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Paraconsistentization and many-valued logics

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Abstract

This paper shows how to transform explosive many-valued systems into paraconsistent logics. We investigate especially the case of three-valued systems showing how paraconsistent three-valued logics can be obtained from them.

1 Introduction

An explosive logic can be transformed into a non-explosive one by means of methods of *paraconsistentization*. There are many ways which can be used to perform this task of converting explosive into non-explosive logics. In general, it is theoretically possible to *paraconsistentize* all non-paraconsistent systems. Paraconsistent logics had the major philosophical impact of distinguishing explosive systems from systems that imply contradictions, making it possible to infer whenever we face contradictory contexts.

In the history of paraconsistency, we can find three traditional accounts used to give a paraconsistent dimension to an explosive system. The first one developed by Stanisław Jaśkowski in [14] focuses in a way to define a paraconsistent discussive logic from a standard modal logic, while the second one proposed by Newton da Costa in [6] introduces a paraconsistent negation using the notion of *well-behavior* in such a way that new properties of a weaker negation not satisfying *ex falso* are generated. The third approach suggested by Graham Priest in [18] considers a new logical designated value besides truth and this gives rise to a paraconsistent logic. These three traditional accounts can be seen as methods for paraconsistentizing classical logic and, in some sense, they are mechanisms of paraconsistentization which can be applied to a wide range of logics.

All previous techniques are not unified by a standard strategy to transform a given logic into a paraconsistent one. Indeed, each one uses a particular procedure to formulate paraconsistency at the formal level. The initial concept and idea of *paraconsistentization* has been proposed in [5]. Afterwards, in two recent papers (cf. [8] and [9]), it has been showed how to turn a given logic into a paraconsistent one by a very precise methodology which works for a great variety of systems. In [8], the idea of *paraconsistentization* is presented in the context of category theory and it uses abstract logic as a main source. In this way, paraconsistentization appears as an endofunctor in the category of logics preserving some basic properties of the initial logic. In [9], considering notions of *axiomatic formal systems* and *valuation structures*, the quest of paraconsistentization is introduced by means of the concept of *paradeduction* in axiomatic formal systems and *paraconsequence* in valuation structures. Thus, it is possible to paraconsistentize proof systems and semantics showing how some properties are invariant under paraconsistentization. In this particular case, this method preserves soundness and completeness.

The above papers settled the basic theory of paraconsistentization, up to this level. Despite the fact the there are many unknown methods of paraconsistentization, this article shows how to turn some explosive many-valued logics into paraconsistent ones using the basic idea initially proposed in [8], that is: given a logic $L = \langle For, \vDash_L \rangle$, the paraconsistentization of L is a logic given by $\mathbb{P}(L) = \langle For, \vDash_L^{\mathbb{P}} \rangle$ such that: $\Gamma \vDash_L^{\mathbb{P}} \alpha$ if, and only if, there exists $\Gamma' \subseteq \Gamma, L$-consistent such that $\Gamma' \vDash_L \alpha$. We deal mainly with systems such as L_3, G_3 and K_3 defined by means of logical matrices, though it is obviously possible to extend the same approach to the whole hierarchies L_n and G_n.[1] We show what are the properties preserved or lost by the paraconsistentized version of these systems. This paper attempts to explore the universe of paraconsistent many-valued logics. Other studies on three-valued paraconsistent logics worth mentioning are those of [1], [2] and [4].

The philosophical relevance of this investigation appears when we take into account motivations and basic intuitions of the many-valued systems considered here.

The logic L_3, studied and examined by Łukasiewicz in [16], was created to deal with the problem of future contingents. Łukasiewicz argued that certain sentences, especially those about the future, are now neither true nor false. In this sense, the third truth-value 1/2 receives an ontological interpretation. Future contingents are indeed neither true nor false now. But this is not the only possible interpretation for 1/2.

K_3 was developed with another goal in mind. It was designed to deal with the possibility of partial information concerning the truth value of propositions. According to Kleene in [15], the truth-value 1/2 can be interpreted as "undefined" or as "unknown". The important concern here is that it can be seen as a lack of information, that is, we say that a propo-

[1] In the case of the logic L_3, the question of how to paraconsistentize it has been posed by Walter Carnielli (personal communication, 2005) to the second author of this paper.

sition has the value 1/2 when we do not know whether it is true or false. Given this interpretation, it is possible that a proposition with an undefined truth-value might became true or false, according to future information. So we have now an epistemological account of the value 1/2.

Nonetheless, notice that both proposals are not, at least in principle, philosophically committed with the consistency of formulas with value 1/2. But since explosion holds in these systems, a philosophical investigation on the possibility of contradictory information or contradictory future contingents is *a priori* excluded. By paraconsistentizing these logics, we provide a formal framework in which these scenarios - contradictory information and contradictory future contingents - can be investigated. In this sense, paraconsistentization of logics widens the scope of investigation of these logics.

Gödel intermediate logics have been developed with the aim of showing that intuitionistic logic cannot be defined using "finitely many elements (truth values)" (Gödel, page 225, [12]). Gödel discovered an infinite hierachy of consistent systems between intuitionistic and classical logic. Paraconsistentizing this hierarchy shows that there exists a whole new hierarchy of paraconsistent many-valued systems which does not need, necessarily, to be consistent. The problem of determining where is this new hierarchy (between which logics?) is still open.

It is important to stress two aspects of this proposal. First, since paraconsistentization preserves the consistent sets of formulas from the original logic, if no contradiction arises, then the paraconsistent version of a logic is equivalent to the original one. Second, paraconsistentization merely allows reasoning with inconsistent sets, but does not impose a specific interpretation of this inconsistency. In particular, paraconsistentization is neutral with respect to the existence of true (or real) contradictions.

In what follows, we apply the standard methodology of paraconsistentization to precise many-valued systems. Using this technique we are able to build paraconsistent many-valued systems on a very large scale.

2 Preliminaries

These preliminaries set the main terminology and concepts which are used throughout the text. Standard notions and results on many-valued logics can be found in [13], [17], [11], [3] and [19].

