Universidade de Brasília

Instituto de Ciências Exatas

Departamento de Matemática

Asymptotics for Empirical Processes and Similarity Tests for Regenerative Sequences

by

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Por

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Dedication

Without their love, inspiration and support I would not be the person I am today. I dedicate this work to my parents Libia Hoyos and Fausto Calvache.

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Abstract

In this work we address the asymptotic behavior of regenerative sequences. For stabilized partial sum we establish convergence in Mallows distance to a Gaussian random variable. For the associated empirical process and the empirical quantile process we show the weak convergence to functionals of a mean-zero Gaussian process with continuous sample paths $\tilde{B}$, being $\tilde{B}$ a modified Brownian motion. As a by product asymptotic null distributions are derived for the classical statistics of Kolmogorov-Smirnov and Crámer-von Mises. And, applications include similarity tests of location-scale families for Harris Markov chain with atom.

Keywords: Mallows Distance; Empirical Process; Regenerative Process; Invariance Principle; Goodness-of-Fit.

Resumo

Neste trabalho abordamos o comportamento assintótico de sequências regenerativas. Para somas parciais estabilizadas mostramos a convergência em distância Mallows para uma variável aleatória Gaussiana. Para o processo empírico e o processo quantil empirico associados provamos a convergência fraca para um processo Gaussiano de média zero e com trajetórias continuas $\tilde{B}$, sendo $\tilde{B}$ uma variante da ponte Browniana. Como subproduto obtemos a distribuição assintótica nula para as estatísticas clássicas de Kolmogorov-Smirnov e Crámer-von Mises. Além disso, propomos testes de similaridade relativo a familias de escala-locação para cadeias de Markov Harris com átomo.

Palavras-chave: Distância Mallows; Processo Empírico; Processo Regenerativo; Princípio de Invariância; Qualidade de Ajuste.

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Introduction

Regenerative processes have acquired a major importance in applied probability studies. From its original formulation by Doeblin (1938), it has grown to play a central role in applied fields as varied as queueing theory, telecommunications, finance, production, inventory, biology, computer science and physics, all of which use models that sometimes rely on regenerative structures for their analysis. For regenerative sequences and their applications we refer the reader to Asmussen (2003), Haas (2002), Sigman and Wolff (1993), Smith (1955, 1958) and references therein.

The essence of regeneration is that the evolution of the process between any two successive regeneration times is an independent probabilistic replica of the process in any other "cycle". Thus, under mild regularity conditions, the time-average limits, the existence of a limiting distribution and others basic results about the asymptotic behavior are well-defined for a regenerative process. More specifically, we say that a stochastic process $\{X_n\}_{n\geq 0}$ is regenerative if there exists a sequence of random times $T_0 < T_1 < T_2 < ...$ at which the process can be split into i.i.d. "cycles"

$$\eta_0 = \{X_n, 0 \leq n < T_1\}, \eta_1 = \{X_n, T_1 \leq n < T_2\}, \ \eta_2 = \{X_n, T_2 \leq n < T_3\} ....$$

Irreducible, aperiodic and positive recurrent Markov chains with countable state space constitute a basic example of a regenerative process with $\{T_n\}_{n\geq 0}$ being the times of successive returns to a given state. Chains with general state space also exhibit regenerative structures when Harris recurrent chains with atom are considered. In the Markov chain setting, regenerative analysis has simplified many complicated analytical arguments associated with the limit theory of such processes. Significant results that detail the connection between regeneration and Markov chains can be found in the works by Athreya and Ney (1978) and Nummelin (1978). For a systematic study of the splitting technique and regeneration phenomena in the theory of Harris Markov processes, the reader is refered to two excellent books written by Nummelin (1984) and Meyn and Tweedie (1993).

Aiming at Goodness-of-Fit type statistics for Markov chains with general state space and that possess limiting distribution with a continuous and strictly positive density function we first develop some asymptotic results for regenerative processes. A key element to be considered is the concept of the associated canonical(or occupational) probability measure $\tilde{\pi}$. As pointed out in Athreya and Lahiri (2006) the expected time that the regenerative sequence $\{X_n\}_{n\geq 0}$ spends in A over the expected inter-regeneration time provides,

$$\tilde{\pi}(A) = \frac{1}{\mu_T} E\left\{\sum_{j=T_1}^{T_2-1} I_A(X_j)\right\}, \quad \mu_T = E\left\{T_2 - T_1\right\}. \tag{1}$$

In fact, if φ is a measurable function then, under mild conditions, the strong law of large numbers (SLLN) holds and the "time average" $\sum_{j=0}^{n} \varphi(X_j)/n$ converges almost surely (a.s.) to the "space average" $\int \varphi d\tilde{\pi}$. Indeed, the canonical measure determines the limiting distribution of the process. And, from the Markov chains point of view, the Kac's Theorem will allow us to identify the canonical measure $\tilde{\pi}$ with the limiting measure of Harris recurrent chain that possesses an atom (see Bertail and Clémençon (2006) or Meyn and Tweedie (1993)). Mixing conditions and geometrically ergodic chains will also play a role in this matter (Dehling et al. (2009) and Shao and Yu (1996)).

Thus, to study the asymptotic properties of the process one needs to analyse the partial sum $S_n = \sum_{j=0}^{n} \varphi(X_n)$. Our approach will rely on the dissection formula used by Chung (1967) to achieve Central Limit Theorem (CLT) for aperiodic, irreducible and positive recurrent Markov chains,

$$S_n = A_n + \sum_{k=1}^{N_n-1} Y_k + B_n, \tag{2}$$

where N_n is conveniently chosen,

$$A_n = \sum_{j=0}^{T_1-1} \varphi(X_j), \;\; Y_k = \sum_{j=T_k}^{T_{k+1}-1} \varphi(X_j) \;\; and \;\; B_n = \sum_{T_{N_n}}^{n} \varphi(X_j).$$

Based on the dissection formula we prove a version of CLT for aperiodic and positive recurrent regenerative sequence (Theorem 2.3.2). As compared to similar results such as CLT from Glynn and Whitt (1993) our hypotheses are somehow weaker. Also, in Chapter 2 induced by the successfull use of Mallows distance to derive CLT type results for stable laws (see, e.g., Johnson and Samworth (2005) or Dorea and Oliveira (2014)) as well as to characterize domains of attraction for extreme values (Mousavinasr et al. (2020)) we will introduce Mallows distance in our work.

Mallows distance $d_r(F, G)$ measures the discrepancy between two distribution functions F and G. For $r > 0$ define

$$d_r(F, G) = \inf_{(X,Y)} \{E(|X - Y|)^r\}^{1/r}, \;\; X \overset{d}{=} F, Y \overset{d}{=} G.$$

where the infimum is taken over all random vectors (X, Y) with marginal distributions F and G ($\overset{d}{=}$: equality in distribution). Convergence in Mallows distance is closely related to convergence in distribution ($\xrightarrow{d}$). From Bickel and Freedman (1981) : for distributions with finite r-th moments and for $r \geq 1$,

$$d_r(F_n, G) \to 0 \Longleftrightarrow F_n \xrightarrow{d} G \;\; and \;\; \int |x|^r dF_n(x) \to \int |x|^r dG(x).$$

Via Mallows distance, we will present several variants of the CLT for regenerative sequences. Some related results concerning strong approximation and their rates of convergence are also included in the last section of Chapter 2.

Next, consider the canonical distribution function

$$\tilde{F}(x) = \frac{1}{\mu_T} E\left(\sum_{j=T_1}^{T_2-1} I_{(-\infty,x]}(X_j)\right) \tag{3}$$

and the associated empirical process and the empirical quantile process

$$\beta_n(x) = \sqrt{n}(F_n(x) - \tilde{F}(x)),\ x \in \mathbb{R}, \tag{4}$$

$$q_n(t) = \sqrt{n}(F_n^{-1}(t) - \tilde{F}^{-1}(t)),\ t \in (0,1). \tag{5}$$

Where F_n is the empirical distribution function

$$F_n(x) = \frac{1}{n}\sum_{j=1}^{n} I_{(-\infty,x]}(X_j) \quad, x \in \mathbb{R},\ n \geq 1,$$

F_n^{-1} and $\tilde{F}^{-1}$ are the generalized inverse of F_n and $\tilde{F}$, respectively.
Note that from the above results we have for $r \geq 1$,

$$d_r\left(\frac{\sqrt{n}(F_n(x) - \sqrt{n}\mu)}{\sigma}, Z\right) \to 0$$

where the constants μ and σ are conveniently chosen. Interpret above as the Mallows distance between the corresponding distributions with Z having $N(0,1)$ distribution. For the i.i.d. case Donsker's Theorem (cf. Billingsley (1968)) states that the empirical process β_n converges weakly ($\Rightarrow$) to a Brownian bridge process B. The dependent case is far more complex, see, for example, the works of Berkes and Philipp (1977, 1978), Doukhan et al. (1995), Borovkova et al. (2001), Dedecker and Prieurd (2007), Shao and Yu (1996) and Dehling et al. (2009).

