1-U} - \Phi_1^{-1}(U)\big|\big\}\Big|. \end{aligned}$$

We will show that the first expectation is finite but not the second. Thus we can not have $d_1(F,\Phi_1) < \infty$. *Let* $Y \stackrel{d}{=} \Phi_1$ *and* $X \stackrel{d}{=} F$. *Then* $\Phi_1^{-1}(U) \stackrel{d}{=} Y$

and $F^{-1}(U) \stackrel{d}{=} X$.

$$
\begin{aligned}
E\big\{\big|\frac{1}{1-U} - \Phi_1^{-1}(U)\big|1_{(U\ge u)}\big\} &= E\big\{\big|\frac{1}{1-U} - \frac{1}{1-U}(1+\frac{O(1)}{{\Phi_1}^{-1}(U)})\big|1_{(U\ge u)}\big\} \\
&= E\big\{\big|\frac{1}{1-U}\frac{O(1)}{{\Phi_1}^{-1}(U)}\big|1_{(U\ge u)}\big\} \\
&= E\big\{\big|\frac{1}{1-\Phi_1(Y)}\frac{O(1)}{Y}\big|1_{(Y\ge\Phi_1^{-1}(u))}\big\} \\
&= E\big\{\frac{|O(1)|}{1+b_{\Phi_1}(Y)}1_{(Y\ge\Phi_1^{-1}(u))}\big\} \\
&< \infty\ .
\end{aligned}
$$

But

$$
\begin{aligned}
E\big\{\big|\frac{1}{1-U} - F^{-1}(U)\big|1_{(U\ge u)}\big\} &= E\big\{\big|\frac{1}{1-U}b_F(F^{-1}(U))\big|1_{(U\ge u)}\big\} \\
&= E\big\{\big|\frac{1}{1-F(X)}b_F(X)\big|1_{(X\ge F^{-1}(u))}\big\} \\
&= E\big\{\big|X(1+2\log X)\big|1_{(X\ge F^{-1}(u))}\big\} \\
&= \infty.
\end{aligned}
$$

The above example shows that we can not expect the convergence of (2.3.5) to hold for all $F \in \mathcal{D}_{max}(\Phi_\alpha)$ and (2.3.6) suggests the subdomain to be considered. For constants $c > 0$ and $\beta > 0$

$$\mathcal{C}(\Phi_\alpha) = \{F : 1 - F(x) = cx^{-\alpha}(1 + x^{-\beta}O(1)),\ x > x_0 > 0\}.$$

By analogy to stable laws we may call $\mathcal{C}(\Phi_\alpha)$ max domain of strong normal attraction of Φ_α.

Theorem 2.3.8 *Let* $\alpha \geq 1$ *and* $X \stackrel{d}{=} F \in \mathcal{C}(\Phi_\alpha)$. *Assume that*

$$E\{|X|^\alpha 1_{(X\le 0)}\} < \infty.$$

Then

$$d_\alpha(F_{M_n}, \Phi_\alpha) \underset{n}{\to} 0 \quad \Longleftrightarrow \quad F_{M_n} \xrightarrow{d} \Phi_\alpha.$$

Proof. By Remark 2.3.4 (a) and Theorem 2.3.1 it is enough to prove $d_\alpha(F, \Phi_\alpha) < \infty$. Let

$$F(x) = 1 - cx^{-\alpha}(1 + x^{-\beta}O(1)) \ , \ x > x_0 > 0.$$

Define the following auxiliary distribution

$$H(x) = \begin{cases} 0 \ , & x < (2c)^{1/\alpha} \\ 1 - cx^{-\alpha} \ , & x > (2c)^{1/\alpha}. \end{cases}$$

First, we show for this auxiliary distribution H that

$$d_\alpha^\alpha(F, H) = E\{|F^{-1}(U) - H^{-1}(U)|^\alpha\} < \infty \ , \ U \overset{d}{=} U[0, 1]. \qquad (2.3.7)$$

Let $x_* > x_0$ such that $F(x_*) \geq \frac{1}{2}$ and $|x_*^{-\beta}O(1)| < \frac{1}{3}$. Then for $x > x_*$ we have $F(x) \geq \frac{1}{2}$ and for $u \geq \frac{1}{2}$,

$$H^{-1}(u) = \big(\frac{c}{1-u}\big)^{1/\alpha}.$$

Now,

$$\begin{aligned} x - H^{-1}(F(x)) &= \big[x - (\frac{x^\alpha}{1 + x^{-\beta}O(1)})^{1/\alpha}\big] \\ &= x\big[1 - (\frac{1}{1 + x^{-\beta}O(1)})^{1/\alpha}\big]. \end{aligned}$$

Since

$$|\frac{x^{-\beta}O(1)}{1 + x^{-\beta}O(1)}| \leq \frac{1}{2}$$

we can use the inequality

$$|1 - z^\gamma| \leq |1 - z|^\gamma \quad \text{for} \quad |1 - z| \leq \frac{1}{2} \quad \text{and} \quad 0 < \gamma \leq 1$$

and

$$\begin{aligned}
\left|1 - \left(\frac{1}{1 + x^{-\beta}O(1)}\right)^{1/\alpha}\right| &\leq \left|\frac{x^{-\beta}O(1)}{1 + x^{-\beta}O(1)}\right|^{1/\alpha} \\
&\leq \left(\frac{1}{2}\right)^{1/\alpha} |x^{-\beta}O(1)|^{1/\alpha}.
\end{aligned}$$

Now,

$$\begin{aligned}
& E\left\{\left|F^{-1}(U) - H^{-1}(U)\right|^{\alpha} 1_{(U \geq F(x))}\right\} \\
= \; & E\left\{\left|X - H^{-1}(F(X))\right|^{\alpha} 1_{(X \geq x)}\right\} \\
= \; & E\left\{|X|^{\alpha}\left|1 - \left(\frac{1}{1 + x^{-\beta}O(1)}\right)^{1/\alpha}\right|^{\alpha} 1_{(X \geq x)}\right\} \\
\leq \; & E\left\{|X|^{\alpha}\left(\frac{1}{2}\right)^{1/\alpha}|X|^{-\beta/\alpha}O(1)\right\} \\
< \; & \infty.
\end{aligned}$$

Clearly if $u < \frac{1}{2}$ then $H^{-1}(u) = 0$. We may assume $F(x) < \frac{1}{2}$ if $x < 0$,

$$E\left\{\left|F^{-1}(U) - H^{-1}(U)\right|^{\alpha} 1_{(U < \frac{1}{2})}\right\} \leq E\left\{|X|^{\alpha} 1_{(X < 0)}\right\} < \infty.$$

By Theorem 1.4.2 we have $d_\alpha(F, H) < \infty$. Similarly write

$$1 - \Phi_\alpha(x) = x^{-\alpha} + x^{-2\alpha}O(1)$$

and proceed analogously to show that $d_\alpha(H, \Phi_\alpha) < \infty$. Since d_α, for $\alpha \geq 1$, is a metric we have

$$d_\alpha(F, \Phi_\alpha) \leq d_\alpha(F, H) + d_\alpha(H, \Phi_\alpha) < \infty$$

This concludes the proof. $\square$

2.4 The General Case

In the i.i.d. case we explored the connection between maxima and the partial sums, for latter results are known when $X_1, X_2, \ldots$ are not identically distributed or when a dependency structure is assumed, see for example , Johnson and Samworth (2005) or Barbosa and Dorea (2010) . One should expect to inherit some of these results for the maxima. On the other hand, it is intuitive that for the extremes the dependency structure would not play a central role as in the case of partial sums. Indeed, our Theorem 2.4.2 shows that under Linderberg's type conditions we have the desired Mallows convergence.