Let us consider an usual *propositional language* L with $\neg$ (negation symbol), $\vee$ (disjunction symbol), $\wedge$ (conjunction symbol), $\rightarrow$ (implication symbol) and propositional letters: $p, q, r, ..., p_1, q_1, r_1, ...$ and so on. The set of propositional letters is denoted by $Prop$ and the set of formulas of L, defined as usual, is denoted by For. We use α, β, γ,..., and Γ, Δ,... as syntactical variables for formulas and sets of formulas, respectively.

A *(logical) matrix* for L is a 6-uple

$$M = \langle Val, D, f_\neg, f_\vee, f_\wedge, f_\to \rangle$$

such that Val is a non-empty set, D is a non-empty proper subset of Val, $f_\neg$ is an unary function $f_\neg : Val \to Val$ and $f_\vee, f_\wedge, f_\to$ are binary functions of the type $f_\vee, f_\wedge, f_\to : Val \times Val \to Val$. Elements of Val are called *truth-values* (or simply *values*) and elements of D are called *designated values*.

A *M-valuation* for L is a function

$$v : Prop \to Val.$$

Every M-valuation v can be extended, in an unique way, to all elements of For by the following recursive clauses:

i. $v(\neg\alpha) = f_\neg(v(\alpha))$;
ii. $v(\alpha \vee \beta) = f_\vee(v(\alpha), v(\beta))$;
iii. $v(\alpha \wedge \beta) = f_\wedge(v(\alpha), v(\beta))$;
iv. $v(\alpha \to \beta) = f_\to(v(\alpha), v(\beta))$.

We denote by $\mathcal{V}_M$ the set of all M-valuation for L.

Let Γ be a subset of For and M a matrix for L. An element $v \in \mathcal{V}_M$ is a *M-model* of Γ iff (if and only if) $v(\gamma) \in D$, for all $\gamma \in \Gamma$. We denoted by $Mod_M(\Gamma)$ the set of all M-models for Γ. If $\Gamma = \{\alpha\}$ is an unitary set, we denote the set $Mod_M(\{\alpha\})$ by $Mod_M(\alpha)$.

Let Γ be a subset of For and α an element of For. We say that α is a *M-consequence* of Γ, in symbols $\Gamma \vDash_M \varphi$, if every M-model of Γ is a M-model of α, that is, $Mod_M(\Gamma) \subseteq Mod_M(\alpha)$.

We have the following immediate properties:

I. If $\alpha \in \Gamma$, then $\Gamma \vDash_M \alpha$;
II. If $\Gamma \vDash_M \alpha$, then $\Gamma \cup \Delta \vDash_M \alpha$;
III. If $\Gamma \vDash_M \alpha$ and $\Delta \vDash_M \gamma$, for all $\gamma \in \Gamma$, then $\Delta \vDash_M \alpha$.

If Γ is a set of formulas, the set of *M-consequences* of Γ, denoted by $Cn_M(\Gamma)$, is such that:

$$Cn_M(\Gamma) := \{\alpha \in FOR : \Gamma \vDash_M \alpha\}.$$

Cn_M can be seen, therefore, as an operator on $\wp(For)$, the set of all subsets of For, and Cn_M satisfies the Tarskian axioms (cf. [20] and [21]) :

I'. $\Gamma \subseteq Cn_M(\Gamma)$;
II'. $Cn_M(\Gamma) \subseteq Cn_M(\Gamma \cup \Delta)$;
III'. $Cn_M(Cn_M(\Gamma)) = Cn_M(\Gamma)$.

The pair $\langle For, Cn_M \rangle$ is, therefore, a normal consequence structure in the sense of [8] (see below).

If α is an element of For, we say that α is a *M-tautology* iff $Mod_M(\alpha) = \mathcal{V}_M$, that is, every M-valuation v is such that $v(\alpha) \in D$. Thus, every M-valuation is a M-model of α. We use the notation $\vDash_M \alpha$ to indicate that α is a M-tautology.

It is easy to see that: $\vDash_M \alpha$ iff $\emptyset \vDash_M \alpha$. Therefore, the set of all M-tautologies is the set $Cn_M(\emptyset)$. By monotonicity, the property (II') above, for all $\Gamma \subseteq For$, $Cn_M(\emptyset) \subseteq Cn_M(\Gamma)$, that is, the set of M-consequences of a set $\Gamma \subseteq For$ always contains all the M-tautologies.

3 Paraconsistentization of logics

We review some notions in the domain of paraconsistentization and we follow the presentations in [8] and [9].

A pair $L = \langle X, Cn_L \rangle$ is a *consequence structure* iff X is a non empty set and Cn_L is a mapping in $\wp(X)$. (We have no axioms at all!)

In a consequence structure of the type $L = \langle X, Cn_L \rangle$, $\Gamma \subseteq X$ is *L-consistent* if and only if $Cn_L(\Gamma) \neq X$. Otherwise, Γ is said to be *L-inconsistent*.

We say that a consequence structure $L = \langle X, Cn_L \rangle$ is *normal* iff Cn_L satisfies the properties (I')-(III') above.

If $L = \langle X, Cn \rangle$ is a consequence structure, we define a $\mathbb{P}$-transformation of L, called a *paraconsistentization* of L, as a consequence structure $\mathbb{P}(L) = \langle X, Cn_L^{\mathbb{P}} \rangle$ such that[2]: For all subset Γ of X we have

$$Cn_L^{\mathbb{P}}(\Gamma) = \bigcup \{Cn_L(\Gamma') \in \wp(X) : \Gamma' \subseteq \Gamma \text{ and } \Gamma' \text{ is } L\text{-consistent}\}.$$

Therefore, we have that $\alpha \in Cn_L^{\mathbb{P}}(\Gamma)$ iff there exists $\Gamma' \subseteq \Gamma$, L-consistent, such that $\alpha \in Cn(\Gamma')$.