In our case, under regularity conditions, we will show that the empirical process β_n converges weakly to a zero-mean and continuous sample paths Gaussian process $\tilde{B}_{\tilde{F}}$ with covariance function given by

$$\begin{aligned} E(\tilde{B}_{\tilde{F}}(x), \tilde{B}_{\tilde{F}}(y)) &= \tilde{F}(x \wedge y) - \tilde{F}(x)\tilde{F}(y) \\ &+ \sum_{j=1}^{\infty} E\left\{I_{(-\infty,x]}(X_0) - \tilde{F}(x), I_{(-\infty,y]}(X_j) - \tilde{F}(y)\right\} \\ &+ \sum_{j=1}^{\infty} E\left\{I_{(-\infty,y]}(X_0) - \tilde{F}(y), I_{(-\infty,x]}(X_j) - \tilde{F}(x)\right\}. \end{aligned} \tag{6}$$

Unlikely as in the i.i.d. case, the well-known Delta Method cannot be used directly to show the weak convergence of the empirical quantile process q_n. Different set of arguments such as the Skorokhod Theorem and properties of locally uniformly aproximation of monotone functions were needed to establish the desired convergence

$$q_n(\cdot) \Rightarrow -\frac{\tilde{B}(\cdot)}{\tilde{f}(\tilde{F}^{-1}(\cdot))}.$$

As by product of these weak convergences, for the statistics

$$D_n = \sqrt{n}\left\|F_n - \tilde{F}\right\|_\infty \qquad \text{(Kolmogorov-Smirnov)}$$

$$W_n^2 = n\int_{-\infty}^{\infty}(F_n(x) - \tilde{F}(x))^2 d\tilde{F}(x) \qquad \text{(Cramér-von Mises)}$$

we obtain the asymptotic null distributions

$$D_n \xrightarrow{d} \left\|\tilde{B}_{\tilde{F}}\right\|_\infty \quad \textit{and} \quad W_n^2 \xrightarrow{d} \int_0^1 \tilde{B}_{\tilde{F}}(t)^2 dt.$$

On the other hand, del Barrio et al. (1990,2000) proposed a set of similarity tests of location-scale families based on the empirical distribution and the 2nd-order Mallows distance. We extend its use for our regenerative settings by considering the statistics

$$\sqrt{n}d_2(F_n, \tilde{F}) = \left(n\int_0^1 (F_n^{-1}(t) - \tilde{F}^{-1}(t))^2 dt\right)^{1/2}$$

and

$$R_n = 1 - \frac{\left(\int_0^1 F_n^{-1}(t)G^{-1}(t)dt\right)^2}{\hat{\sigma}_n^2}.$$

The latter tests whether $\tilde{F} \in \mathcal{G}_G$ and G is a standard member of the location-scale family $\mathcal{G}_G$.

We will provide conditions that guarantee the convergences

$$\sqrt{n}d_2(F_n, \tilde{F}) \xrightarrow{d} \left(\int_0^1 \frac{\tilde{B}_{\tilde{F}}^2(t)}{\tilde{f}^2(\tilde{F}^{-1}(t))}dt\right)^{1/2} \tag{7}$$

and the convergence in distribution of the statistics nR_n to

$$\int_0^1 \frac{\tilde{B}_{\tilde{F}}^2(t)}{\tilde{f}^2(\tilde{F}^{-1}(t))}dt - \left(\int_0^1 \frac{\tilde{B}_{\tilde{F}}(t)}{\tilde{f}(\tilde{F}^{-1}(t))}dt\right)^2 - \left(\int_0^1 \frac{\tilde{B}_{\tilde{F}}(t)\tilde{F}^{-1}(t)}{\tilde{f}(\tilde{F}^{-1}(t))}dt\right)^2. \tag{8}$$

After this brief description of our objectives, motivations and tools used, we now detail how this work is organized. A better characterization of each chapter will be provided in the introduction of each one.

In Chapter 1, we present preliminary concepts and results that are fundamental for the understanding of the subsequent chapters. It includes some details on Markov chains, renewal theory, Mallows distance, moment inequalities, uniform integrability, empirical processes and weak convergence.

In Chapter 2 we will focus on the convergence of the partial sum $S_n = \sum_{j=1}^{n} \varphi(X_j)$ to a Gaussian random variable. First, we present some basic concepts concerning regenerative processes and explore the role of the canonical measure $\tilde{\pi}$. Illustrative examples and results such as the existence of a limiting distribution, conditions for SLLN to hold as well as Glivenko-Cantelli type theorem are gathered in Section 2.2. Our Theorem 2.3.2 provides a variant of the CLT for regenerative sequences and in 2.3.1 our hypotheses are compared to known conditions for CLT to hold. Theorem 2.4.6, under r-th moment conditions on blocks n_j's, we obtain convergence in Mallows distance and moments convergence for $\dfrac{S_n - a_n}{b_n}$ to a standard normal random variable. In Section 2.5 we discuss the approximation of the partial sum S_n by a Brownian motion with rate of convergence $O(\log n)$.

In Chapter 3 we establish the weak convergence in the Skorokhod space D for the empirical and empirical quantile processes. Basic assumptions include aperiodicity and positive recurrency of the regenerative sequence. Theorem 2.3.2 and $q_n(t) \xrightarrow{d} -\dfrac{\tilde{B}(t)}{\tilde{f}(\tilde{F}^{-1}(t))}$ lead to the convergence of finite dimensional distributions of the process $\beta_n(\cdot)$ and $q_n(\cdot)$. Our Theorem 3.3.5 shows that the empirical process $\beta_n(x)$ converges weakly to the zero-mean Gaussian process $\tilde{B}_{\tilde{F}}$. For its proof Shao and Yu's tightness criterion (1996) and α-mixing properties of the sequence $\{X_n\}_{n\geq 0}$ are used. Our Theorems 3.4.4 and 3.4.5 establish the weak convergence of the uniform quantile process and of the process $q_n(\cdot)$, respectively. For its proof our approach makes use of the Skorokhod's Representation Theorem and properties of locally uniformly aproximation of monotone functions.

Finally, in Chapter 4, we study the asymptotic null distribution for statistics associated to a regenerative sample. In Section 4.3, our Lemma 4.3.2 provides sufficient conditions to obtain the asymptotic null distribution for the classic statistics of Kolmogorov-Smirnov D_n and Cramér-von Mises W_n^2 . In Section 4.4 we use the 2nd-order Mallows distance between the empirical distribution and the canonical measure $\tilde{F}$ to study the statistics $\sqrt{n}d_2(F_n, \tilde{F})$ and R_n defined by (4.2) and (4.4), respectively. The Lemma 4.4.2 provides sufficient conditions to obtain the convergence (4.3) and Lemma 4.4.3 establishes the limiting distribution of the statistics nR_n under the null hypothesis that the canonical measure $\tilde{F}$ belongs to the tested location-scale family. The results derived in this chapter are directly related to the weak convergence of the empirical and quantile process associated to X_n. Since any Harris chains $\{X_n\}_{n\geq 1}$ on a general state space that possess an atom A is a regenerative process with limiting distribution F_{lim}, by Kac's Theorem we have $F_{lim} = \tilde{F}$ where $\tilde{F}$ is the canonical distribution given by

$$\tilde{F}(x) = \frac{1}{E_A(T_A)} E_A \left\{ \sum_{j=0}^{T_A - 1} I_{(-\infty,x]}(X_j) \right\}, \, x \in \mathbb{R},$$

where $T_A = \inf\{n \geq 1, X_n \in A\}$ the hitting time on A. So, our invariance principle is valid for Harris Markov chains and then we can use the statistics described above to test $H_0 : \tilde{F} = F_0$ or $\tilde{F} \in \mathcal{G}_G$. On the other hand, in Subsection 3.2.1, we established that the empirical process associated with a $\mathcal{L}$-geometrically ergodic Markov chain $\{X_n\}_{n\geq 0}$ under some assumptions on the Markov transition function satisfies the invariance principle of Theorem 3.2.2. Thus, as stated in 4.3.1 the proposed statistics are applicable to a class of Markov chains that includes $\mathcal{L}$-geometrically ergodic chains and positive Harris recurrent chains with an atom.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter we gather the necessary concepts and known results to be used in the subsequent chapters. As basic references we refer the reader to Chung (1967) and Meyn and Tweedie (1993) for Markov chains, Serfozo (2009) and Athreya and Lahiri (2006) for Renewal Theory, Mallows (1972), Bickel and Friedman (1981) and Dorea and Ferreira (2012) for Mallows distance, Shorack and Wellner (1986) and Csörgő and Révész (1981) for empirical processes and Billinsgley (1968) for weak convergence.