Let $\{X_n\}_{n\geq 1}$ be a general sequence of random variables. We will be using the following well-known inequalities,

$$E\{\left|X_1 + \cdots + X_n\right|^r\} \leq \sum_{j=1}^{n} E\{|X_j|^r\} , \qquad 0 < r \leq 1 \tag{2.4.1}$$

and

$$E\{\left|X_1 + \cdots + X_n\right|^r\} \leq n^{r-1} \sum_{j=1}^{n} E\{|X_j|^r\} , \qquad r > 1. \tag{2.4.2}$$

Lemma 2.4.1 *Let $\alpha > 0$ and let $\{A_n\}_{n\geq 1}$ be any sequence of events. Then we have the following inequalities*

$$\big(\max\{|X_1|, \ldots, |X_n|\}\big)^{\alpha} \leq \max\left\{|X_1|^{\alpha}, \ldots, |X_n|^{\alpha}\right\} , \tag{2.4.3}$$

$$\begin{aligned}\max\{X_1, \ldots, X_n\} \;\; \leq \;\; & \max\left\{|X_1| 1_{A_1}, \ldots, |X_n| 1_{A_n}\right\} + \\ & \max\left\{|X_1| 1_{A_1^c}, \ldots, |X_n| 1_{A_n^c}\right\}\end{aligned} \tag{2.4.4}$$

and

$$\max\{|X_1|, \ldots, |X_n|\} \leq \sum_{j=1}^{n} |X_j|. \tag{2.4.5}$$

Proof. For any sequence of real number (2.4.3) and (2.4.5) hold trivially. Let $A_n \subset \Omega$ and assume that for some $\omega \in \Omega$ we have

$$\max\{X_1(\omega), \ldots, X_n(\omega)\} = X_i(\omega).$$

If $\omega \in A_i$ then

$$\begin{aligned} \max\{X_1, \ldots, X_n\}(\omega) = X_i(\omega) &\leq |X_i(\omega)| 1_{A_i}(\omega) \\ &\leq \max\big\{|X_1| 1_{A_1}, \ldots, |X_n| 1_{A_n}\big\}(\omega) \end{aligned}$$

and (2.4.4) holds.
If $\omega \in A_i^c$ then

$$\begin{aligned} \max\{X_1, \ldots, X_n\}(\omega) = X_i(\omega) &\leq |X_i(\omega)| 1_{A_i^c}(\omega) \\ &\leq \max\big\{|X_1| 1_{A_1^c}, \ldots, |X_n| 1_{A_n^c}\big\}(\omega) \end{aligned}$$

and (2.4.4) follows. □

Now we can present the following theorem in general case.

Theorem 2.4.2 *Let M_n be defined by (2.1.2) . Let $\{Y_n\}_{n \geq 1}$ be a sequence of i.i.d. random variables with common distribution Φ_α. Assume that for all $b > 0$*

$$\frac{1}{n} \sum_{j=1}^{n} E\big\{|X_j - Y_j|^\alpha 1_{(|X_j - Y_j| > bn^{1/\alpha})}\big\} \underset{n}{\to} 0. \tag{2.4.6}$$

Then

$$\lim_{n\to\infty} d_\alpha(F_{M_n}, \Phi_\alpha) = 0$$

.

Proof. Since Φ_α is a max-stable distribution we have

$$\frac{\max\{Y_1, \ldots, Y_n\}}{n^{1/\alpha}} \stackrel{d}{=} \Phi_\alpha.$$

By inequalities (2.3.2), (2.4.1)–(2.4.5) we can write,

$$\begin{aligned}
d_\alpha^\alpha(F_{M_n}, \Phi_\alpha) &\leq E\big\{\big|M_n - \frac{\max\{Y_1,\ldots,Y_n\}}{n^{1/\alpha}}\big|^\alpha\big\} \\
&\leq \frac{1}{n}E\big\{\big|\max\{X_1,\ldots,X_n\} - \max\{Y_1,\ldots,Y_n\}\big|^\alpha\big\} \\
&\leq \frac{1}{n}E\big\{\big[\max\{|X_1-Y_1|,\ldots,|X_n-Y_n|\}\big]^\alpha\big\} \\
&\leq \frac{1}{n}E\big\{\big[\max_{1\leq j\leq n}|X_j-Y_j|1_{(|X_j-Y_j|\leq bn^{1/\alpha})} + \qquad (2.4.7) \\
&\qquad \max_{1\leq j\leq n}|X_j-Y_j|1_{(|X_j-Y_j|>bn^{1/\alpha})}\big]^\alpha\big\} \\
&\leq \frac{c(\alpha)}{n}\bigg(\big[E\big\{\max_{1\leq j\leq n}|X_j-Y_j|1_{|X_j-Y_j|\leq bn^{1/\alpha})}\big\}\big]^\alpha + \qquad (2.4.8) \\
&\qquad \big[E\big\{\max_{1\leq j\leq n}|X_j-Y_j|1_{(|X_j-Y_j|>bn^{1/\alpha})}\big\}\big]^\alpha\bigg) \\
&\leq \frac{c(\alpha)}{n}\bigg(E\big\{\max_{1\leq j\leq n}|X_j-Y_j|^\alpha 1_{|X_j-Y_j|\leq bn^{1/\alpha})}\big\} + \qquad (2.4.9) \\
&\qquad E\big\{\max_{1\leq j\leq n}|X_j-Y_j|^\alpha 1_{(|X_j-Y_j|>bn^{1/\alpha})}\big\}\bigg) \\
&\leq c(\alpha)b^\alpha + \frac{c(\alpha)}{n}\sum_{j=1}^n E\big\{|X_j-Y_j|^\alpha 1_{(|X_j-Y_j|>bn^{1/\alpha})}\big\}
\end{aligned}$$

where $c(\alpha) = 1$ if $0 < \alpha < 1$ and $c(\alpha) = 2^{\alpha-1}$ if $\alpha \geq 1$. In (2.4.7), (2.4.8) and

(2.4.9) we have used (2.4.4), (2.4.1) or (2.4.2) and (2.4.3) , respectively. Since b is arbitrary, it can be chosen sufficiently small. Using (2.4.6) conclusion follows. □

Remark 2.4.3 *(a) By reviewing the proof of Theorem 2.4.2 we can see that if we replace the condition (2.4.6) by a weaker condition*

$$E\{\max_{1\leq j\leq n}|X_j - Y_j|^\alpha 1_{(|X_j - Y_j| > bn^{1/\alpha})}\} \underset{n}{\to} 0.$$

the result still holds.

(b) Theorem 2.4.2 dispenses the condition of an i.i.d. sequence $\{X_n\}_{n\geq 1}$, while provides a mode of convergence stronger than convergence in distribution.

(c) Though our Theorem 2.4.2 does not requires independency or same distribution for the X_n's, the Lindeberg condition imposes a closeness with respect to a sequence of i.i.d. random variables. This suggests that processes $\{X_n\}_{n\geq 0}$ that admit a decomposition into independent blocks could well be studied via Mallows distance. That is the case of Markov chains and more generally the regenerative processes that we treat it in the following chapter.

The following proposition shows that when $\{X_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables the Linderberg condition (2.4.6) reduces to the requirement that $d_\alpha(F, \Phi_\alpha) < \infty$.