Now, consider a matrix M for L. Thus, M yields a consequence structure $\langle For, Cn_M \rangle$ as seen above. (We also will use M for the pair $\langle For, Cn_M \rangle$, in order to fit the notation above.) In this way, the paraconsistentization ($\mathbb{P}$-transformation) of $M = \langle For, Cn_M \rangle$ is a consequence structure $\mathbb{P}(M) = \langle For, Cn_M^{\mathbb{P}} \rangle$ such that: For all $\Gamma \subseteq For$ and $\alpha \in For$, we have that:

$$\Gamma \vDash_M^{\mathbb{P}} \alpha \text{ iff there exists } \Gamma' \subseteq \Gamma,\ M\text{-consistent, such that } \Gamma' \vDash_M \alpha.$$

Note that if M is a matrix for L such that $M = \langle For, Cn_M \rangle$ is the consequence structure associated to M, then we have: If $\Gamma \subseteq For$ is M-consistent, then $Mod_M(\Gamma) \neq \emptyset$. Because, if $Mod_M(\Gamma) = \emptyset$, then $Mod_M(\Gamma) \subseteq Mod_M(\alpha)$, for all $\alpha \in For$. Therefore, $\Gamma \vDash_M \alpha$, for all $\alpha \in For$, that is, $Cn_M(\Gamma) = For$. Thus, Γ is M-inconsistent.

On the other hand, the converse is valid only in special cases.

Lemma 1 *Let $M = \langle Val, D, f_\neg, f_\vee, f_\wedge, f_\to \rangle$ be a matrix such that $f_\neg$ satisfies the following condition:*

$$(*) \qquad \text{if } x \in D, \text{ then } f_\neg(x) \notin D.$$

Then, we have that if $Mod_M(\Gamma) \neq \emptyset$, then Γ is M-consistent.

[2] Notice that the domains of L and $\mathbb{P}(L)$ are the same set X.

Proof. Suppose that $Mod_M(\Gamma) \neq \emptyset$ and let $v \in Mod_M(\Gamma)$. Suppose, on the contrary, that Γ is M-inconsistent. Then, $Cn_M(\Gamma) = For$, that is, $Mod_M(\Gamma) \subseteq Mod_M(\alpha)$, for all $\alpha \in For$. Thus,

$$Mod_M(\Gamma) \subseteq \bigcap_{\alpha \in For} Mod_M(\alpha).$$

Let p be propositional letter. Since $f_\neg$ satisfies (*), we have $Mod_M(p) \cap Mod_M(\neg p) = \emptyset$. Therefore,

$$Mod_M(\Gamma) \subseteq \bigcap_{\alpha \in For} Mod_M(\alpha) = \emptyset.$$

But, $v \in Mod_M(\Gamma)$ (contradiction!). □

4 Paraconsistentizing the logic L_3 of Łukasiewicz

The logic L_3 of Łukasiewicz initially proposed in [16], and extensively studied in the literature, is characterized by the matrix

$$L_3 = \langle \{0, 1/2, 1\}, \{1\}, f_\neg, f_\vee, f_\wedge, f_\to \rangle$$

such that:

i. $f_\neg(x) = 1 - x$;
ii. $f_\vee(x, y) = Max\{x, y\}$;
iii. $f_\wedge(x, y) = Min\{x, y\}$;
iv. $f_\to(x, y) = Min\{1, (1 - x + y)\}$.

These conditions give rise to the following three-valued truth-tables:

x	y	$f_\neg(y)$	$f_\vee(x,y)$	$f_\wedge(x,y)$	$f_\to(x,y)$
1	1	0	1	1	1
1	1/2	1/2	1	1/2	1/2
1	0	1	1	0	0
1/2	1		1	1/2	1
1/2	1/2		1/2	1/2	1
1/2	0		1/2	0	1/2
0	1		1	0	1
0	1/2		1/2	0	1
0	0		0	0	1

The consequence relation for L_3 is defined as follows: $\Gamma \vDash_{L_3} \alpha$ iff for all L_3-valuation v such that $v(\gamma) = 1$ for all $\gamma \in \Gamma$, we have $v(\alpha) = 1$. (Notice also that 1 is the only designated value.)

In this precise way, if we apply a $\mathbb{P}$-transformation on L_3, we obtain the following relation:

$\Gamma \vDash^{\mathbb{P}}_{L_3} \alpha$ iff there exists $\Gamma' \subseteq \Gamma$, $Mod_{L_3}(\Gamma') \neq \emptyset$, such that $\Gamma' \vDash_{L_3} \alpha$.

Notice that, by Lemma 1, we have that Γ is L_3-consistent iff $Mod_{L_3}(\Gamma) \neq \emptyset$, that is, there is a L_3-valuation v such that $v(\gamma) = 1$ for all $\gamma \in \Gamma$. We also have that $\Gamma \vDash_{L_3} \alpha$ iff $Mod_{L_3}(\Gamma) \subseteq Mod_{L_3}(\alpha)$, by definition.

The paraconsistentization of L_3 using the standard procedure was first developed in [10]. In [8], we have presented a sufficient condition to the $\mathbb{P}$-transformation of a given logic L to be a paraconsistent logic. We have to consider the following essential concepts (see [8], definition 4.1, p.246).

Let $L = \langle X, Cn_L \rangle$ be a consequence structure. We assume that X is endowed with a negation operator $\neg$.

(a) L is *explosive* iff for all $A \subseteq X$, if $x \in X$ is such that $\{x, \neg x\} \subseteq Cn_L(A)$, then A is *L-inconsistent* ($Cn_L(A) = X$). Otherwise, L is called *paraconsistent*;

(b) L satisfies *joint consistency* iff there exists $x \in X$ such that $\{x\}$ and $\{\neg x\}$ are both L-consistent but $\{x, \neg x\}$ is L-inconsistent;

(c) L satisfies *conjunctive property* iff for all $x, y \in X$, there is a $z \in X$ such that $Cn_L(\{x, y\}) = Cn_L(\{z\})$.

Given the above definitions, we have some results.