1.2 Some Notation and Terminology

i.i.d.	:	Independent and identically distributed
CLT	:	Central limit theorem
SLLN	:	Strong law of large numbers
$\xrightarrow{d}$	:	Convergence in distribution
$\stackrel{d}{=}$	:	Equality in distribution
$\xrightarrow{p}$	:	Convergence in probability
a.s.	:	Almost surely, with probability 1

$\xrightarrow{a.s.}$	:	Almost sure convergence
$\Rightarrow$	:	Weak Convergence
$d_r(F,G)$	:	Mallows distance of r-th order
$a \wedge b$	:	Minimum of a and b
$\lfloor a \rfloor$	:	the integer part of a, i.e.,$\lfloor a \rfloor = k$ if $k \leq a < k+1$
$a_n = o(b_n)$	:	$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$
$b_n = O(b_n)$	:	$\limsup_{n\to\infty} \left\lvert \frac{a_n}{b_n} \right\rvert < \infty$
$N(\mu, \sigma^2)$	:	Normal distribution with mean μ and variance σ^2
$C[0,1]$	:	Space of continuous real-valued functions on $[0,1]$
$D[0,1]$	:	Space of functions on $[0,1]$ that are right-continuous and have left-hand limits.
$\sigma(X)$	:	Sigma algebra generated by X.
$E(Y\vert\mathcal{F})$	:	Conditional expectation of Y given $\mathcal{F}$
$P(A\vert\mathcal{F})$	:	Probability of A given $\mathcal{F}$
$I_A(\cdot)$ or $I(A)$	:	The indicator function of a set A

1.3 Markov Chains

Classical Markov chains possess a denumerable state space S and a transition probability matrix $\mathbf{P} = ((P_{ij}))_{i\in S, j\in S}$. For any set $A \subset S$, the first hitting (passage or visit or return) time to the set A of a chain $\{X_n\}_{n\geq 0}$ is defined by

$$T_1(A) = \inf\{n : X_n \in A, n \geq 1\}.$$

For a fixed state $i \in S$ the r-th visit to state i is given by

$$T_r(i) = \inf\{n : n > T_{r-1}(i), X_n = i\}, \ r \geq 1$$

and

$$0 = T_0(i) < T_1(i) < T_2(i) < \cdots T_r(i) < \cdots.$$

The state i is said to be recurrent if

$$P(T_1(i) < \infty | X_0 = i) = 1 \ \text{ or } \ P(X_n = i \text{ for some } 1 \leq n < \infty | X_0 = i) = 1.$$

and is positive recurrent if $E\{T_2(i) - T_1(i)\} < \infty$. The chain $\{X_n\}_{n\geq 0}$. is said to be irreducible if

$$P(T_1(j) < \infty | X_0 = i) > 0 \text{ or } P(X_n = j \text{ for some } 1 \leq n < \infty | X_0 = i) > 0 \ \forall i \ \forall j.$$

The following result states that we can break the time evolution of a Markov chain into i.i.d. cycles.

Theorem 1.3.1. *Let $\eta_r = \{X_j, T_r(i) \leq j < T_{r+1}(i); T_{r+1}(i) - T_r(i)\}$ for $r = 0, 1, 2, ...$ Let i be a positive recurrent state. Given $X_0 = i$, the sequence $\{\eta_r\}_{r\geq 0}$ are i.i.d. as random vectors with a random number of components. More precisely, for any $k \in \mathbb{N}$,*

$$\begin{aligned} P_i & \left(\eta_r = (x_{r_0}, x_{r_1}, ..., x_{r_{j_r}}), T_{r+1}(i) - T_r(i) = j_r, r = 0, 1, ..., k\right) \\ = & \prod_{r=0}^{k} P_i\left(\eta_1 = (x_{r_0}, x_{r_1}, ..., x_{r_{j_r}}), T_1(i) = j_r\right) \end{aligned}$$

for any $x_{r_0}, x_{r_1}, ..., x_{r_{j_r}}$, $r = 0, 1, ..., k$.

In the regenerative context, the visit times $T_0(i) < T_1(i) < T_2(i) < \cdots T_r(i)$ are the regeneration times and the η_r's are the cycles or excursions.

We will be interested in Markov chains with general state space. Let $(S, \mathcal{G})$ be a measurable space and let $\{X_n\}_{n\geq 0}$ be a stochastic process taking values on S and equipped with a transition probability kernel

$$P = \{P(x, A) : x \in S, A \in\}.$$

Where $P(x, \cdot)$ is a probability measure on $(S, \mathcal{G})$ for all $x \in S$, $P(\cdot, A)$ is an $\mathcal{G}$-measurable function for all $A \in \mathcal{G}$ and P satisfies

$$P((X_{n+1} \in A) | X_0, X_1, ..., X_n) = P((X_{n+1} \in A) | X_n) \ a.s. \ for \ all \ n \geq 0$$

and for any initial distribution of X_0. It follows that for $A_0, A_1, ..., A_n \in \mathcal{G}$ and any initial $\mu_0(A) = P(X_0 \in A_0)$ we can write

$$P(X_0 \in A_0, X_1 \in A_1, ..., X_n \in A_n) = \int_{A_0} \mu_0(dx_0) \int_{A_1} P(x_0, dx_1) \cdots \int_{A_n} P(x_{n-1}, dx_n).$$

The concepts of irreducibility, recurrence or aperiodicity can all be carried out to general state space by making use of an auxiliary measure ϕ. In the case of discrete space S, the measure ϕ is just the counting measure on S. The following notation will be used : $P_x(\cdot)$ for the probability of chain started at x; and $P_\mu(\cdot)$ for the chain with initial distribution μ.

Definition 1.3.1. *Let ϕ be a non-zero σ-finite measure on $(S, \mathcal{G})$.*

(i) The Markov chain $\{X_n\}_{n\geq 0}$ (or equivalently, its transition function $P(\cdot,\cdot)$) is said to be ϕ-irreducible (or irreducible in the sense of Harris with respect to measure ϕ) if for any $A \in \mathcal{G}$ and all $x \in S$ we have

$$\phi(A) > 0 \Rightarrow P_x(T_1(A) < \infty) > 0.$$

(ii) The Markov chain $\{X_n\}_{n\geq 0}$ that is Harris irreducible with respect to ϕ is said to be Harris recurrent if for all $x \in S$ we have

$$A \in \mathcal{G},\ \phi(A) > 0 \Rightarrow P_x(T_1(A) < \infty) = 1.$$

(iii) The set $A \in \mathcal{G}$ is an atom if there exists a probability measure ν such that $P(x, B) = \nu(B)$, $x \in A$ and $A \in \mathcal{G}$. The set A is an accessible atom for a $\phi-$irreducible Markov chain if $\phi(A) >$ 0 and for all $x \in S$ and $y \in S$ we have $P(x, \cdot) = P(y, \cdot)$.

Remark 1.3.1. *If a chain has an accessible atom then the times at which the chain enters the atom are regeneration times.*

For $A \in \mathcal{G}$ define the successive return times to A by

$$T_k(A) = \inf \{n : n \geq T_{k-1}(A), X_n \in A\}\,,\ k \geq 2.$$

When the chain is Harris recurrent then, for any initial distribution, the probability of returning infinitely often to the atom A is equal to one. By the strong Markov property it follows that, for any initial distribution μ, the sample paths of the chain can be divided into i.i.d. blocks of random length corresponding to consecutive visits to A. The cycles can be defined by

$$\eta_1 = (X_{T_1(A)}, X_{T_1(A)+1}, ..., X_{T_2(A)-1}), ..., \eta_k = (X_{T_k(A)}, X_{T_k(A)+1}, ..., X_{T_{k+1}(A)-1}).$$

The previous remark is a consequence of the following result.

Theorem 1.3.2 (Athreya and Lahiri (2006); Theorem 14.2.9)**.** *Let $\{X_n\}_{n\geq 0}$ be a Harris Markov chain with transition function $P(\cdot,\cdot)$ and state space $(S,\mathcal{G})$, where $\mathcal{G}$ is countably generated and. Then there exists a set $A_0 \in \mathcal{G}$, a constant $0 < \alpha < 1$ and a probability measure $\nu(\cdot)$ on $(S,\mathcal{G})$ such that for all $x \in A_0$,*

$$P(x,A) \geq \alpha\nu(A), \;\; \forall A \in \mathcal{G}, \tag{1.1}$$

and for all $x \in S$,

$$P_x(T_1(A_0) < \infty) = 1.$$

Besides, for any initial distribution μ, there exists a sequence of random times $\{T_i\}_{i\geq 1}$ such that under P_μ, the sequence of excursions $\eta_j \equiv \left\{X_{T_{j+r}}, 0 \leq r < T_{j+1} - T_j, T_{j+1} - T_j\right\}_{j\geq 1}$ are i.i.d. with $X_{T_j} \overset{d}{=} \nu(\cdot)$.

1.4 Renewal Processes

The results that we will present in this section are important tools for characterizing the limiting behavior of probabilities and expectations of regenerative processes. Basic references are Athreya and Lahiri (2006) and Serfozo (2009).

Suppose $0 = T_0 < T_1 < T_2 < ....$ are finite random times at which a certain event occurs. The number of the times T_n in the interval $(0,t]$ is given by

$$N(t) = \sum_{n=1}^{\infty} I_{\{T_n \leq t\}} \;\; t \geq 0, \;\; N(0) \equiv 0.$$

Definition 1.4.1. *A point process $N(t)$ is a renewal process if the inter-occurrence times $\tau_n = T_n - T_{n-1}$, for $n \geq 1$, are independent with a common distribution F and $\tau_0 = 0$. The T_n's are called renewal times, referring to the independent or renewed stochastic information* *at these times. The τ_n are the inter-renewal times, and $N(t)$ is the number of renewals in* $(0, t]$.

Note that

$$T_n = \tau_1 + \tau_2 + \cdots + \tau_n, \ n \geq 1.$$

Also note that for each $t \geq 0$ and $n = 0, 1, 2, ...$

$$\{N(t) = n\} = \{T_n \leq t, T_{n+1} > t\} = \{T_n \leq t < T_{n+1}.\} \tag{1.2}$$

These equations state, loosely speaking, that $t \to N(t)$ is the inverse function of $n \to T_n$, and suggest that classical results on $\{T_n\}_{n\geq 0}$ could be converted to results on $\{N(t)\}_{t\geq 0}$.