Proposition 2.4.4 *Under hypothesis of Theorem 2.4.2 if $\{X_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables with common distribution F then condition (2.4.6) is equivalent to $d_\alpha(F, \Phi_\alpha) < \infty$.*

Proof. (i)($\Rightarrow$) For $j = 1, 2, \ldots$ we have $d_\alpha^\alpha(F, \Phi_\alpha) \leq E\{|X_j - Y_j|^\alpha\}$. So we can write

$$\begin{aligned}
d_\alpha^\alpha(F, \Phi_\alpha) &\leq \frac{1}{n}\sum_{j=1}^{n} E\big\{|X_j - Y_j|^\alpha\big\} \\
&\leq \frac{1}{n}\bigg(E\big\{|X_j - Y_j|^\alpha 1_{(|X_j - Y_j| \leq bn^{1/\alpha})}\big\} + \\
&\qquad E\big\{|X_j - Y_j|^\alpha 1_{(|X_j - Y_j| > bn^{1/\alpha})}\big\}\bigg) \\
&\leq b^\alpha n + \frac{1}{n}\sum_{j=1}^{n} E\big\{|X_j - Y_j| 1_{(|X_j - Y_j| > bn^{1/\alpha})}\big\}.
\end{aligned}$$

Condition (2.4.6) follows that for each $\epsilon > 0$ exists $n_0 = n_0(\epsilon)$ such that, $\forall n \geq n_0$

$$\frac{1}{n}\sum_{j=1}^{n} E\big\{|X_j - Y_j|^\alpha 1_{(|X_j - Y_j| > bn^{1/\alpha})}\big\} < \epsilon$$

that follows $\forall\ n \geq n_0$

$$d_\alpha^\alpha(F, \Phi_\alpha) \leq b^\alpha n_0 + \epsilon < \infty\ .$$

(ii) ($\Leftarrow$)Assume $d_\alpha(F, \Phi_\alpha) < \infty$. For some $X \stackrel{d}{=} F$ and $Y \stackrel{d}{=} \Phi_\alpha$ we have $d_\alpha^\alpha(F, \Phi_\alpha) = E\big\{|X - Y|^\alpha\big\}$. We may take $(X_j, Y_j) \stackrel{d}{=} (X, Y)\ ,\ \ j = 1, 2, \ldots$. Thus

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{n} & \quad E \quad \big\{|X_j - Y_j|^\alpha 1_{(|X_j - Y_j| > bn^{1/\alpha})}\big\} \\
= & \quad E \quad \big\{|X_j - Y_j|^\alpha 1_{(|X_j - Y_j| > bn^{1/\alpha})}\big\} \underset{n}{\rightarrow} 0
\end{aligned}$$

and (2.4.6) follows. $\square$

This will allow us to extend Theorem 2.3.1 for $\alpha > 0$.

Corollary 2.4.5 *Let $\alpha > 0$. Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with common distribution F. If $d_\alpha(F, \Phi_\alpha) < \infty$ then*

$$d_\alpha(F_{M_n}, \Phi_\alpha) \underset{n}{\to} 0.$$

As a corollary of Theorem 2.4.2 we also have the moment convergence for M_n.

Corollary 2.4.6 *Under hypothesis of Theorem 2.4.2 if $1 \leq \alpha' < \alpha$ we have for $Y \stackrel{d}{=} \Phi_\alpha$,*

$$E\{|M_n|^{\alpha'}\} \underset{n}{\to} E\{|Y|^{\alpha'}\} \quad \textit{and} \quad F_{M_n} \xrightarrow{d} \Phi_\alpha. \tag{2.4.10}$$

Proof. From Proposition 1.2.11 we have $E\{|Y|^{\alpha'}\} < \infty$. Using the same notation as in the proof of Theorem 2.4.2 we have

$$E\{|\frac{\max\{Y_1, \ldots, Y_n\}}{n^{1/\alpha}}|^{\alpha'}\} < \infty.$$

By Liapounov's inequality

$$\begin{aligned} d_{\alpha'}^{\alpha'}(F_{M_n}, \Phi_\alpha) &\leq E\{|\frac{\max\{X_1, \ldots, X_n\} - \max\{Y_1, \ldots, Y_n\}}{n^{1/\alpha}}|^{\alpha'}\} \\ &\leq \left(E\{|\frac{\max\{X_1, \ldots, X_n\} - \max\{Y_1, \ldots, Y_n\}}{n^{1/\alpha}}|^{\alpha}\}\right)^{\alpha'/\alpha} \underset{n}{\to} 0. \end{aligned}$$

Thus $F_{M_n} \in \mathcal{L}_{\alpha'}$ and $\Phi_\alpha \in \mathcal{L}_{\alpha'}$. Result follows as a direct application of Theorem 1.4.2. □

As noted in the Preliminaries, since

$$\min\{X_1, \ldots, X_n\} = -\max\{-X_1, \ldots, -X_n\}$$

analogous results for minima are derived. We just remember

$$\Phi'_\alpha(x) = \begin{cases} 1 - \exp\{-(-x)^{-\alpha}\} & x < 0, \\ 1 & x > 0. \end{cases}$$

Theorem 2.4.7 *Let* $W_n = \min\{X_1, \ldots, X_n\}/n^{\frac{1}{\alpha}}$ *and* $W_n \stackrel{d}{=} F_{W_n}$. *Let* $\{Y_n\}_{n\geq 1}$ *be a sequence of i.i.d. random variables with common distribution* Φ'_α. *Assume that for all* $b > 0$

$$\frac{1}{n}\sum_{j=1}^{n} E\big\{|X_j - Y_j|^\alpha 1_{(|X_j-Y_j|>bn^{1/\alpha})}\big\} \underset{n}{\to} 0. \tag{2.4.11}$$

Then

$$\lim_{n\to\infty} d_\alpha(F_{W_n}, \Phi'_\alpha) = 0 \ .$$

Proof. Since $\{Y_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables with common distribution Φ'_α we have

$$\frac{\min\{Y_1, \ldots, Y_n\}}{n^{1/\alpha}} \stackrel{d}{=} \Phi'_\alpha.$$

Thus

$$\begin{aligned} d_\alpha^\alpha(F_{W_n}, \Phi'_\alpha) &\leq E\big\{\big|\frac{\min\{X_1, \ldots, X_n\}}{n^{\frac{1}{\alpha}}} - \frac{\min\{Y_1, \ldots, Y_n\}}{n^{\frac{1}{\alpha}}}\big|^\alpha\big\} \\ &= E\big\{\big|\frac{-\max\{-X_1, \ldots, -X_n\}}{n^{\frac{1}{\alpha}}} + \frac{\max\{-Y_1, \ldots, -Y_n\}}{n^{\frac{1}{\alpha}}}\big|^\alpha\big\} \\ &= \frac{1}{n}E\big\{\big| - \max\{-X_1, \ldots, -X_n\} + \max\{-Y_1, \ldots, -Y_n\}\big|^\alpha\big\} \\ &\leq \frac{1}{n}E\big\{\big[\max\{|X_1 - Y_1|, \ldots, |X_n - Y_n|\}\big]^\alpha\big\} \end{aligned}$$

with the same steps of the proof of Theorem 2.4.2 follows

$$\leq c(\alpha)b^\alpha + \frac{c(\alpha)}{n} \sum_{j=1}^{n} E\big\{|X_j - Y_j|^\alpha 1_{(|X_j - Y_j| > bn^{1/\alpha})}\big\}$$

where $c(\alpha) = 1$ if $0 < \alpha < 1$ and $c(\alpha) = 2^{\alpha-1}$ if $\alpha \geq 1$. Using (2.4.6) and since b is arbitrary conclusion follows. $\square$

Chapter 3

Mallows Distance Convergence for Extremes: Regeneration Approach

3.1 Introduction

In this chapter, we treat all three types of extreme distributions Φ_α (Fréchet), Ψ_α (Weibull) and Λ (Gumbel). We present results that, for $r \geq 1$, exhibit sufficient conditions for the convergence

$$d_r(F_{M_n}, \phi_\alpha) \underset{n}{\rightarrow} 0, \quad d_r(F_{M_n}, \Psi_\alpha) \underset{n}{\rightarrow} 0 \quad \text{and} \quad d_r(F_{M_n}, \Lambda) \underset{n}{\rightarrow} 0.$$

Where for given random variables $X_1, X_2, \ldots$ we define

$$M_n = \frac{\max\{X_1, \ldots, X_n\} - b_n}{a_n} \quad \text{and} \quad M_n \stackrel{d}{=} F_{M_n}. \tag{3.1.1}$$

First, making use of moment convergence results from Lemma 3.2.1 and under the framework of i.i.d. random variables we derive the desired Mallows convergence, Theorem 3.2.2. A key assumption is the proper moment control relative to the left tail,

$$\int_{-\infty}^{0} |x|^r dF(x) < \infty,$$

being F the common distribution of the i.i.d. sequence.