Proposition 2 *L_3 satisfies explosion, joint consistency and it has the conjunctive property.*

Proof. (a) If $\{\alpha, \neg\alpha\} \subseteq \Gamma$, we have that $v(\alpha) = 1$ iff $v(\neg\alpha) = 0$, for all L_3-valuation v. So, $Mod_{L_3}(\Gamma) = \emptyset$ and $Cn_L(\Gamma) = For$, that is, Γ is L_3-inconsistent and L_3 is explosive.
(b) In L_3, for all propositional letters p we have that $v_1(p) = 1$ and $v_2(\neg p) = 1$ for some $v_1, v_2 \in \mathcal{V}_{L_3}$, but $Mod_{L_3}(\{p, \neg p\}) = \emptyset$, that is, $\{p, \neg p\}$ is L_3-inconsistent.
(c) In L_3, $Cn_{L_3}(\{\alpha, \beta\}) = Cn_{L_3}(\{\alpha \wedge \beta\})$. □

Theorem 3 *If a consequence structure $L = \langle X, Cn_L \rangle$ is normal, explosive, satisfies joint consistency and also the conjunctive property, then $\mathbb{P}(L)$ is paraconsistent.*

Proof. See [8], theorem 4.2, p.246. □

Corolary 4 *$\mathbb{P}(L_3)$ is paraconsistent.*

Let us verify the idempotency of the $\mathbb{P}$-transformation with respect to L_3. Notice that, in L_3, we have that $Mod_{L_3}(\neg(\alpha \to \alpha)) = \emptyset$ and, then, $\{\neg(\alpha \to \alpha)\} \vDash_{L_3} \beta$, for all $\beta \in For$.

Proposition 5 *If a consequence structure $L = \langle X, Cn_L \rangle$ is normal and there is $\alpha \in For$ such that $\{\alpha\}$ is L-inconsistent, then in $\mathbb{P}(L)$ there is no $\mathbb{P}(L)$-inconsistent sets.*

Proof. See [8], proposition 3.6, p.245. □

Corolary 6 $\mathbb{P}(L_3) = \mathbb{P}(\mathbb{P}(L_3))$.

Proof. Since L_3 is a monotonic logic (that is,$A \subset Cn_{L_3}(\Gamma)$), if $\Gamma \subseteq For$ is L_3-consistent, then $Cn_{L_3}(\Gamma) = Cn^{\mathbb{P}}_{L_3}(\Gamma)$. As $Mod_{L_3}(\neg(\alpha \to \alpha)) = \emptyset$, by the proposition above, every $\Gamma \subset For$ is $\mathbb{P}(L_3)$-consistent. Therefore, $Cn^{\mathbb{P}}_{L_3}(Cn^{\mathbb{P}}_{L_3}(\Gamma)) = Cn^{\mathbb{P}}_{L_3}(\Gamma)$, for all $\Gamma \subseteq For$. □

We recall that a consequence structure $L = \langle X, Cn_L \rangle$ is *normal* iff L satisfies the following properties:

i. *Inclusion*: $A \subseteq Cn_L(A)$ for all $A \subseteq X$;
ii. *Monotonicity*: $Cn_L(A) \subseteq Cn_L(A \cup B)$, for all $A, B \subseteq X$;
iii. *Idempotency*: $Cn_L(A) = Cn_L(Cn_L(A))$, for all $A \subseteq X$.

Proposition 7 $\mathbb{P}(L_3)$ *does not satisfy inclusion.*

Proof. It is enough to see that $\{\neg(\alpha \to \alpha)\} \nvDash^{\mathbb{P}}_{L_3} \neg(\alpha \to \alpha)$. □

In L_3, the *transitivity property* holds: If $\Delta \vDash_{L_3} \alpha$ and $\Gamma \vDash_{L_3} \delta$, for all $\delta \in \Delta$, then $\Gamma \vDash_{L_3} \alpha$.

Proposition 8 $\mathbb{P}(L_3)$ *does not satisfy transitivity.*

Proof. In L_3, we have that $\{\alpha\} \vDash_{L_3} \alpha \vee \beta$. Let p, q be propositional letters. Since $\{p\}$ and $\{\neg p\}$ are L_3-consistent, it follows that $\{p\} \vDash^{\mathbb{P}}_{L_3} p \vee q$ and also $\{\neg p\} \vDash^{\mathbb{P}}_{L_3} \neg p$. Thus, if $\Delta = \{p \vee q, \neg p\}$ and $\Gamma = \{p, \neg p\}$, then it follows that $\Gamma \vDash^{\mathbb{P}}_{L_3} \delta$, for all $\delta \in \Delta$. On the other hand, if $v \in \mathcal{V}_{L_3}$ is such that $v(p \vee q) = v(\neg p) = 1$, we have that $v(p) = 0$ and then $v(q) = 1$. Therefore, $\Delta \vDash_{L_3} q$. Thus, $\Delta \vDash^{\mathbb{P}}_{L_3} q$ since that Δ is L_3-consistent. Moreover, the only L_3-consistent subsets of Γ are: $\emptyset, \{p\}$ and $\{\neg p\}$. But $\emptyset \nvDash_{L_3} q$, $\{p\} \nvDash_{L_3} q$ and $\{\neg p\} \nvDash_{L_3} q$. So, $\Gamma \nvDash^{\mathbb{P}}_{L_3} q$. □

Corolary 9 $\mathbb{P}(L_3)$ *does not satisfy idempotency.*

Proof. Consider $\Gamma = \{p, \neg p\}$ (p is a propositional letter). Then, $p \vee q, \neg p \in Cn^{\mathbb{P}}_{L_3}(\Gamma)$ and $q \in Cn^{\mathbb{P}}_{L_3}(Cn^{\mathbb{P}}_{L_3}(\Gamma))$ but $q \notin Cn^{\mathbb{P}}_{L_3}(\Gamma)$. □

Proposition 10 $\mathbb{P}(L_3)$ *is monotonic.*

Proof. In fact, if $L = \langle X, Cn_L \rangle$ is a consequence structure, then $\mathbb{P}(L)$ is monotonic. See [8], proposition 3.4, p.245. (Paraconsistentization enforces monotonicity.) □

Despite the fact that $\mathbb{P}(L_3)$ does not satisfy transitivity in its full form, a weak form of transitivity holds.