Theorem 1.4.1 (Renewal Theorem)**.** *Let $\mu_T = E\{T_2 - T_1\}$ be the mean of the inter-renewal distribution. Then*

$$\lim_{t\to\infty} \frac{N(t)}{t} \overset{a.s.}{=} \frac{1}{\mu_T}. \tag{1.3}$$

$$\lim_{t\to\infty} \frac{E\{N(t)\}}{t} = \frac{1}{\mu_T}. \tag{1.4}$$

We are interest in discrete renewal process. So, let $\{\tau_j\}_{j\geq 0}$ be independent positive integer valued random variables such that $\{\tau_j\}_{j\geq 1}$ are i.i.d. with distribution $\{p_j\}_{j\geq 1}$. Let $T_0 = 0$, $T_n = \sum_{j=0}^{n} \tau_j$, $n \geq 0$ and

$$u_n = P(\text{there is a renewal at time } n) = P(T_k = n \text{ for some } k \geq 0).$$

Theorem 1.4.2. *[Lindvall (1992); Theorem 1.4.2] Let g.c.d.$\{k : p_k > 0\} = 1$ and $\mu = \sum_{j=1}^{\infty} jp_j \in (0, \infty)$. Then*

i) $u_n \longrightarrow \dfrac{1}{\mu}$ *as* $n \to \infty$.

ii) If $0 < \sum_{j=1}^{\infty} j^k p_j < \infty$ *some* $k > 1$ *then* $|u_n - \mu^{-1}| = o(n^{-(k-1)})$.

Consider the discrete renewal equation

$$a_n = b_n + \sum_{j=1}^{n} a_{n-j} p_j, \ \ n = 0, 1, 2, ... \tag{1.5}$$

In the general case, it can be shown that the unique solution to (1.5) is given by

$$a_n = \sum_{j=0}^{n} b_{n-j} u_j.$$

Theorem 1.4.3 (Discrete Renewal Equation). *Let $\{b_j\}_{j\geq 0}$ be a such that $\sum_{j=1}^{\infty}|b_j| < \infty$. Let $\{a_n\}_{n\geq 0}$ with $a_0 = b_0$ and*

$$a_n = b_n + \sum_{j=1}^{\infty} a_{n-j}p_j, \ \ n \geq 1.$$

If $0 < \mu = \sum_{j=1}^{\infty} jp_j < \infty$ and assume the g.c.d. $\{k : p_k > 0\} = 1$. Then

$$a_n = \sum_{j=0}^{\infty} b_j u_{n-j}, \ n \geq 0 \ \text{and} \ \lim_{n\to\infty} a_n = \frac{1}{\mu}\sum_{j=0}^{\infty} b_j.$$

For a renewal process N_n, the following processes provide more information about renewal times.

Definition 1.4.2. *i) $A_n = t - T_{N_n}$, the backward recurrence time at n (or the age), which is the time since the last renewal prior to n.*

ii) $B_n = T_{N_n+1} - n$, the forward recurrence time at n (or the residual renewal time), which is the time to the next renewal after n.

Then

$$\lim_{n\to\infty} P\{A_n \leq k\} = \lim_{n\to\infty} P\{B_n \leq k\} = \frac{1}{\mu_T}\sum_{j=0}^{k} P(\tau_1 > j) \tag{1.6}$$

(cf. Example 48 - Chapter 2 from Serfozo (2009)).

1.5 Mallows distance

The Mallows distance (1972) between two distributions functions F and G generalizes the "Wasserstein distance" appeared for the first time in 1970 (case r = 1). Thus, in the literature, the name distance of Wasserstein has also been used instead of Mallows.

Definition 1.5.1. *For $r > 0$, the Mallows r-distance between distributions F and G is given by*

$$d_r(F,G) = \inf_{(X,Y)} \{E(|X-Y|)^r\}^{1/r}, \ X \overset{d}{=} F, Y \overset{d}{=} G. \tag{1.7}$$

where the infimum is taken over all random vectors (X,Y) with marginal distributions F and G, respectively.

For $r \geq 1$ the Mallows distance represents a metric on the space of distribution functions

$$L_r = \left\{ F : \int_{\mathbb{R}} |x|^r dF(x) < \infty \right\}.$$

The following metric relationships are valid

$$d_r(F,G) \leq d_r(F,F_0) + d_r(F_0,G), \tag{1.8}$$

where F_0 is a distribution function.

There is a close connection between convergence in Mallows distance convergence and the convergence in distribution.

Theorem 1.5.1 (Bickel and Freedman (1981))**.** *For $r \geq 1$ and for distributions $G \in L_r$ and $\{F_n\}_{n\geq 1} \subset L_r$ we have, as $n \to \infty$*

$$d_r(F_n,G) \to 0 \Longleftrightarrow F_n \xrightarrow{d} G \text{ and } \int |x|^r dF_n(x) \to \int |x|^r dG(x). \tag{1.9}$$

Theorem 1.5.2 (Dorea and Ferreira (2012))**.** *Let $r \geq 1$, $X^* \stackrel{d}{=} F$, $Y^* \stackrel{d}{=} G$ and $(X^*, Y^*) \stackrel{d}{=} H$, where $H(x,y) = F(x) \wedge G(y) = \min\{F(x), G(y)\}$. Then the following representation holds*

$$\begin{aligned} d_r^r(F,G) &= E\left\{|F^{-1}(U) - G^{-1}(U)|^r\right\} = \int_0^1 |F^{-1}(u) - G^{-1}(u)|^r du \\ &= E\left\{|X^* - Y^*|^r\right\} = \int_{\mathbb{R}^2} |x-y|^r dH(x,y) \end{aligned}$$

where U is uniformly distributed on the interval $(0,1)$ and $0 < u < 1$.

Theorem 1.5.3 (Johnson and Samworth (2005))**.** *Let $X, X_1, X_2, ...,$ i.i.d. random variables. Assume $var(X) > 0$ and for some $r \geq 2$ we have $d_r(X,Z) < \infty$ where Z has normal distribution with mean 0. Then as $n \longrightarrow \infty$*

$$d_r\left(\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{nvar(X)}}, Z_0\right) \longrightarrow 0, \tag{1.10}$$

where $Z_0 \stackrel{d}{=} N(0,1)$.

1.6 Moment Inequalities, Mixing and Uniform Integrability

We gather below some known moment inequalities. They can be found in the books of Billinsley (1968), Gut (2005), Hall and Heyde (1960). For easier referencing purpose we have stated the inequalities as Lemmas and Theorems.

Lemma 1.6.1. *a) Let $Y_1, Y_2, ...Y_n$ random variables. Then*

$$E\left\{\left|\sum_{i=1}^{n} Y_i\right|^p\right\} \leq \sum_{i=1}^{n} E\left\{|Y_i|^p\right\} \ if\ 0 < p \leq 1. \tag{1.11}$$

$$E\left\{\left|\sum_{i=1}^{n} Y_i\right|^p\right\} \leq n^{p-1}\sum_{i=1}^{n} E\left\{|Y_i|^p\right\} \ if\ p \geq 1. \tag{1.12}$$

b) Let $\{\sum_{i=1}^{n} Y_i, \mathcal{F}_n\}_{n\geq 1}$ be a martingale. Then there exists a constant c_p such that

$$E\left\{\left|\sum_{i=1}^{n} Y_i\right|^p\right\} \leq c_p E\left\{\left(\sum_{i=1}^{n} Y_i^2\right)^{p/2}\right\} \ if\ p > 1. \tag{1.13}$$

If $p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$ we have

$$\left(E\left\{\left|\sum_{i=1}^{n} Y_i\right|^p\right\}\right)^{1/p} \leq \left(E\left\{\max_{1\leq m\leq n}\left|\sum_{i=1}^{m} Y_i\right|^p\right\}\right)^{1/p} \leq q\left(E\left\{\left|\sum_{i=1}^{n} Y_i\right|^p\right\}\right)^{1/p}. \tag{1.14}$$

Rosenthal (1970) proved the following inequality which is a extension of the classical convexity inequity:

$$E\left\{\left|\sum_{i=1}^{n} Y_i\right|^p\right\} \leq \sigma^p n^{p/2} \ if\ 1 \leq p \leq 2.$$

Lemma 1.6.2 (Rosenthal(1970))**.** *Let $Y_1, Y_2, ...Y_n$ i.i.d. random variables with $E(Y_i) = 0$, $\sigma^2 = E(Y_i^2)$, then exists a constant c_p such that*

$$E\left\{\left|\sum_{i=1}^{n} Y_i\right|^p\right\} \leq c_p\left\{\sigma^p n^{p/2} + E|Y_1|^p n\right\} \ if\ p > 2. \tag{1.15}$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_1$ and $\mathcal{F}_2$ be two σ-algebras contained in $\mathcal{F}$. Define the following measures of dependence between $\mathcal{F}_1$ and $\mathcal{F}_2$:

$$\alpha(\mathcal{F}_1, \mathcal{F}_2) = \sup_{A\in\mathcal{F}_1, B\in\mathcal{F}_2} |P(A\cap B) - P(A)P(B)|.$$

Let $\{X_n\}_{n\geq 1}$ be a sequence of real-valued random variables on $(\Omega, \mathcal{F}, P)$ and let $\mathcal{F}_n^m = \sigma(X_i, n \leq i \leq m)$ be σ-algebras generated by the indicated random variables and put

$$\alpha(n) = \sup_{k\geq 1} \alpha(\mathcal{F}_1^k, \mathcal{F}_{n+k}^{\infty}).$$

Definition 1.6.1. *The sequence $\{X_n\}_{n\geq 1}$ is said to be $\alpha-$mixing (or strong mixing), if*

$$\alpha(n) \to \infty \text{ as } n \to \infty.$$

The following result is a Rosenthal-type inequality for $\alpha-$mixing.