In section 3.3, we borrow some of the arguments from Rootzén (1988) by considering the submaxima over the cycles,

$$\xi_j = \max\{X_n : T_{j-1} \le n < T_j\} \quad j \ge 1.$$

Then approximate $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$ by $\max\{\xi_0, \ldots, \xi_{v_n}\}$ where v_n is conveniently chosen. Corollary 3.3.2 characterizes the max-domain of attraction for Φ_α , Ψ_α and Λ. The Lemma 3.3.4 provides moments convergence. Finally Theorem 3.3.5 summarizes the main results for regenerative processes.

3.2 Convergence for I.I.D Random Variable Sequence

Throughout this section we will assume that $\{X_n\}_{n\ge1}$ is a sequence of i.i.d. random variable with common distribution F. If $F \in \mathcal{D}_{\max}(H)$ for an extreme value distribution H, then there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that $X_{(n)} = max\big\{X_1, X_2, \cdots, X_n\big\}$ satisfies

$$P\Big(\frac{X_{(n)} - b_n}{a_n} \le x\Big) = F^n(a_n x + b_n) \underset{n}{\to} H(x). \tag{3.2.1}$$

As mentioned before one way to achieve convergence in Mallows distance is exploring its close relation to the convergence in distribution and corresponding moment convergence results. Now we may ask for which value of $r > 0$ it is true that

$$\lim_{n\to\infty} E\Big\{\Big(\frac{X_{(n)} - b_n}{a_n}\Big)^r\Big\} = \int_{\mathbb{R}} |x|^r dH(x) \quad ?$$

The tail conditions which comprise the domain of attraction criteria are only a control on the right tail. As mentioned in Proposition 1.2.11 this implies if $F \in \mathcal{D}_{\max}(\Phi_\alpha)$ then

$$\int_0^{+\infty} x^r dF(x) < \infty, \quad \textit{for all} \;\; r \in (0, \alpha),$$

and if $F(x) \in \mathcal{D}_{\max}(\Lambda)$ then

$$\int_0^{+\infty} x^r dF(x) < \infty, \quad \textit{for all} \;\; r \in (0, \infty).$$

but no control is provided over the left tail and it is possible $\int_{-\infty}^{0} |x|^r dF(x) = \infty$ for any $r > 0$. Thus, it is necessary to impose some condition on the left tail. Regarding convergence of moments, Proposition 2.1 from Resnick (1987) provides the answer.

Lemma 3.2.1 *Suppose $F \in \mathcal{D}_{\max}(H)$. Let M_n be defined by (3.1.1). For an extreme value distribution H,*

(i) If $H = \Phi_\alpha$ and for some integer $0 < r < \alpha$

$$\int_{-\infty}^{0} |x|^r dF(x) < \infty$$

then

$$\lim_{n \to \infty} E\{(M_n)^r\} = \int_0^{\infty} x^r d\Phi_\alpha(x) = \Gamma(1 - \alpha^{-1} r),$$

where $a_n = (1/(1-F))^{-1}(n)$ and $b_n = 0$.

(ii) If $H = \Psi_\alpha$ and for some integer $r > 0$,

$$\int_{-\infty}^{\omega(F)} |x|^r dF(x) < \infty$$

then

$$\lim_{n\to\infty} E\big\{(M_n)^r\big\} = \int_{-\infty}^{0} |x|^r d\Psi_\alpha(x) = (-1)^r\Gamma(1+\alpha^{-1}r),$$

where $a_n = \big[\omega(F) - (1/(1-F))^{-1}(n)\big]$ *and* $b_n = \omega(F)$.

(iii) If $H = \Lambda$ *and for some integer* $r > 0$,

$$\int_{-\infty}^{0} |x|^r dF(x) < \infty$$

then

$$\lim_{n\to\infty} E\big\{(M_n)^r\big\} = \int_{-\infty}^{+\infty} |x|^r d\Lambda(x) = (-1)^r\Gamma^{(r)}(1),$$

where $b_n = (1/(1-F))^{-1}(n)$ *and* $a_n = g(b_n)$.

$\Gamma^{(r)}(1)$ *is the* r*-th derivative of the gamma function at* $x = 1$.

Using Lemma 3.2.1 and Theorem 1.4.2 we easily prove the following Mallows distance convergence for M_n.

Theorem 3.2.2 *Let* M_n *be defined by (3.1.1).*

(i) If for some integer $1 \le r < \alpha$

$$\int_{-\infty}^{0} |x|^r dF(x) < \infty \tag{3.2.2}$$

then

$$F \in \mathcal{D}_{\max}(\Phi_\alpha) \quad \Longleftrightarrow \quad d_r(F_{M_n}, \Phi_\alpha) \underset{n}{\to} 0,$$

where $a_n = (1/(1-F))^{-1}(n)$ *and* $b_n = 0$.

(ii) *If for some integer* $r \geq 1$,

$$\int_{-\infty}^{\omega(F)} |x|^r dF(x) < \infty \tag{3.2.3}$$

then

$$F \in \mathcal{D}_{\max}(\Psi_\alpha) \quad \Longleftrightarrow \quad d_r(F_{M_n}, \Psi_\alpha) \underset{n}{\rightarrow} 0$$

where $a_n = \left[\omega(F) - (1/(1-F))^{-1}(n)\right]$ *and* $b_n = \omega(F)$.

(iii) *If for some integer* $r \geq 1$,

$$\int_{-\infty}^{0} |x|^r dF(x) < \infty \tag{3.2.4}$$

then

$$F \in \mathcal{D}_{\max}(\Lambda) \quad \Longleftrightarrow \quad d_r(F_{M_n}, \Lambda) \underset{n}{\rightarrow} 0$$

where $b_n = (1/(1-F))^{-1}(n)$ *and* $a_n = g(b_n)$.

Proof.

i) $\Rightarrow$) Since $F \in \mathcal{D}_{\max}(\Phi_\alpha)$ then by Proposition 1.2.11

$$\int_0^{+\infty} x^r dF(x) < \infty, \quad \textit{for all} \;\; r \in (0, \alpha).$$

This together with condition (3.2.2) ensures that $F \in \mathcal{L}_r$. Even more, $F_{M_n} \in \mathcal{L}_r$ since by Lemma 3.2.1 we have

$$\lim_{n\to\infty} E\{(M_n)^r\} = \int_{-\infty}^{\infty} |x|^r d\Phi_\alpha(x) < \infty.$$

Now applying Proposition 1.4.2 we have convergence in Mallows r-th distance for $1 \leq r < \alpha$.