Proposition 11 (Weak transitivity) *If* $\{\alpha\} \vDash^{\mathbb{P}}_{L_3} \beta$ *and* $\{\beta\} \vDash^{\mathbb{P}}_{L_3} \gamma$*, then* $\{\alpha\} \vDash^{\mathbb{P}}_{L_3} \gamma$.

We postponed the proof and consider a previous lemma. We say that $\alpha \in For$ is a L_3-*contradiction* iff for every $v \in \mathcal{V}_{L_3}$ we have that $v(\alpha) = 0$. (We use this definition always when the matrix has a truth-value $0 \in Val$ such that $0 \notin D$.) Moreover, we recall that a formula $\alpha \in For$ is a L_3-tautology iff for every $v \in \mathcal{V}_{L_3}$ we have that $v(\alpha) = 1$. (1 is the only designated value.)

Lemma 12 *It holds that:*

i. If β *is a* L_3*-contradiction, then for all* $\Gamma \subseteq For$, $\Gamma \nvDash^{\mathbb{P}}_{L_3} \beta$*;*

ii. If β *is a* L_3*-tautology and* $\{\beta\} \vDash^{\mathbb{P}}_{L_3} \gamma$*, then* γ *is a* L_3*-tautology and for every* $\Gamma \subseteq For$, $\Gamma \vDash^{\mathbb{P}}_{L_3} \gamma$*;*

iii. If $\{\alpha\} \vDash^{\mathbb{P}}_{L_3} \beta$*, then* β *is a* L_3*-tautology or* $\{\alpha\}$ *is* L_3*-consistent and* $\{\alpha\} \vDash_{L_3} \beta$.

Proof. i. Suppose that β is a L_3-contradiction and there is a $\Gamma \subseteq For$ such that $\Gamma \vDash^{\mathbb{P}}_{L_3} \beta$. So, there is a $\Gamma' \subseteq \Gamma$ with $Mod_{L_3}(\Gamma') \neq \emptyset$ such that $\Gamma' \vDash_{L_3} \psi$. But, for $v \in Mod_{L_3}(\Gamma')$ we have $v(\beta) = 0$ (contradiction!).
ii. Suppose that β is a L_3-tautology and there is a $v \in \mathcal{V}_{L_3}$ such that $v(\gamma) \neq 1$. Since $\{\beta\} \vDash^{\mathbb{P}}_{L_3} \gamma$, we have $\emptyset \vDash_{L_3} \gamma$ or $\{\beta\} \vDash_{L_3} \gamma$. In both cases, γ has to be a L_3-tautology. Further, if γ is a L_3-tautology, then it is clear that $\Gamma \vDash^{\mathbb{P}}_{L_3} \gamma$, for all $\Gamma \subseteq For$.
iii. Immediate from i. and ii. □

Now, we prove weak transitivity.

Proof of proposition. Suppose that $\{\alpha\} \vDash^{\mathbb{P}}_{L_3} \beta$ and $\{\beta\} \vDash^{\mathbb{P}}_{L_3} \gamma$. By Lemma (i), β is not a L_3-contradiction. We have two cases:
(a) β is a L_3-tautology. In this case, since $\{\beta\} \vDash^{\mathbb{P}}_{L_3} \gamma$, by Lemma (ii), we have that $\{\alpha\} \vDash^{\mathbb{P}}_{L_3} \gamma$.
(b) ψ is not a L_3-tautology. In this case, by Lemma (iii), $\{\alpha\}$ is L_3-consistent and $\{\alpha\} \vDash_{L_3} \beta$. On the other hand, since $\{\beta\} \vDash^{\mathbb{P}}_{L_3} \gamma$ we have that γ is a L_3-tautology, and we have $\{\alpha\} \vDash^{\mathbb{P}}_{L_3} \gamma$; or $\{\beta\}$ is L_3-consistent and $\{\beta\} \vDash_{L_3} \gamma$. Since $\{\alpha\} \vDash_{L_3} \beta$ and $\{\beta\} \vDash_{L_3} \gamma$, by transitivity in L_3, we have $Mod_{L_3}(\{\alpha\}) \subseteq Mod_{L_3}(\{\beta\}) \subseteq Mod_{L_3}(\{\gamma\})$ and then $\{\alpha\} \vDash_{L_3} \gamma$. Therefore, $\{\alpha\} \vDash^{\mathbb{P}}_{L_3} \gamma$. □

It is a well known fact that the semantic deduction theorem does not hold in L_3. There is, however, the following version of the deduction theorem:

$$\Gamma \cup \{\alpha\} \vDash_{L_3} \beta \text{ iff } \Gamma \vDash_{L_3} \alpha \to (\alpha \to \beta). \qquad (*)$$

In $\mathbb{P}(L_3)$ we have only that:

Proposition 13 *If* $\Gamma \cup \{\alpha\} \vDash^{\mathbb{P}}_{L_3} \beta$*, then* $\Gamma \vDash^{\mathbb{P}}_{L_3} \alpha \to (\alpha \to \beta)$.