Theorem 1.6.3. *[Shao and Yu (1996), Theorem 4.1] Let $2 < p < r \leq \infty$, $2 < v \leq r$ and $\{X_n\}$ be and α-mixing sequence of random variables with $E\{X_n\} = 0$ and $\|X_n\|_r := (E|X_n|^r)^{1/r} < \infty$. Assume that*

$$\alpha(n) = O(n^{-\theta}), \quad for \ \ some \ \ \theta > 0.$$

If $\theta > v/(v-2)$ and $\theta \geq (p-1)r/(r-p)$ then for any $\epsilon > 0$ there exists $K = K(\epsilon, r, p, v, \theta, \alpha)$ such that

$$E\{|S_n|^p\} \leq K\left(n^{p/2}\max_{i\leq n}\|X_i\|_v^p + n^{1+\epsilon}\max_{i\leq n}\|X_i\|_r^p\right). \tag{1.16}$$

Now, let $S_n = \sum_{j=1}^{n} Y_j$ where $\{Y_j\}_{j\geq 1}$ is an i.i.d. sequence of random variables and let N be a stopping time, we will need estimates of the moments of S_N in terms of moments of N and Y_j. For this we recall the definition of stopping time.

Definition 1.6.2. *A positive integer valued random variable N is called a stopping time with respect to $\{Y_j\}_{j\geq 1}$ if for every $j \geq 1$, the event $\{N = j\} \in \sigma(Y_1, ..., Y_j)$.*

Theorem 1.6.4. *Suppose that $E|Y_k|^p$ for some $r \geq 0$ and that $EY_k = 0$ when $p \geq 1$. Then for a stopping time N we have*

i) $E|S_N|^p \leq E|Y_1|^p \cdot EN$ *for* $0 < p \leq 1$.

ii) $E|S_N|^p \leq c_p E|Y_1|^p \cdot EN$ *for* $1 \leq p \leq 2$.

iii) $E|S_N|^p \leq c_p(\{E(Y_1^2)\}^{p/2} \cdot E\{N^{p/2}\} + E|Y_1|^r \cdot EN) \leq 2c_p \cdot E|Y_1|^p \cdot E\{N^{p/2}\}$ *for* $p \geq 2$, *where c_p is a numerical constant depending on p only.*

In Chapter 2 we will need to establish moment convergence. Convergence in distribution by itself simply cannot ensure convergence of any moments. An extra condition that ensures convergence of appropriate moments is the uniform integrability.

Definition 1.6.3. *A sequence of random variables $\{Y_n\}_{n\geq 1}$ is said to be uniformly integrable if*

$$\lim_{\alpha\to\infty}\sup_n \int_{(|Y_n|\geq\alpha)} |Y_n| dP = 0.$$

Theorem 1.6.5. *Suppose $Y_n \xrightarrow{d} Y$. If $\left\{|Y_n|^k, n \geq 1\right\}$ is uniformly integrable, then*

$$E\left\{|Y_n|^r\right\} \longrightarrow E\left\{|Y|^r\right\} \ \ for\ \ every\ \ 0 < r \leq k.$$

Theorem 1.6.6. *Let $Y_1, Y_2, ..., X_1, X_2, ...$ be random variables.*

i) If $|Y_n| \leq X$ a.s. for all n, where X is a positive integrable random variable. Then $\{Y_n\}_{n\geq 1}$ is uniformly integrable.

ii) Let $|Y_n| \leq X_n$ a.s. for all n, where $X_1, X_2, ...$ are positive integrable random variable. If $\{X_n\}_{n\geq 1}$ is uniformly integrable, then so is $\{Y_n\}_{n\geq 1}$.

iii) If $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ are uniformly integrable, then so is $\{Y_n + X_n\}_{n\geq 1}$.

1.7 Empirical Processes

Now we will present some definitions and basic results on empirical processes. For references on this section see for example, Csörgő and Révész (1981) or Shorack and Wellner (1986).

Definition 1.7.1. *Let $X_1, X_2, ..., X_n$ be random variables. The empirical distribution function associated with $X_1, X_2, ..., X_n$ is defined as*

$$F_n(x,\omega) = \frac{\sum_{j=1}^n I_{\{X_j\leq x\}}}{n}, x \in \mathbb{R}, \tag{1.17}$$

where I_A is the indicator of event A.

Definition 1.7.2. *The empirical process associated with* $X_1, X_2, ..., X_n$ *with distribution function* F *is defined as*

$$\beta_n(x) = \sqrt{n}(F_n(x) - F(x)), \ x \in \mathbb{R}, \tag{1.18}$$

and the uniform empirical processes is given by

$$u_n(t) = \sqrt{n}(U_n(x) - t), \ 0 \le t \le 1,$$

where $U_n(t)$ *is the uniform empirical distribution.*

Note that if F is a continuous function then

$$\beta_n(x) = u_n(F(x)).$$

For every fixed $x \in \mathbb{R}$, $E(F_n(x)) = F(x)$ and $VarF_n(x) = n^{-1}F(x)(1 - F(x))$, because $nF_n(x)$ is binomial $(n, p = F(x))$ random variable. Hence, by the classical law of large numbers, we get

$$F_n(x) \xrightarrow{a.s.} F(x) \ \ as \ \ n \to \infty.$$

On the other hand, viewing $\{F_n(x) : x \in \mathbb{R}, n = 1, 2, 3, ...\}$ as a stochastic process in x and n, its sample functions in x are distributions functions and we have

Theorem 1.7.1. *[Glivenko- Cantelli Theorem]*

$$\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \xrightarrow{a.s.} 0 \ \ as \ \ n \to \infty.$$

and also, we have the CLT for empirical processes:

Theorem 1.7.2. *Let* $x \in \mathbb{R}$ *such that* $0 < F(x) < 1$, *then*

$$\beta_n(x) \xrightarrow{d} Z(x) \stackrel{d}{=} N(0, F(x)(1 - F(x))). \tag{1.19}$$

Observe that a Brownian bridge $B(t)$ has a normal distribution $N(0, t(1 - t))$.

Definition 1.7.3. *A zero-mean Gaussian process* $\{B(t) : 0 \le t \le 1\}$ *is called a Brownian bridge if the covariance is given by* $Cov(B(t), B(s)) = \min(s, t) - st$. *Or, equivalently,*

$$\left\{B(t) \stackrel{d}{=} W(t) - tW(1) : 0 \le t \le 1\right\},$$

where $\{W(t) : t \ge 0\}$ *is the standard Brownian motion.*

Now we will present the concept of quantile empirical process which can be considered as the inverse of the empirical process $\beta_n(t)$.

Definition 1.7.4. *Let $X_1, X_2, ..., X_n$ be random variables with distribution function F. Then*

a) The inverse distribution function (or quantile function) of F is given by

$$F^{-1}(t) = \inf\{x : F(x) \geq t\}, \; F^{-1}(0) = F^{-1}(0^+).$$

b) The inverse empirical distribution function (or empirical quantile function) is given by

$$F_n^{-1}(t) = \inf\{x : F_n(x) \geq t\}, \; 0 \leq t \leq 1.$$

As for F_n, we associate the empirical quantile function F_n^{-1} a stochastic process.

Definition 1.7.5. *The empirical quantile process associated with $X_1, X_2, ..., X_n$ with distribution function F is defined as*

$$q_n(t) = \sqrt{n}(F_n^{-1}(t) - F^{-1}(t)), \; 0 \leq t \leq 1. \tag{1.20}$$

and the uniform quantile processes is given by

$$u_n(t) = \sqrt{n}(U_n^{-1}(t) - t), \; 0 \leq t \leq 1,$$

where $U_n^{-1}(t)$ is the uniform empirical quantile function.

Observe that for random variable X with a continuous distribution F we have that $F(X) \stackrel{d}{=} U$, where U is a uniform $[0, 1]$ random variable and consequently, if $F'(x)$ exists

$$q_n(t) = \sqrt{n}(F^{-1}(U_n^{-1}(t)) - F^{-1}(t))$$

and using the mean value theorem, we can write

$$q_n(t) = \sqrt{n}(U_n^{-1}(t) - t)\left(F^{-1}(\xi_n)\right)', \; for \; t \wedge U_n^{-1}(t) \leq \xi_n \leq t \vee U_n^{-1}(t).$$

Moreover, if $(F^{-1}(t))' = \dfrac{1}{f(F^{-1}(t))} < \infty$ for $t \in (0, 1)$ and $f = F'$, then we have

$$q_n(t) = \frac{u_n(t)}{f(F^{-1}(\xi_n))}.$$

Now, not so immediately as for empirical process $\beta_n(t)$ we have the following quantile CLT.