$\Leftarrow$) Let $d_r(F_{M_n}, \Phi_\alpha) \underset{n}{\rightarrow} 0$. By Theorem 1.4.2 there are a sequence of random variable Y_n and Y such that $Y_n \stackrel{d}{=} F_{M_n}$, $Y \stackrel{d}{=} \Phi_\alpha$, $(Y_n, Y) \stackrel{d}{=} F_{M_n} \wedge \Phi_\alpha$ and

$$d_r^r(F_{M_n}, \Phi_\alpha) = E\{|Y_n - Y|^r \underset{n}{\to} 0$$

Now from the r-mean convergence we have $Y_n \xrightarrow{d} Y$ or equivalently, $F_{M_n} \xrightarrow{d} \Phi_\alpha$.

The proof of (ii) and (iii) are similar to item (i). □

3.3 Convergence for Regenerative Process

In this section we consider a regenerative process $\{X_n\}_{n\geq 0}$ with values in a measurable space $(E, \mathcal{E})$. As described in section 1.5 this means there exist integer-valued random variables $0 < T_0 < T_1 < \dots$ such that the cycles ,

$$C_0 = \{X_n, 0 \leq n < T_0\} \ , \ C_1 = \{X_n, T_0 \leq n < T_1\} \ , \ \dots$$

are independent and, in addition, $C_1, C_2, \dots$ have the same distribution. In what follows we will denote $v_n = \inf\{k; T_k > n\}$ and $\mu = E[Y_1]$ where $Y_1 = T_1 - T_0$.

Let $\xi_0 = \max\limits_{0 \leq n < T_0}(X_n)$ and for $j \geq 1$, define the submaximum over the j-th cycle by

$$\xi_j = \max_{T_{j-1} \leq n < T_j}(X_n).$$

Rootzén (1988) in Theorem 3.1 show that $X_{(n)} = \max\{X_1, X_2, \dots, X_n\}$ is approximated by $max\{\xi_0, \dots, \xi_{v_n}\}$, which in turn can be approximated by $\max\{\xi_0, \dots, \xi_{[\frac{n}{\mu}]}\}$. Since the distribution of the first cycle, C_0 , is in general arbitrary, a condition is needed to assure that the first block does not affect the extremal behavior.

Theorem 3.3.1 *(Rootzén (1988)). Let $\{X_n\}_{n\geq 0}$ be a regenerative process with renewal sequence $\{T_k\}_{k\geq 0}$ and let $\mu = E[Y_1] < \infty$. Under the assumption that the first block does not affect the extremal behavior, that is to say that*

$$P\big(\xi_0 > \max\{\xi_1, \ldots, \xi_k\}\big) \longrightarrow 0 \quad \text{as} \quad k \to \infty, \tag{3.3.1}$$

then we have

$$\sup_{x\in\mathbb{R}} |P(X_{(n)} \leq x) - G^n(x)| \underset{n}{\to} 0, \tag{3.3.2}$$

where $G(x) = P(\xi_1 \leq x)^{\frac{1}{\mu}}$.

It is trivial to see that (3.3.1) holds if $\{X_n\}_{n\geq 0}$ is zero-delayed, since $\xi_0, \xi_1, \ldots$ then are i.i.d. Since G is a distribution function it follows that the only possible limit laws for M_n defined by (3.1.1), are the three extreme value distributions. For cases where the tail of the distribution of ξ_1 can be controlled we can derive detailed information on M_n.

Corollary 3.3.2 *Let $\{X_n\}_{n\geq 0}$ be a regenerative process with renewal sequence $\{T_k\}_{k\geq 0}$ and let $\mu = E[Y_1] < \infty$. Let $G(x) = P(\xi_1 \leq x)^{\frac{1}{\mu}}$ where $\xi_1 = \max_{T_0\leq n<T_1}(X_n)$. Then under assumption (3.3.1) we have*

(i) $1 - G \in RV_{-\alpha}$ if and only if

$$F_{M_n} \xrightarrow{d} \Phi_\alpha \tag{3.3.3}$$

where $a_n = (1/(1-G))^{-1}(n)$ and $b_n = 0$.

(ii) $F_{M_n} \xrightarrow{d} \Psi_\alpha$, if and only if $\omega(G) < \infty$ and $1 - G\big(\omega(G) - \frac{1}{x}\big) \in RV_{-\alpha}$. In this case $a_n = \omega(G) - (1/(1-G))^{-1}(n)$ and $b_n = \omega(G)$.

(iii) $F_{M_n} \xrightarrow{d} \Lambda(x)$ *if and only if there exists a Von Mises function* G^* *such that for* $x \in (z_0, \omega(G))$

$$1 - G(x) = c(x)(1 - G^*) = c(x)\exp\{-\int_{z_0}^{x} \frac{1}{g(y)}dy\}, \qquad (3.3.4)$$

and

$$\lim_{x \to \omega(G)} c(x) = c > 0.$$

In this case $b_n = (1/(1-G))^{-1}(n)$ *and* $a_n = g(b_n)$.

Proof. (i) ($\Rightarrow$) By Theorem 3.3.1 we have

$$\sup_{x \in \mathbb{R}} |P(X_{(n)} \le x) - G^n(x)| \underset{n}{\to} 0,$$

where $G(x) = P(\xi_1 \le x)^{\frac{1}{\mu}}$. Since G is a distribution function and $1 - G \in RV_{-\alpha}$ it follows from Theorem 1.2.8

$$G^n(a_n x) = P(\xi_1 \le a_n x)^{\frac{n}{\mu}} \underset{n}{\to} \Phi_\alpha(x)$$

where $a_n = (1/(1-G))^{-1}(n)$. Now (3.3.3) follows from (3.3.2).

($\Leftarrow$) Conversely suppose

$$F_{M_n} \xrightarrow{d} \Phi_\alpha$$

This, combined with (3.3.2) , yields

$$G^n(a_n x) \underset{n}{\to} \Phi_\alpha(x)$$

and so by Theorem 1.2.8 we have $1 - G \in RV_{-\alpha}$.

The proofs of (ii) and (iii) are in a similar way using Theorems 1.2.9 and 1.2.10, respectively, along with (3.3.2). □

Next, we extend moment convergence results for i.i.d. sequences to regenerative process. For that the following upper bounds will be needed. The proof makes use of some ideas from the proof of Lemma 2.2 from Resnick (1987).

Lemma 3.3.3 *Let $\{X_n\}_{n\geq 0}$ be a regenerative process with renewal sequence $\{T_k\}_{k\geq 0}$ and let $\mu = E[Y_1] < \infty$. Let (3.3.1) hold. Assume that $F_{M_n} \xrightarrow{d} \Lambda$. Then for $G(x) = P(\xi_1 \leq x)^{\frac{1}{\mu}}$ we have*

(i) Given $\epsilon > 0$, we have for $y > 0$ and all sufficiently large n

$$1 - G^n(a_n y + b_n) \leq (1+\epsilon)^3(1+\epsilon y)^{-\epsilon^{-1}}.$$

(ii) Let z_0 be the value in the representation (3.3.4). Given ϵ choose $z_1 \in (z_0, \omega(G))$ such that $|g'(t)| < \epsilon$ if $t > z_1$. Then for $s \in (\frac{z_1 - b_n}{a_n}, 0)$ and for large n we have

$$G^n(a_n s + b_n) \leq e^{-(1-\epsilon)^2(1+\epsilon|s|)^{\epsilon^{-1}}}.$$

Proof. By Corollary 3.3.2 (iii) (3.3.4) holds. We recall that $a_n = g(b_n)$.
(i) Since g is absolutely continuous function on $(z_0, \omega(G))$ with density g' and $\lim_{u\to x_0} g'(u) = 0$, we choose n such that $|g'(t)| < \epsilon$ if $t \geq b_n$ and we can write for $s > 0$

$$\begin{aligned} \frac{g(a_n s + b_n)}{a_n} - 1 &= \int_{b_n}^{a_n s + b_n} \frac{g'(u)}{a_n} du \\ &= \int_0^s g'(a_n u + b_n) du \\ &\leq \epsilon s. \end{aligned}$$