Proof. Suppose that $\Gamma \cup \{\alpha\} \vDash^{\mathbb{P}}_{L_3} \beta$. Then, there is $\Gamma' \subseteq \Gamma \cup \{\alpha\}$, L_3-consistent, such that $\Gamma' \vDash_{L_3} \beta$. We have then three cases:
(a) $\alpha \in \Gamma$. In this case, $\Gamma' \subseteq \Gamma$ such that $\Gamma' \cup \{\alpha\} \vDash_{L_3} \beta$ (L_3 is monotonic). Then, by (*), $\Gamma \vDash_{L_3} \alpha \to (\alpha \to \beta)$. Since Γ' is L_3-consistent, we have $\Gamma \vDash^{\mathbb{P}}_{L_3} \alpha \to (\alpha \to \beta)$.
(b) $\alpha \notin \Gamma$ and $\alpha \notin \Gamma'$. In this case, we have again that $\Gamma' \subseteq \Gamma$ and the same argument shows that $\Gamma \vDash^{\mathbb{P}}_{L_3} \alpha \to (\alpha \to \beta)$.
(c) $\alpha \notin \Gamma$ and $\alpha \in \Gamma'$. In this case, $\Gamma' - \{\alpha\} \subseteq \Gamma$ is L_3-consistent and $\Gamma' - \{\alpha\} \cup \{\alpha\} \vDash_{L_3} \beta$. Given (*), $\Gamma' - \{\alpha\} \vDash_{L_3} \alpha \to (\alpha \to \beta)$. However, $\Gamma' - \{\varphi\} \subseteq \Gamma$ is L_3-consistent, and we have $\Gamma \vDash^{\mathbb{P}}_{L_3} \alpha \to (\alpha \to \beta)$. □

In order to see that the converse of the above proposition is not valid, we consider $\alpha = \beta = \neg(p \to p)$ (p is a propositional letter). In this case, we have that

$$\Gamma \vDash^{\mathbb{P}}_{L_3} \neg(p \to p) \to ((\neg(p \to p) \to \neg(p \to p)))$$

because $\neg(p \to p) \to ((\neg(p \to p) \to \neg(p \to p)))$ is a L_3-tautology. But

$$\Gamma \cup \{\neg(p \to p)\} \nvDash^{\mathbb{P}}_{L_3} \neg(p \to p)$$

because $\neg(p \to p)$ is a L_3-contradiction.

5 Paraconsistentizing the system G_3 of K. Gödel

The logic G_3 of Gödel (originally developed in [12]) is characterized by the matrix

$$G_3 = \langle \{0, 1/2, 1\}, \{1\}, f_\neg, f_\vee, f_\wedge, f_\to \rangle$$

such that:

i. $f_\neg(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0; \end{cases}$
ii. $f_\vee(x, y) = Max\{x, y\}$;
iii. $f_\wedge(x, y) = Min\{x, y\}$;
iv. $f_\to(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y; \end{cases}$.

These conditions give rise to the following three-valued truth-tables:

x	y	$f_\neg(y)$	$f_\vee(x,y)$	$f_\wedge(x,y)$	$f_\to(x,y)$
1	1	0	1	1	1
1	1/2	0	1	1/2	1/2
1	0	1	1	0	0
1/2	1		1	1/2	1
1/2	1/2		1/2	1/2	1
1/2	0		1/2	0	0
0	1		1	0	1
0	1/2		1/2	0	1
0	0		0	0	1

The difference between G_3 and L_3 remains in the definition of $f_{\rightarrow}$ and $f_{\neg}$. In G_3, $f_{\rightarrow}(1/2, 0) = 0$, while in L_3, $f_{\rightarrow}(1/2, 0) = 1/2$. On the other hand, $f_{\neg}(1/2) = 0$ in G_3, and $f_{\neg}(1/2) = 1/2$ in L_3. The consequence relations $\vDash_{G_3}$ and $\vDash^{\mathbb{P}}_{G_3}$ are:

$$\Gamma \vDash_{G_3} \alpha \Leftrightarrow Mod_{G_3}(\Gamma) \subseteq Mod_{G_3}(\alpha),$$

and, by Lemma 1,

$$\Gamma \vDash^{\mathbb{P}}_{G_3} \alpha \Leftrightarrow \text{ there exists } \Gamma' \subseteq \Gamma, Mod_{G_3}(\Gamma') \neq \emptyset, \text{ such that } \Gamma' \vDash_{G_3} \alpha.$$

It is easy to see that G_3 satisfies explosive property, because if $\{\alpha, \neg\alpha\} \subseteq Cn_{G3}(\Gamma)$, then $Mod_{G_3}(\Gamma) \subseteq Mod_{G_3}(\{\alpha, \neg\alpha\}) = \emptyset$ and, therefore, $\Gamma \vDash_{G_3} \beta$ for all $\beta \in For$. Moreover, G_3 satisfies joint consistency and conjunctive property. Thus, we have:

Proposition 14 *$\mathbb{P}(G_3)$ is paraconsistent.*

On the other hand, we have that $\alpha \wedge \neg\alpha$ is a G_3-contradiction and then $\{\alpha \wedge \neg\alpha\}$ is G_3-inconsistent. So, by Proposition 5, we have:

Proposition 15 *$\mathbb{P}(\mathbb{P}(G_3)) = \mathbb{P}(G_3)$.*

Proposition 16 *$\mathbb{P}(G_3)$ does not satisfy inclusion.*

Proof. $\{\alpha \wedge \neg\alpha\} \nvDash^{\mathbb{P}}_{G_3} \alpha \wedge \neg\alpha$. □

Proposition 17 *$\mathbb{P}(G_3)$ does not satisfy transitivity.*

Proof. Analogous to the proof of proposition 8. □

Corolary 18 *$\mathbb{P}(G_3)$ does not satisfy idempotency.*

Let us consider the weak transitivity. In this case, the Lemma 12 remains valid for $\mathbb{P}(G_3)$. Thus, we have:

Proposition 19 *In $\mathbb{P}(G_3)$, the weak form of transitivity is valid.*

Proof. Analogous to the proof of proposition 11. □

It remains to consider the deduction theorem. It is well known that the G_3-matrix "involves all assertions of intuitionistic propositional calculus. Gödel's axiomatization coincides with the usual axiom system of the intuitionistic calculus with

$$(\neg\alpha \rightarrow \beta) \rightarrow (((\beta \rightarrow \alpha) \rightarrow \beta) \rightarrow \beta))$$

added" (see Bolc & Borowik [3], p 84). Therefore, the deduction theorem is valid in G_3.

In $\mathbb{P}(G_3)$ we have only that:

Proposition 20 *If* $\Gamma \cup \{\alpha\} \vDash^{\mathbb{P}}_{G_3} \beta$*, then* $\Gamma \vDash^{\mathbb{P}}_{G_3} \alpha \to \beta$.