Theorem 1.7.3. *[Shorack and Wellner(1986), Proposition 1, Chapter 18] Let* $X_1, X_2, ..., X_n$ *be random variables with distribution function* F *with derivate in* $F^{-1}(t)$, $t \in (0,1)$. *Assume that* $F'(F^{-1}(t)) = \frac{1}{f(F^{-1}(t))} > 0$. *Then as* $n \to \infty$

$$q_n(t) \xrightarrow{d} \frac{B(t)}{f(F^{-1}(t))} \stackrel{d}{=} N\left(0, \frac{t(1-t)}{f^2(F^{-1}(t))}\right). \tag{1.21}$$

To obtain a quantile CLT in the $\alpha-$mixing case, we have the following Bahadur representation of sample quantiles.

Theorem 1.7.4. *[Xing, Yang, Liu et al. (2012), Theorem 2.3] Let* $\{X_n\}_{n\geq 1}$ *be an strictly stationary and* $\alpha-$*mixing sequence of random variables with a common distribution function* F*, where* F *is absolutely continuous and has a continuous density function* f *such that* $0 < f(F^{-1}(t)) < \infty, t \in (0,1)$. *If* f' *is bounded in some neighborhood of* $F^{-1}(t)$ *and* $\alpha(n) = O(n^{-\beta})$ *for some* $\beta > 1$. *Then, as* $n \to \infty$,

$$F_n^{-1}(t) = F^{-1}(t) + \frac{t - F(F^{-1}(t))}{f(F^{-1}(t))} + R_n \ a.s., \tag{1.22}$$

where $R_n = O(n^{-3/4} \log n)$ *is such that* $\sqrt{n} R_n \to 0$.

1.8 Weak Convergence

Let S be a metric space. We will present some basic results concerning the weak convergence of sequences of probability measures on the σ-algebra $\mathcal{S}$ of Borel sets in S. For references on this section see Billinsgley (1968).

Definition 1.8.1. *Let* P_n *and* P *be probability measures on* $(S, \mathcal{S})$ *such that*

$$\int_S f dP_n \longrightarrow \int_S f dP$$

for every bounded, continuous real function f *on* S*, we say that* P_n *converges weakly to* P *and write* $P_n \Rightarrow P$.

Let $\{X_n\}$ *be a sequence of random elements on* $(S, \mathcal{S})$*, we say that* $\{X_n\}$ *converges in distribution to the random element* X*, and we write*

$$X_n \xrightarrow{d} X \ (or \ X_n \Rightarrow X),$$

if the distributions P_n of the X_n converge weakly to the distribution P of X.

Suppose that h maps S into another metric space S', with Bore1 σ-field $\mathcal{S}'$. If h is measurable then each probability P on $(S, \mathcal{S})$ induces on $(S', \mathcal{S}')$ a probability Ph^{-1} defined as usual by $Ph^{-1}(A) = P(h^{-1}A)$. If h is continuous then $P_n \Rightarrow P$ implies $P_n h^{-1} \Rightarrow Ph^{-1}$, but the continuity of the mapping h can be a replaced by a weaker condition. Assume only that h is measurable and let D_h be the set of its discontinuities.

Theorem 1.8.1 (Continuous Mapping Theorem)**.** *Let $h : S \longrightarrow S'$ be measurable . If $P_n \Rightarrow P$ and $P(D_h) = 0$, then $P_n h^{-1} \Rightarrow Ph^{-1}$.*

The following notion of tightness proves important both in the theory of weak convergence and in its applications.

Definition 1.8.2. *A family of probability measure $\mathcal{P}$ on $(S, \mathcal{S})$ is tight if for each positive ϵ there exists a compact set K such that $P(K) > 1 - \epsilon$, for all $P \in \mathcal{P}$.*

Now, let $D[0,1]$ be the space of functions $x(t)$ on $[0,1]$ that are right-continuous and have left-hand limits.
The following theorem establishes sufficient conditions for weak convergence in $D[0,1]$.

Theorem 1.8.2. *Let P_n, P be probability measures on $D[0,1]$. If the finite-dimensional distributions of P_n converge weakly to finite-dimensional distributions of P, and if $\{P_n\}$ is tight, then $P_n \Rightarrow P$.*

The following theorem establishes sufficient conditions for the tightness of a sequence X_n. It is a version of Theorem 15.5, Billinsgley (1968).

Theorem 1.8.3. *Let $X_1, X_2, ...$ be a random variables in $D[0,1]$. The sequence $\{X_n\}_{n \geq 1}$ is tight if and only if these two conditions hold:*

1. *For each positive η, there exists an $a \in \mathbb{R}$ such that for each $n \geq 1$*

$$P\{|X_n(t)| > a\} \leq \eta, \;\; for \; every \; t \in [0,1].$$

2. For each positive ϵ and η, there exist a δ, with $0 < \delta < 1$, and an integer n_0 such that

$$P\left\{\sup_{t<s<t+\delta} |X_n(s) - X_n(t)| \geq \epsilon\right\} \leq \delta\eta, \; n \geq n_0.$$

The following result is one of the main tools to prove the convergence of finite-dimensional distributions of a stochastic process.

Theorem 1.8.4 (Cramer-Wold)**.** *Let $X_n = (X_{n1}, X_{n2}, ..., X_{nk})$ and $X = (X_1, X_2, ..., X_k)$ be random vectors of dimension k. Then*

$$X_n \xrightarrow{d} X$$

if and only if:

$$\sum_{j=1}^{k} t_j X_{nj} \xrightarrow{d} \sum_{j=1}^{k} t_j X_j$$

for each $(t_1, ..., t_k) \in \mathbb{R}^k$,that is, if every fixed linear combination of the coordinates of X_n converges in distribution to the correspondent linear combination of coordinates of X.

In order to obtain the weak convergence of the empirical quantile process, object of study of the second section of chapter 3, we will present some important results and definitions.

Definition 1.8.3. *Suppose E is a set and $\{f_n\}_{n\geq1}$ is a sequence of real-valued functions on it. We say the sequence $\{f_n\}_{n\geq1}$ is uniformly convergent on E for f if*

$$\sup_{x\in E} |f_n(x) - f(x)| \longrightarrow 0 \text{ as } n \to \infty. \tag{1.23}$$

For functions defined on $\mathbb{R}$, the sequence $\{f_n\}_{n\geq1}$ is said to be locally uniformly convergent if (1.23) holds for any compact interval.

Remark 1.8.1. *If $x_n(t) \overset{n\to\infty}{\longrightarrow} x(t)$ in the Skorohod topology and $x(t)$ is a continuous function(defined on a compact set), then $x_n(t) \overset{n\to\infty}{\longrightarrow} x(t)$(locally) uniformly.*

In mathematics and statistics, Skorokhod's representation theorem is a result that shows that a weakly convergent sequence of probability measures whose limit measure is sufficiently well-behaved can be represented as the distribution of a pointwise convergent sequence of random variables defined on a common probability space.

Theorem 1.8.5 (Skorokhod's representation theorem)**.** *Let S be a separable space. Suppose that $\{X_n\}_{n\geq 1}$ is a sequence of random elements on $(S, \mathcal{S})$ such that $X_n \xrightarrow{d} X$. Then there is a probability space $(\Omega, \mathcal{F}, P)$ on which are defined S-valued random variables $X'_n, n = 1, 2, ...$ and X' with same distributions of X_n and X respectively, such that $X'_n \xrightarrow{a.s.} X'$.*

In addition, we will present a lemma which together with Skorokhod's representation theorem and with the relation explained in Remark 1.8.1, allows us to obtain the weak convergence of the empirical quantile process. This lemma is an adaptation of Vervaat's Lemma (1972). For more details and their demonstration, see (Resnick, 2007) and (Vervaat, 1971).

Lemma 1.8.6. *Suppose for any n, $x_n(t) \in D[0, 1]$ is a non-decreasing function and $x_0(t) \in C[0, 1]$. If $c_n \to \infty$ and*

$$c_n(x_n(t) - t) \overset{n\to\infty}{\longrightarrow} x_0(t) \tag{1.24}$$

locally uniformly, then

$$c_n(x_n^{-1}(t) - t) \overset{n\to\infty}{\longrightarrow} -x_0(t) \tag{1.25}$$

locally uniformly.

Chapter 2

Asymptotics for Regenerative Sequences.

2.1 Introduction

In this chapter we study the asymptotic behavior of a regenerative sequence $\{X_n\}_{n\geq 0}$ on $(S, \mathcal{G})$ and with regeneration times $\{T_n\}_{n\geq 0}$. More precisely, we consider a partial sum $S_n = \sum_{j=1}^{n} \varphi(X_j)$ with $\varphi : S \to \mathbb{R}$ be a measurable function and then we obtain convergence in distribution, convergence in moments and convergence in Mallows distance. We also present the approximation of the partial sum S_n by a Brownian motion $\{W(t) : t \geq 0\}$ and show that this can be carried out at rate of convergence $O(\log n)$.

As mentioned before, the Mallows distance measures the discrepancy between two distribution functions and has been successfully used to derive Central Limit Theorem type results (see, e.g., Johnson and Samworth (2005) or Dorea and Oliveira (2014)). In this sense, we establish conditions to obtain convergence in Mallows distance of order r and convergence of the moments of order $r \geq 2$ for regenerative process. It is worth pointing out we will apply in the next chapter of our work the results obtained in Chapter 2 to analyze the asymptotic behavior of the empirical process associated with a regenerative sequence.