And we have immediately

$$\frac{a_n}{g(a_n s + b_n)} \geq (1 + \epsilon s)^{-1}. \tag{3.3.5}$$

On the other hand, note that

$$1 - G(b_n) \sim n^{-1}$$

so that for large n and $y > 0$

$$n\big(1 - G(a_n y + b_n)\big) \leq (1 + \epsilon)\frac{1 - G(a_n y + b_n)}{1 - G(b_n)}.$$

From (3.3.4) we have

$$1 - G(a_n y + b_n) \quad = \quad c(a_n y + b_n) e^{-\displaystyle\int_{z_0}^{a_n y + b_n} \frac{1}{g(s)} ds}$$

and

$$1 - G(b_n) \quad = \quad c(b_n) e^{-\displaystyle\int_{z_0}^{b_n} \frac{1}{g(s)} ds}.$$

Thus

$$n\big(1 - G(a_n y + b_n)\big) \leq (1 + \epsilon)\frac{c(a_n y + b_n)}{c(b_n)} e^{-\displaystyle\int_{b_n}^{a_n y + b_n} \frac{1}{g(s)} ds}.$$

Since $c(x) \to c > 0$ as $x \to \omega(G)$ for sufficient large n the preceding is bounded by

$$\begin{aligned}
&\leq \quad (1 + \epsilon)^2 e^{-\displaystyle\int_0^y \frac{a_n}{g(a_n s + b_n)} ds} \\
&\leq \quad (1 + \epsilon)^2 e^{-\displaystyle\int_0^y (1 + \epsilon s)^{-1} ds} \\
&= \quad (1 + \epsilon)^2 e^{-\epsilon^{-1} \ln(1 + \epsilon y)} \\
&= \quad (1 + \epsilon)^2 (1 + \epsilon y)^{-\epsilon^{-1}}.
\end{aligned} \tag{3.3.6}$$

In the last inequality we have used (3.3.5) . Therefore

$$\begin{aligned}
1 - G^n(a_n y + b_n) &= 1 - \exp\big\{n \ln G(a_n y + b_n)\big\} \\
&\leq n(-\ln G(a_n y + b_n)) \\
&\leq (1+\epsilon) n (1 - G(a_n y + b_n))
\end{aligned}$$

In the last inequality we have used $\lim\limits_{z \to 1} \dfrac{-\ln z}{1-z} = 1$. Now with (3.3.6) result follows.

(ii) For $u \in (\dfrac{z_1 - b_n}{a_n}, 0)$ and large n we have

$$\begin{aligned}
1 - \frac{g(a_n u + b_n)}{a_n} &= \int_{a_n u + b_n}^{b_n} \frac{g'(t)}{a_n} dt \\
&= \int_u^0 g'(a_n t + b_n) dt \\
&\geq -\epsilon |u|
\end{aligned}$$

The last inequality holds because $a_n t + b_n > a_n u + b_n > z_1$. Thus we have shown

$$1 + \epsilon|u| \geq \frac{g(a_n u + b_n)}{a_n}. \tag{3.3.7}$$

On the other hand, for large n

$$\begin{aligned}
G^n(a_n s + b_n) &= \Big(1 - (1 - G(a_n s + b_n))\Big)^n \\
&\leq \exp\Big\{ - n(1 - G(a_n s + b_n))\Big\} \\
&\leq \exp\Big\{ - (1-\epsilon)\frac{1 - G(a_n s + b_n)}{1 - G(b_n)}\Big\} \\
&\leq \exp\Big\{ - (1-\epsilon)\frac{c(a_n s + b_n)}{c(b_n)} \exp\big\{ \int_{b_n}^{a_n s + b_n} \frac{1}{g(u)} du\big\}\Big\}.
\end{aligned}$$

Supposing z_1 has been chosen so that $c(z_1)/c(b_n) \geq 1 - \epsilon$, the preceding is bounded by

$$\begin{aligned}
&\leq \exp\Big\{ - (1-\epsilon)^2 \exp\Big\{ \int_s^0 \frac{a_n}{g(a_n u + b_n)} du \Big\}\Big\} \\
&\leq \exp\Big\{ - (1-\epsilon)^2 \exp\Big\{ \int_s^0 (1 + \epsilon|u|)^{-1} du \Big\}\Big\} \\
&= \exp\big\{ - (1-\epsilon)^2 (1 + \epsilon|s|)^{\epsilon^{-1}} \big\}.
\end{aligned}$$

In the last inequality we have used (3.3.7). □

Now we may repeat Lemma 3.2.1 for regenerative processes.

Lemma 3.3.4 *Let $\{X_n\}_{n\geq 0}$ be a regenerative process that satisfies the conditions of Theorem 3.3.1 . Let M_n be defined by (3.1.1). Suppose $F_{M_n} \xrightarrow{d} H$ for an extreme value distribution H.*

(i) If $H = \Phi_\alpha$ and for some X_i with distribution F and some integer $0 < r < \alpha$

$$\int_{-\infty}^{0} |x|^r dF(x) < \infty \tag{3.3.8}$$

then

$$\lim_{n\to\infty} E\{M_n\}^r = \int_0^\infty x^r d\Phi_\alpha(x) = \Gamma(1 - \alpha^{-1} r),$$

where $a_n = (1/(1-G))^{-1}(n)$ and $b_n = 0$.

(ii) If $H = \Psi_\alpha$ and for some X_i with distribution F and for some integer $r > 0$,

$$\int_{-\infty}^{\omega(F)} |x|^r dF(x) < \infty \tag{3.3.9}$$

then

$$\lim_{n\to\infty} E\{M_n\}^r = \int_{-\infty}^{0} |x|^r d\Psi_\alpha(x) = (-1)^r \Gamma(1+\alpha^{-1}r),$$

where $a_n = \omega(G) - (1/(1-G))^{-1}(n)$ *and* $b_n = \omega(G)$.

(iii) *If* $H = \Lambda$ *and for some* X_i *with distribution* F *and for some integer* $r > 0$,

$$\int_{-\infty}^{0} |x|^r dF(x) < \infty \tag{3.3.10}$$

then

$$\lim_{n\to\infty} E\{M_n\}^r = \int_{-\infty}^{+\infty} |x|^r d\Lambda(x) = (-1)^r \Gamma^r(1),$$

where $b_n = (1/(1-G))^{-1}(n)$ *and* $a_n = g(b_n)$

$\Gamma^{(r)}(1)$ *is the* r*-th derivative of the gamma function at* $x = 1$.

Proof. The proof makes use of some ideas from the proof of Proposition 2.1 on page 77 from Resnick (1987).