Proof. Suppose that $\Gamma \cup \{\alpha\} \vDash^{\mathbb{P}}_{G_3} \beta$. Then, there is $\Gamma' \subseteq \Gamma \cup \{\alpha\}$, G_3-consistent, such that $\Gamma' \vDash_{G_3} \beta$. We have then three cases:
(a) $\alpha \in \Gamma$;
(b) $\alpha \notin \Gamma$ and $\alpha \notin \Gamma'$.
In these cases, $\Gamma' \subseteq \Gamma$ and $\Gamma' \vDash_{G_3} \beta$. By monotonicity, $\Gamma' \cup \{\alpha\} \vDash_{G_3} \beta$ and, since deduction theorem is valid in G_3, we have $\Gamma' \vDash_{G_3} \alpha \to \beta$. Thus, $\Gamma \vDash^{\mathbb{P}}_{G_3} \alpha \to \beta$.
(c) $\alpha \notin \Gamma$ and $\alpha \in \Gamma'$. In this case, let $\Gamma^* = \Gamma' - \{\alpha\}$. Since $\Gamma^* \subseteq \Gamma'$, Γ^* is G_3-consistent and $\Gamma' = \Gamma^* \cup \{\alpha\} \vDash_{G_3} \beta$. By deduction theorem for G_3, we have $\Gamma^* \vDash_{G_3} \alpha \to \beta$. Therefore, $\Gamma \vDash^{\mathbb{P}}_{G_3} \alpha \to \beta$. □

The converse of proposition above is not valid. It is enough to see that $(\alpha \wedge \neg\alpha) \to (\alpha \wedge \neg\alpha)$ is a G_3-tautology and we have $\vDash^{\mathbb{P}}_{G_3} (\alpha \wedge \neg\alpha) \to (\alpha \wedge \neg\alpha)$, but $\{\alpha \wedge \neg\alpha\} \nvDash^{\mathbb{P}}_{G_3} \alpha \wedge \neg\alpha$.

6 Paraconsistentizing the system K_3 of S. Kleene

The system K_3 of Kleene (see [15]) is very similar to L_3. There is just one difference in the definition of $f_\to$. In K_3, $f_\to(1/2, 1/2) = 1/2$. Thus, K_3 is characterized by the matrix

$$K_3 = \langle \{0, 1/2, 1\}, \{1\}, f_\neg, f_\vee, f_\wedge, f_\to \rangle$$

such that $f_\neg, f_\vee, f_\wedge$ and $f_\to$ is given by the following truth tables:

x	y	$f_\neg(y)$	$f_\vee(x,y)$	$f_\wedge(x,y)$	$f_\to(x,y)$
1	1	0	1	1	1
1	1/2	1/2	1	1/2	1/2
1	0	1	1	0	0
1/2	1		1	1/2	1
1/2	1/2		1/2	1/2	1/2
1/2	0		1/2	0	1/2
0	1		1	0	1
0	1/2		1/2	0	1
0	0		0	0	1

Let $\alpha(p_1, ...p_n)$ be a formula such that α has $p_1, ...p_n$ as propositional letters. Let $v \in \mathcal{V}_{K_3}$ be a K_3-valuation such that $V(p_1) = ... = v(p_n) = 1/2$. Then, it is easy to see that $v(\alpha) = 1/2$. We conclude that: *In* K_3*, we have neither tautologies nor contradictions.*

We recall that 1 is the unique designated value and if $\Gamma \subseteq For$, then

$$Mod_{K_3}(\Gamma) = \{v \in \mathcal{V}_{K_3} : v(\gamma) = 1 \text{ for all } \gamma \in \Gamma\}.$$

The consequence relations are defined as usual:

$$\Gamma \vDash_{K_3} \alpha \Leftrightarrow Mod_{K_3}(\Gamma) \subseteq Mod_{K_3}(\alpha),$$

and

$$\Gamma \vDash^{\mathbb{P}}_{K_3} \alpha \Leftrightarrow \text{ there exists } \Gamma' \subseteq \Gamma, K_3\text{-consistent, such that } \Gamma' \vDash_{K_3} \alpha.$$

But, by Lemma 1, in K_3 we have: $\Gamma \subseteq For$ is K_3-consistent iff $Mod_{K_3}(\Gamma) \neq \emptyset$.

Although K_3 has no contradictions, the set $\{\alpha \wedge \neg\alpha\}$ is K_3-inconsistent (there is no K_3-valuation v such that $v(\alpha \wedge \neg\alpha) = 1$), that is, $Mod_{K_3}(\{\alpha \wedge \neg\alpha\}) = \emptyset$ and $Cn_{K_3}(\{\alpha \wedge \neg\alpha\}) = For$.

Since K_3 was defined by means of a matrix, K_3 is a normal logic, that is, K_3 satisfies inclusion, monotonicity and idempotency. Moreover, K_3 satisfies explosive property (if $\{\alpha, \neg\alpha\} \subseteq Cn_{K_3}(\Gamma)$, then $Mod_{K_3}(\Gamma) = \emptyset$ and $Cn_{K_3}(\{\Gamma) = For$), joint consistency (by propositional letters) and conjunctive property ($Cn_{K_3}(\{\alpha, \beta\}) = Cn_{K_3}(\{\alpha \wedge \beta\})$).

Proposition 21 $\mathbb{P}(K_3)$ *is paraconsistent.*

Since $\{\alpha\wedge\neg\alpha\}$ is K_3-inconsistent, by Proposition 5, $\mathbb{P}(K_3)$ has no $\mathbb{P}(K_3)$-inconsistent sets, and we have:

Proposition 22 $\mathbb{P}(\mathbb{P}(K_3)) = \mathbb{P}(K_3)$.