In Section 2.2 we present some basic results for regenerative process. First, we define the concept of regenerative sequences and provide some illustrative examples. Next, in Theorem 2.2.3 we state sufficient conditions for the SLLN to hold, for the existence of a limiting distribution and for the Glivenko-Cantelli type results. In Section 2.3 we provide a variant of the CLT for regenerative sequences, Theorem 2.3.2, and in 2.3.1 we show that our hypotheses are weaker than those used by Glynn and Whitt (1993). We will use the dissection formula proposed in Chung (1967)

$$S_n = A_n + \sum_{k=1}^{N_n-1} Y_k + B_n, \tag{2.1}$$

where N_n is conveniently chosen,

$$A_n = \sum_{j=0}^{T_1-1} \varphi(X_j), \;\; Y_k = \sum_{j=T_k}^{T_{k+1}-1} \varphi(X_j) \;\; and \;\; B_n = \sum_{T_{N_n}}^{n} \varphi(X_j).$$

and using renewal theory we show that A_n and B_n in (2.1) are negligible. The use of a CLT for randomized sums of i.i.d. variables allow us to obtain the asymptotic normality of the central term in (2.1). With the control of the tail parts our Theorem 2.3.2 shows that there are constants a_n and $b_n > 0$ such that $\dfrac{S_n - a_n}{b_n}$ converges in distribution to Gaussian variable under second moment conditions on blocks n_j's. Special cases of this result are CLT's for renewal and Markovian processes.

In Section 2.4 we prove convergence in Mallows distance of order $r \geq 2$ for the partial sum S_n. Under regularity conditions we prove that A_n and B_n are negligible and then we study the convergence of the central term in (2.1). In this sense, our Theorem 2.4.2 shows that $\dfrac{\sum_{k=1}^{n} Y_k}{b_n}$ converges in Mallows distance and moments of order r to a standard normal variable Z_0 under the condition $d_r(Y_k, Z) < \infty$ for some k and a normal variable Z. Next, our Theorem 2.4.3 generalizes Theorem 2.4.2 taking the random variable N_n instead of n. This result is important because it establishes conditions under which randomly indexed partial sums preserve convergence in Mallows distance. Finally, as a consequence of these result our Theorem 2.4.6 provides sufficient conditions for convergence in Mallows distance and moments of order r of $\dfrac{S_n - a_n}{b_n}$ to standard normal variable Z_0.

In the last section, we study the approximating of the partial sum S_n by a Brownian motion with rate $O(\log n)$. In Theorem 2.5.1 we obtain a version for regenerative sequences of KMT (Komlós, Major and Tusnády) strong approximation obtained in the paper "Strong approximation for additive functionals of geometrically ergodic Markov chains" by Merlevede and Rio (2015). This adaptation was possible because the authors used regenerative methods.

2.2 Regenerative Sequences

For references on this subsection see Athreya and Lahiri (2006), Asmuseen (2003) or Serfozo (2009).

A sequence of random variables is regenerative if it probabilistically restarts itself at random times and thus can be broken up into i.i.d. pieces. Below is the formal definition of regenerative sequences.

Definition 2.2.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(S, \mathcal{G})$ be a measurable space. A sequence of random variables $\{X_n\}_{n\geq 0}$ defined on $(\Omega, \mathcal{F}, P)$ with values in $(S, \mathcal{G})$ is called regenerative if there exists a sequence of random times $0 = T_0 < T_1 < T_2 < T_3 < \cdots$ such that the "cycles " or "excursions"*

$$\begin{aligned}
\eta_0 = &\quad (X_0, X_1, X_2, ..., X_{T_1 - 1}, T_1 - T_0)\\
\eta_1 = &\quad (X_{T_1}, X_{T_1+1}, ..., X_{T_2 - 1}, T_2 - T_1)\\
. &\qquad\qquad\qquad .\\
. &\qquad\qquad\qquad .\\
\eta_k = &\quad (X_{T_k}, X_{T_k+1}, ..., X_{T_{k+1} - 1}, T_{k+1} - T_k)\\
. &\qquad\qquad\qquad .\\
. &\qquad\qquad\qquad .
\end{aligned}$$

are i.i.d. as random vectors with a random number of components. More precisely,

$$\begin{aligned}
&P\left(T_{j+1} - T_j = k_j, X_{T_j+l} \in A_{l,j}, 0 \leq l < k_j, j = 1, 2, ..., r\right)\\
&= \prod_{j=1}^{r} P(T_1 = k_j, X_{T_1+l} \in A_{l,j}, 0 \leq l < k_j). \qquad (2.2)
\end{aligned}$$

$\forall k_1, k_2, ..., k_r \in \mathbb{N}$ *e* $A_j \in \mathcal{S}, 1 \leq l \leq k_j, j = 1, ..., r,$ *and* $r \geq 1$. *A regenerative sequence* $\{X_n\}_{n\geq 0}$ *is called delayed when the first cycle,* $\eta_0 := \{X_j : 0 \leq j < T_1\}$ *has different distribution than all the other cycles.*

The random times $\{T_n\}_{n\geq 0}$ are called regeneration times and clearly, $\{T_n\}_{n\geq 0}$ is a renewal process i.e.,

$$\tau_1 = T_1 - T_0, \ \tau_2 = T_2 - T_1, \ \tau_3 = T_3 - T_2, \ \cdots ,$$

are i.i.d. random variables and $T_n = \tau_1 + \tau_2 + \cdots \tau_n$. So we can define the counting process N_n by the relation

$$N_n = k \ \ if \ \ T_k \leq n < T_{k+1} \ \ for \ \ k = 0, 1, 2, ...,$$

i.e., N_n counts the number of regenerations up to time n.

Example 2.2.1. *Any independently and identically distributed sequence* $\{X_n\}_{n\geq 0}$ *of random variables is regenerative with* $T_k = k$ *as the embedded renewal process.*

Example 2.2.2. *By Theorem 1.3.1, any Markov chain* $\{X_n\}_{n\geq 0}$ *with a countable state space* S *that is irreducible and recurrent is regenerative with* $\{T_n\}_{n\geq 1}$ *being the times of successive returns to a given state.*

Example 2.2.3. *Any Harris recurrent chain satisfying (1.1) is regenerative by Theorem 1.3.2.*

Example 2.2.4 (The GI/GI/1 Queue)**.** *This is a model where the n-th customer arrives at time* t_n*, waits in a common queue (in a first in first out manner) that has one server, and when served, has service time* S_n*. Arrival times form a renewal process with independently and identically distributed interarrival times* $T_n = t_{n+1} - t_n$*. The delay sequence* $\{D_n\}_{n\geq 1}$ *defined by the recursion* $D_{n+1} = \max(0, D_n + S_n - T_n)$*, which denotes how long each customer waits in the queue before entering service form a positive recurrent regenerative process.*

Example 2.2.5. *Let* $\{Y_n\}_{n\geq 0}$ *be a Harris recurrent Markov chain as in Example 2.2.3. Given* $\{Y_n = y_n\}_{n\geq 0}$*, let* $\{A_n\}_{n\geq 0}$ *be independent positive integer valued random variables.*

Set

$$X_n = \begin{cases} y_0 & 0 \leq n < A_0 \\ y_1 & A_0 \leq n < A_0 + A_1 \\ y_2 & A_0 + A_1 \leq n < A_0 + A_1 + A_2 \\ . & \\ . & \\ . & \end{cases}$$

Then $\{X_n\}_{n\geq 1}$ *is called a semi Markov chain with embedded Markov chain* $\{Y_n = y_n\}_{n\geq 1}$ *and sojourn times* $\{A_n\}_{n\geq 0}$. *Since* $\{Y_n\}_{n\geq 0}$ *is regenerative, it follows that* $\{X_n\}_{n\geq 1}$ *is regenerative. See Example 14.2.14 in Athreya and Lahiri (2006) for more details.*

Example 2.2.5 presents a regenerative sequence that is not a Markov chain. So, we have that a regenerative sequence $\{X_n\}_{n\geq 0}$, in general, need not be a Markov chain. Now we present some elementary properties of regenerative sequences.

Proposition 2.2.1. *[Asmuseen (2003), Proposition 1.1] Let* $\{X_n\}_{n\geq 0}$ *be a regenerative sequence with regeneration times* $\{T_n\}_{n\geq 0}$. *If* $\varphi : S \to T$ *is any measurable mapping, then* $\{\varphi(X_n)\}_{n\geq 0}$ *is regenerative sequence with the same regeneration times.*

The above proposition means that the regenerative property is preserved under arbitrary mappings. For instance, take $\{X_n\}_{n\geq 0}$ to be a regenerative sequence with regeneration times $\{T_n\}_{n\geq 0}$ and consider the function $\bar{X}_n = I_A(X_n)$ for some set $A \in \mathcal{G}$. Since $\{X_n\}_{n\geq 0}$ is regenerative, then so is $\left\{\bar{X}_n\right\}_{n\geq 0}$ with the same regeneration times. In this sense, the following result is an immediate consequence of the definition of regenerative sequence.