Since $F_{M_n} \xrightarrow{d} H$ we have from weak convergence theory (Helly-Bray lemma) that for any $L > 0$

$$\lim_{n\to\infty} E\{\left(M_n\right)^r 1_{(|M_n|\le L)}\} = \int_{-L}^{L} x^r dH(x).$$

Thus, it is enough to show

$$\lim_{L\to\infty} \limsup_{n\to\infty} E\{|M_n|^r 1_{(|M_n|>L)}\} = 0, \tag{3.3.11}$$

because

$$
\begin{aligned}
\left|E\{(M_n)^r\}\right. &- \left.\int_{-\infty}^{\infty} x^r dH(x)\right| \\
&\leq \left|E\{(M_n)^r\} - E\{(M_n)^r 1_{(|M_n|\leq L)}\}\right| \\
&+ \left|E\{(M_n)^r 1_{(|M_n|\leq L)}\} - \int_{-L}^{L} x^r dH(x)\right| \\
&+ \left|\int_{-L}^{L} x^r dH(x) - \int_{-\infty}^{\infty} x^r dH(x)\right|. \qquad (3.3.12)
\end{aligned}
$$

and with (3.3.11) we will have

$$
\lim_{L\to\infty} \limsup_{n\to\infty} \left|E\{(M_n)^r\} - E\{(M_n)^r 1_{(|M_n|\leq L)}\}\right| = 0
$$

and so the right side of (3.3.12) will have $\lim_{L\to\infty} \limsup_{n\to\infty} = 0$ and since the left side of (3.3.12) does not depend on L, the desired result follows. We use Fubini's theorem to justify an integration by parts:

$$
\begin{aligned}
E\{|M_n|^r 1_{(|M_n|>L)}\} &= E\{\int_0^{|M_n|} rs^{r-1} ds 1_{(|M_n|>L)}\} \\
&= E\{\int_0^{L} rs^{r-1} ds 1_{(|M_n|>L)}\} + \\
&\quad E\{\int_L^{\infty} rs^{r-1} 1_{(|M_n|>L,|M_n|>s)} ds\} \\
&= L^r P\big(|M_n| > L\big) + \int_L^{\infty} rs^{r-1} P\big(|M_n| > s\big) ds \\
&= A + B.
\end{aligned}
$$

(i) When $H = \Phi_\alpha$ and $r < \alpha$,

$$
\begin{aligned}
\lim_{L\to\infty} \limsup_{n\to\infty} A &= \lim_{L\to\infty} L^r(1 - \Phi_\alpha(L) + \Phi_\alpha(-L)) \\
&= \lim_{L\to\infty} L^r(1 - e^{-L^{-\alpha}}) = \lim_{L\to\infty} L^r L^{-\alpha} \\
&= 0.
\end{aligned}
$$

As for the term B write

$$
\begin{aligned}
B &= \int_L^\infty rs^{r-1} P\big(|\frac{X_{(n)}}{a_n}| > s\big) ds \\
&= \int_L^\infty rs^{r-1} P(X_{(n)} > a_n s) ds + \\
&\quad \int_L^\infty rs^{r-1} P(X_{(n)} < -a_n s) ds \\
&= B_1 + B_2.
\end{aligned}
$$

For B_1, by Theorem 3.3.1, the uniform convergence in (3.3.2) hold and we have

$$
\begin{aligned}
\limsup_{n\to\infty} B_1 &= \limsup_{n\to\infty} \int_L^\infty rs^{r-1} P(X_{(n)} > a_n s) ds \\
&= \limsup_{n\to\infty} \int_L^\infty rs^{r-1} (1 - G^n(a_n s)) ds.
\end{aligned}
$$

We can write for large n

$$
\begin{aligned}
1 - G^n(a_n s) &= 1 - \exp\{n \ln G(a_n s)\} \\
&\leq n(-\ln G(a_n s)) \\
&\leq (1+\epsilon) n (1 - G(a_n s)) \\
&\leq (1+\epsilon)^2 \frac{1 - G(a_n s)}{1 - G(a_n)}.
\end{aligned}
$$

By Corollary 3.3.2 (i) we have $1 - G \in RV_{-\alpha}$. Now apply Theorem 1.2.7, which tells us, given $\epsilon > 0$, if n is large and $L > 1$ then we have the following upper bound

$$
\frac{1 - G(a_n s)}{1 - G(a_n)} \leq (1+\epsilon) s^{-\alpha+\epsilon}.
$$

So that in this case if we choose ϵ such that $r < \alpha - \epsilon$ or equivalently $r - 1 - \alpha + \epsilon < -1$ then

$$\begin{aligned}
\lim_{L\to\infty} \limsup_{n\to\infty} B_1 &\leq \lim_{L\to\infty} (1+\epsilon)^3 \int_L^\infty rs^{r-1} s^{-\alpha+\epsilon} ds \\
&= 0
\end{aligned}$$

Note that since r is assumed less than α, we can choose ϵ small enough such that $r < \alpha - \epsilon$.

Now for B_2 we have

$$\begin{aligned}
B_2 &= \int_L^\infty rs^{r-1} P(X_{(n)} < -a_n s) ds \\
&= \int_{-\infty}^{-L} r|s|^{r-1} P(X_{(n)} < a_n s) ds \\
&= \frac{1}{a_n^r} \int_{-\infty}^{-a_n L} r|s|^{r-1} P(X_{(n)} < s) ds \\
&\leq \frac{1}{a_n^r} \int_{-\infty}^{-a_n L} r|s|^{r-1} F(s) ds.
\end{aligned}$$

By (3.3.8) and the fact that $a_n \to \infty$ we have for all $L > 0$

$$\limsup_{n\to\infty} B_2 = 0.$$

This complete the proof of the part (i).

(ii) When $H = \Psi_\alpha$ and $k > 0$, we have

$$\begin{aligned}
\lim_{L\to\infty} \limsup_{n\to\infty} A &= \lim_{L\to\infty} L^r (1 - \Psi_\alpha(L) + \Psi_\alpha(-L)) \\
&= \lim_{L\to\infty} L^r (e^{-L^\alpha}) \\
&= 0.
\end{aligned}$$

And for B

$$\begin{aligned}
B &= \int_L^\infty rs^{r-1}P\Big(|\frac{X_{(n)}-\omega(G)}{a_n}|>s\Big)ds \\
&= \int_L^\infty rs^{r-1}P\big(X_{(n)}-\omega(G)>a_ns\big)ds \\
&+ \int_L^\infty rs^{r-1}P\big(X_{(n)}-\omega(G)<-a_ns\big)ds \\
&= B_1+B_2.
\end{aligned}$$

Since $\omega(F)<\infty$ we have obviously

$$\lim_{L\to\infty}\limsup_{n\to\infty} B_1 = 0$$

It remains to check for B_2. By uniform convergence in (3.3.2) we can write

$$\begin{aligned}
\limsup_{n\to\infty} B_2 &= \limsup_{n\to\infty}\int_L^\infty rs^{r-1}P\big(X_{(n)}<-a_ns+\omega(G)\big)ds \\
&= \limsup_{n\to\infty}\int_L^\infty r|s|^{r-1}G^n(-a_ns+\omega(G))ds \\
&= \limsup_{n\to\infty}\int_{-\infty}^{-L} r|s|^{r-1}G^n(a_ns-\omega(G))ds \\
&= \limsup_{n\to\infty}\frac{1}{a_n^r}\int_{-\infty}^{-a_nL-\omega(G)} r|s|^{r-1}G^n(s)ds \\
&\le \limsup_{n\to\infty}\frac{1}{a_n^r}G^{n-\mu}(-a_nL-\omega(G))\int_{-\infty}^{-a_nL-\omega(G)} r|s|^{r-1}F(s)ds \\
&\le \limsup_{n\to\infty}\frac{1}{a_n^r}G^{n-\mu}(-a_nL-\omega(G))\int_{-\infty}^{\omega(F)} r|s|^{r-1}F(s)ds.
\end{aligned}$$

By (3.3.9) and the fact that

$$a_n^{-r}G^{n-\mu}(-a_nL-\omega(G))\to 0$$

we have for all $L>0$

$$\limsup_{n\to\infty} B_2 = 0$$

This complete the proof of the part (ii).