Since K_3 has no tautology, $Mod_{K_3}(\emptyset) = \mathcal{V}_{K_3}$ and $Cn_{K_3}(\emptyset) = \emptyset$. Thus, the unique K_3-consistent subset of $\{\alpha \wedge \neg\alpha\}$ is $\emptyset$. So, $\{\alpha \wedge \neg\alpha\} \nvDash^{\mathbb{P}}_{K_3} \alpha \wedge \neg\alpha$, and we have:

Proposition 23 $\mathbb{P}(K_3)$ *does not satisfy inclusion.*

The same counter-example that shows that $\mathbb{P}(L_3)$ does not satisfy transitivity (Proposition 8) can be used in $\mathbb{P}(K_3)$. For propositional letters p and q, $\{p\vee q, \neg p\} \vDash^{\mathbb{P}}_{K_3} q$, $\{p, \neg p\} \vDash^{\mathbb{P}}_{K_3} p\vee q$, $\{p, \neg p\} \vDash^{\mathbb{P}}_{K_3} p\vee\neg p$, but $\{p, \neg p\} \nvDash^{\mathbb{P}}_{K_3} q$. Thus, we have:

Proposition 24 $\mathbb{P}(K_3)$ *does not satisfy transitivity.*

Corolary 25 $\mathbb{P}(K_3)$ *does not satisfy idempotency.*

Now, we consider the weak transitivity.

Lemma 26 *In* $\mathbb{P}(K_3)$*, if* $\{\alpha\} \vDash^{\mathbb{P}}_{K_3} \beta$*, then* $\{\alpha\}$ *is* K_3*-consistent and* $\{\alpha\} \vDash_{K_3} \beta$.

Proof. Suppose that $\{\alpha\} \vDash^{\mathbb{P}}_{K_3} \beta$. Then, there exists $\Gamma \subseteq \{\alpha\}$, K_3-consistent, such that $\Gamma \vDash_{K_3} \beta$. But Γ cannot be $\emptyset$ because $Mod_{K_3}(\emptyset) = \mathcal{V}_{K_3}$ and, in this case, $Mod_{K_3}(\beta) = \mathcal{V}_{K_3}$ and β would be a tautology. Therefore, $\{\alpha\}$ has to be K_3-consistent and $\{\alpha\} \vDash_{K_3} \beta$. □

Proposition 27 $\mathbb{P}(K_3)$ *satisfies weak transitivity.*

Proof. Suppose that $\{\alpha\} \vDash^{\mathbb{P}}_{K_3} \beta$ and $\{\beta\} \vDash^{\mathbb{P}}_{K_3} \gamma$. By Lemma, we have that $\{\alpha\}$ and $\{\beta\}$ are K_3-consistent, $\{\alpha\} \vDash_{K_3} \beta$ and $\{\beta\} \vDash_{K_3} \gamma$. By transitivity in K_3, we have $\{\alpha\} \vDash_{K_3} \gamma$. Therefore, $\{\alpha\} \vDash^{\mathbb{P}}_{K_3} \gamma$. □

It is easy to see that deduction theorem is not valid in K_3. In this system, we have $\{\alpha\} \vDash_{K_3} \alpha$ (by inclusion), but $\nvDash_{K_3} \alpha \to \alpha$. The same example, for propositional letters, shows that deduction theorem is not valid in $\mathbb{P}(K_3)$ also.

Below we present a table containing a logic and its paraconsistentized version. The reader can compare properties which the initial logic has and what is preserved(or lost) in the paraconsistentized version of it.

Summary of results

	L_3	$\mathbb{P}(L_3)$	G_3	$\mathbb{P}(G_3)$	K_3	$\mathbb{P}(K_3)$
explosive property	✓	×	✓	×	✓	×
joint consistency	✓	✓	✓	✓	✓	✓
conjunctive property	✓	×	✓	×	✓	×
paraconsistent	×	✓	×	✓	×	✓
inconsistent sets	✓	×	✓	×	✓	×
$\mathbb{P}(\mathbb{P}(L)) = \mathbb{P}(L)$	✓	✓	✓	✓	✓	✓
inclusion	✓	×	✓	×	✓	×
monotonicity	✓	✓	✓	✓	✓	✓
idempotency	✓	×	✓	×	✓	×
transitivity	✓	×	✓	×	✓	×
weak transitivity	✓	✓	✓	✓	✓	✓
modus ponens	✓	×	✓	×	✓	×
full deduction theorem	×	×	✓	×	×	×
modified full deduction theorem	✓	✓	✓	×	×	×
weak deduction theorem (⇒)	×	×	✓	✓	×	×
modified weak deduction theorem (⇒)	✓	✓	✓	✓	×	×

A tick (✓) means *yes*; a cross (×) means *no*.

7 Conclusion

The procedure used here to paraconsistentize some many-valued logics can be applied to a wide range of logics independent of the fact that they are many-valued or not. Other procedures of paraconsistentization could also be developed generating different systems.

One interesting case that we intend to investigate is what happens when the $\mathbb{P}$-transformation is applied to an already paraconsistent logic. Although it seems unuseful to apply it to a logic which is already paraconsistent, it is interesting to check whether the paraconsistent logic remains the same or not. The case of the logic of paradox (designed by Priest in [18]) seems to be of special interest as it is paraconsistent and many-valued. We let this as an open research topic which we intend to pursue in the future, given that the logic of paradox has very special and unique characteristics which make it a rather complicated case.

Gödel in his note [12] showed that there is an infinite hierarchy between Heyting's intuitionistic logic H and classical logic, as we mentioned in the introduction. This infinite hierarchy gives rise to Gödel intermediate logics. We have produced a paraconsistentization of G_3. Of course, the whole hierarchy of Gödel's logic could be paraconsistentized and a natural question would be: are these paraconsistentized intermediate logics between a paraconsistentized version of H and a paraconsistentized version of classical logic? This topic will be developed in the future, when we intend to study paraconsistentization of Heyting's intuitionistic logic.

In [8], paraconsistentization has been studied in the realm of abstract logic. In [9], it has been applied to axiomatic formal systems and valuation structures, which are still general and abstract. In this paper, specific many-valued systems were studied applying techniques and methodologies of paraconsistentization.

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