(iii) When $H = \Lambda$ for A we have

$$\begin{aligned}\lim_{L\to\infty} \limsup_{n\to\infty} A &= \lim_{L\to\infty} L^r\big(1-\Lambda(L)+\Lambda(-L)\big)\\ &= \lim_{L\to\infty} L^r\big(1-\exp(-e^{-L})+\exp(-e^{L})\big)\\ &= \lim_{L\to\infty} L^r\big(e^{-L}+\exp(-e^{L})\big)=0.\end{aligned}$$

Now, for B

$$\begin{aligned}B &= \int_L^\infty rs^{r-1}P\big(|\frac{X_{(n)}-b_n}{a_n}|>s\big)ds\\ &= \int_L^\infty rs^{r-1}P\big(X_{(n)}-b_n>a_ns\big)ds\\ &+ \int_L^\infty rs^{r-1}P\big(X_{(n)}-b_n<-a_ns\big)ds\\ &= B_1+B_2\end{aligned}$$

For the case of B_1 by uniform convergence in (3.3.2) and applying Lemma 3.3.3 (i) we have for sufficiently large n

$$\begin{aligned}B_1 &\le \int_L^\infty rs^{r-1}(1-G^n(a_ns+b_n))ds\\ &\le (1+\epsilon)^3\int_L^\infty rs^{r-1}(1+\epsilon s)^{-\epsilon^{-1}}ds\end{aligned}$$

On the other hand, we have

$$rs^{r-1}(1+\epsilon s)^{-\epsilon^{-1}} \sim r\epsilon^{-\epsilon^{-1}}s^{r-1-\epsilon^{-1}}.$$

We choose $\epsilon < r^{-1}$ and so

$$r-1-\epsilon^{-1}<-1$$

Thus for some constant C (hereafter C will denote a positive constant, not necessarily the same one)

$$\lim_{L\to\infty} \limsup_{n\to\infty} B_1 \le C \lim_{L\to\infty} \int_L^\infty s^{r-1-\epsilon^{-1}} ds = 0.$$

For B_2 we have

$$\limsup_{n\to\infty} B_2 = \limsup_{n\to\infty} \int_L^\infty rs^{r-1} G^n(-a_n s + b_n) ds$$

Let z_1 is chosen as Lemma 3.3.3 (ii). Since $(z_1 - b_n)/a_n \to -\infty$ so eventually $(z_1 - b_n)/a_n < -L$ and we can write

$$\begin{aligned}
\int_L^\infty rs^{r-1} G^n(-a_n s + b_n) ds &= \int_{-\infty}^{-L} rs^{r-1} G^n(a_n s + b_n) ds \\
&= \int_{-\infty}^{(z_1-b_n)/a_n} rs^{r-1} G^n(a_n s + b_n) ds \\
&+ \int_{(z_1-b_n)/a_n}^{-L} rs^{r-1} G^n(a_n s + b_n) ds \\
&= B_{21} + B_{22}
\end{aligned}$$

For B_{21}, setting $y = a_n s + b_n$ we have

$$B_{21} = \frac{1}{a_n^r} \int_{-\infty}^{z_1} r|y - b_n|^{r-1} G^n(y) dy$$

So for some constant C

$$\begin{aligned}
B_{21} &\le \frac{r}{a_n^r} G^{n-\mu}(z_1) C \int_{-\infty}^{z_1} (|y|^{r-1} + b_n^{r-1}) F(y) dy \\
&= \frac{r}{a_n^r} G^{n-\mu}(z_1) C \int_{-\infty}^{z_1} |y|^{r-1} F(y) dy \\
&+ \frac{r}{a_n^r} G^{n-\mu}(z_1) C \int_{-\infty}^{z_1} b_n^{r-1} F(y) dy.
\end{aligned}$$

Since a_n and b_n are slowly varying functions of n, and $G^{n-\mu}(z_1)$ geometrically fast we get as $n \to \infty$

$$G^{n-\mu}(z_1)a_n^{-r} \longrightarrow 0 \quad , \quad G^{n-\mu}(z_1)a_n^{-r}b_n^{r-1} \longrightarrow 0.$$

On the other hand from (3.3.10) we have

$$\int_{-\infty}^{z_1} |y|^{r-1}F(y)dy < \infty$$

and so for some constant C

$$\int_{-\infty}^{z_1} b_n^{r-1}F(y)dy \leq C \int_{-\infty}^{z_1} b_n^{r-1}|y|^{r-1}F(y)dy < \infty$$

that follows $\limsup_{n\to\infty} B_{21} = 0$.

Finally, applying Lemma 3.3.3 (ii) we have

$$\begin{aligned} B_{22} &= \int_{(z_1-b_n)/a_n}^{-L} r|s|^{r-1}G^n(a_n s + b_n)ds \\ &\leq \int_{(z_1-b_n)/a_n}^{-L} r|s|^{r-1}e^{(1-\epsilon)^2(1+\epsilon|s|)^{\epsilon^{-1}}}ds. \end{aligned}$$

Since $|s|^{r} - 1 e^{-(1-\epsilon)^2(1+\epsilon|s|)^{\epsilon^{-1}}}$ is integrable on $(-\infty, 0)$ for some constant C we get

$$\begin{aligned} \lim_{L\to\infty} \limsup_{n\to\infty} B_{22} &\leq \lim_{L\to\infty} C \int_{-\infty}^{-L} |s|^{r-1}e^{-(1-\epsilon)^2(1+\epsilon|s|)^{\epsilon^{-1}}}ds \\ &= 0. \end{aligned}$$

This completes the proof. $\square$

Theorem 3.3.5 *Let $\{X_n\}$ be a regenerative process that satisfies the conditions of Theorem 3.3.1. Let M_n be defined by (3.1.1).*

(i) If for some X_i with distribution F and some integer $1 \leq r < \alpha$

$$\int_{-\infty}^{0} |x|^r dF(x) < \infty$$

then

$$F_{M_n} \xrightarrow{d} \Phi_\alpha \quad \Longleftrightarrow \quad d_r(F_{M_n}, \Phi_\alpha) \underset{n}{\to} 0,$$

where $a_n = (1/(1-G))^{-1}(n)$.

(ii) If for some X_i with distribution F and some integer $r \geq 1$,

$$\int_{-\infty}^{\omega(F)} |x|^r dF(x) < \infty$$

then

$$F_{M_n} \xrightarrow{d} \Psi_\alpha \quad \Longleftrightarrow \quad d_r(F_{M_n}, \Psi_\alpha) \underset{n}{\to} 0,$$

where $a_n = \omega(G) - (1/(1-G))^{-1}(n)$ and $b_n = \omega(G)$.

(iii) If for some X_i with distribution F and some integer $r \geq 1$,

$$\int_{-\infty}^{0} |x|^r dF(x) < \infty$$

then

$$F_{M_n} \xrightarrow{d} \Lambda \quad \Longleftrightarrow \quad d_r(F_{M_n}, \Lambda) \underset{n}{\to} 0,$$

where $b_n = (1/(1-G))^{-1}(n)$ and $a_n = g(b_n)$.

Proof. Using Lemma 3.3.4, we can repeat the same idea of the proof of Theorem 3.2.2 for this proof.

As we already said in the preliminary any Markov chain $\{X_n\}_{n\geq 0}$ with a countable state space S that is irreducible and recurrent is regenerative with $\{T_i\}_{i\geq 1}$ being the times of successive returns to a given state $\{x\}$. Harris chains on a general state space that possess an atom, are regenerative processes too. So they are applications of the above regenerative methods.

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