Proposition 2.2.2. *[Glynn (1982), Proposition 2.7] Let* $\{X_n\}_{n\geq 0}$ *be a regenerative sequence with regeneration times* $\{T_n\}_{n\geq 0}$. *Assume that* $\varphi_k : S^k = S \times S \times \cdots \times S \longrightarrow \mathbb{R}$ *is a sequence of real-valued functions such that* φ_k *is measurable for every* k. *Let* $Y_n = \varphi_{\tau_n}(X_{T_n}, X_{T_n+1}, \cdots, X_{T_{n+1}-1})$ *for* $n \geq 1$, *where* $\tau_n = T_n - T_{n-1}$. *Then the sequence* $\{(Y_n, \tau_n)\}_{n\geq 1}$ *is i.i.d.*

From the previous proposition we have that if $\{X_n\}_{n\geq 0}$ is a regenerative sequence with regeneration times $\{T_n\}_{n\geq 0}$ and $\varphi : S \to \mathbb{R}$ is a measurable function then the sequence

$Y_n = \sum_{j=T_n}^{T_{n+1}-1} \varphi(X_j)$ is i.i.d. We use this fact in most proofs of our results.

2.2.1 Some results for regenerative sequences

A regenerative sequence $\{X_n\}_{n\geq 0}$ on $(S, \mathcal{G})$ with regeneration times $\{T_n\}_{n\geq 0}$ have independent and identically distributed cycles and cycle lengths, so ergodic theorems are elementary consequences of the Renewal Theory, the Strong Law of Large Numbers and the Central Limit Theorem in the i.i.d. case. To study the asymptotic behavior of the partial sum $S_n = \sum_{j=0}^{n} \varphi(X_j)$ the main tool is the decomposition

$$S_n = \sum_{k=0}^{T_1-1} \varphi(X_k) + \sum_{k=1}^{N_n-1} Y_k + \sum_{j=T_{N_n}}^{n} \varphi(X_j) \tag{2.3}$$

where

$$Y_k = \sum_{j=T_k}^{T_{k+1}-1} \varphi(X_j) \quad \textit{are i.i.d. and } N_n = \sup\{k : T_k \leq n < T_{k+1}\}$$

and then to analyse the asymptotic behavior of the central term in (2.3), since the other two terms are negligible.

In analogy with Markov chains, we need some notation and terminology. $\{X_n\}_{n\geq 0}$ is said to be positive recurrent if $\mu_T = E\{T_2 - T_1\} < \infty$ and null recurrent otherwise. Also, we say that $\{X_n\}_{n\geq 0}$ is aperiodic if $gcd\{j : p_j > 0\} = 1$ where $p_j = P(T_2 - T_1 = j)$. In the remainder of this work, we will assume that $\{X_n\}_{n\geq 0}$ is an aperiodic and a positive recurrent regenerative sequence with $\mu_T > 0$.

For a regenerative sequence $\{X_n\}_{n\geq 0}$ on $(S, \mathcal{G})$ with regeneration times $\{T_n\}_{n\geq 0}$ we can define the occupation probability measure

$$\tilde{\pi}(A) = \frac{1}{\mu_T} E\left\{\sum_{j=T_1}^{T_2-1} I_A(X_j)\right\},\ A \in \mathcal{G}, \tag{2.4}$$

and for a measurable function $\varphi : S \longrightarrow \mathbb{R}$ we can define the distribution function by

$$\tilde{F}_\varphi(x) = \tilde{\pi}(s : \varphi(s) \leq x).$$

Observe that $\tilde{\pi}(A)$ defined by (2.4) is equal to the expected time that the sequence spent in A, $A \in \mathcal{G}$, over the expected inter-regeneration time. And for any function $\varphi \in L_1(S, \mathcal{G}, \tilde{\pi})$ the integral de φ with respect to $\tilde{\pi}$ (denoted $E_{\tilde{\pi}}\{\varphi\}$) represents the expectation of the function φ summed along the path of the process $\{X_n\}_{n\geq 0}$ from T_1 to $T_2 - 1$, over the expected inter-regeneration time:

$$\frac{1}{\mu_T} E\left\{\sum_{j=T_1}^{T_2-1} \varphi(X_j)\right\} = \int \varphi d\tilde{\pi}. \tag{2.5}$$

It can be easily seen by taking a simple function $\varphi(s) = \sum_{i=1}^{n} a_i I_{A_i}(s)$ with $A_i \in \mathcal{G}$ and $a_i \geq 0$. By definition of $\tilde{\pi}$ we have

$$\sum_{i=1}^{n} a_i \tilde{\pi}(A_i) = \sum_{i=1}^{n} \frac{a_i}{\mu_T} E\left\{\sum_{j=T_1}^{T_2-1} I_{A_i}(X_j)\right\} = \frac{1}{\mu_T} E\left\{\sum_{j=T_1}^{T_2-1}\sum_{i=1}^{n} a_i I_{A_i}(X_j)\right\} = \frac{1}{\mu_T} E\left\{\sum_{j=T_1}^{T_2-1} \varphi(X_j)\right\}.$$

And can be easily extended to any function $\varphi \in L_1(S, \mathcal{G}, \tilde{\pi})$.

Now, we present the strong law of large numbers and a limiting distribution for regenerative sequences under conditions that are mild and usually easy to verify. In the following result we will see that the value of a time-average limit is determined by the expected behavior of the process in a single regenerative cycle, this fact has important applications.

Theorem 2.2.3. *[Athreya and Lahiri (2006), Theorem 14.2.10] Let $\{X_n\}_{n\geq 0}$ be a regenerative sequence with regeneration times $\{T_n\}_{n\geq 0}$. Let $\tilde{\pi}$ given by (2.4) . Then, as $n \to \infty$,*

(i) (SLLN for regenerative sequences)

$$\frac{1}{n}\sum_{j=0}^{n} \varphi(X_j) \xrightarrow{a.s.} \tilde{\mu}_\varphi = \int \varphi d\tilde{\pi} \ \ \forall \varphi \in L_1(S, \mathcal{G}, \tilde{\pi}). \tag{2.6}$$

(ii) If the distribution of $T_2 - T_1$ is aperiodic, then $X_n \xrightarrow{d} X$ where $X \stackrel{d}{=} \tilde{\pi}$. In the real-valued case, this amounts to showing that $P(X_n \leq x) \longrightarrow \tilde{F}(x) = \tilde{\pi}(-\infty, x]$ for all continuity points of the cumulative distribution function $\tilde{F}$.

Remark 2.2.1. *Let i be a positive recurrent state for a Markov chain $\{X_n\}_{n\geq 0}$ on a countable space S with transition probability matrix $\boldsymbol{P}$. Let the occupation probability measure*

$$\tilde{\pi}_j = \frac{1}{E_i(T_1(i))} E_i\left\{\sum_{k=0}^{T_1(i)-1} I(X_k = j)\right\}, \tag{2.7}$$

where $T_1(i) = \inf\{n : X_n = i, n \geq 1\}$ *Then* $\tilde{\pi} = \{\tilde{\pi}_j\}_{j\in S}$ *is a stationary distribution for* $\boldsymbol{P}$. *Besides, if* $\{X_n\}_{n\geq 0}$ *is a positive recurrent irreducible Markov chain there is a unique invariant measure* π *given by*

$$\pi = \left\{\tilde{\pi}_j \equiv (E_j\{T_1(j)\})^{-1}, j \in S\right\}.$$

On the other hand, let $\{X_n\}_{n\geq 0}$ *be an aperiodic Harris Markov chain on a countably generated state space* $(S, \mathcal{G})$, *with transition probability* $P(\cdot,\cdot)$, *and initial probability distribution* ν. *Assume that* A *is an accessible atom. Then* $\{X_n\}_{n\geq 0}$ *is positive recurrent if and only if* P *has a unique invariant probability measure* π, *(See Kac's theorem in [46]) in which case* π *coincides with* $\tilde{\pi}$ *given by (2.4), i.e.,*

$$\pi(B) = \frac{1}{E_A(T_1(A))} E_A\left\{\sum_{j=0}^{T_1(A)-1} I_B(X_j)\right\}, \ B \in \mathcal{G} \tag{2.8}$$

where $T_1(A) = \inf\{n \geq 1, X_n \in A\}$ *the hitting time on* A. *Therefore, Theorem 2.2.3 holds for Markov chains with enumerable state space and for Harris Markov chains with occupancy measure given by (2.7) and (4.5), respectively. (See, Athreya and Lahiri (2006), Theorem 14.1.20 and Theorem 14.2.11). Thus a Harris ergodic chain converges in distribution to a unique invariant probability measure.*

A Glivenko-Cantelli theorem is a fundamental result in statistics. It says that an empirical distribution function uniformly approximates the true distribution function for a sufficiently large sample size. Athreya and Roy (2016) proved a general Glivenko-Cantelli theorems for three types of sequences of random variables: regenerative, stationary and exchangeable. In particular, these results hold for irreducible Harris recurrent Markov chains that admit a stationary probability distribution.

Theorem 2.2.4. *[Athreya and Roy (2016), Theorem 3] Let* $\{X_n\}_{n\geq 0}$ *be a regenerative sequence with regeneration times* $\{T_n\}_{n\geq 0}$ *such that* $0 < \mu_T < \infty$. *Suppose that* $\varphi : S \longrightarrow \mathbb{R}$ *is a mensurable function and let*

$$F_n(x) = \frac{1}{n}\sum_{j=1}^{n} I_{(-\infty,x]}(\varphi(X_j)) \ \textit{and} \ \tilde{F}_\varphi(x) = \tilde{\pi}(s : \varphi(s) \leq x), x \in \mathbb{R}, \ n \geq 1. \tag{2.9}$$

Then

$$\sup_{x\in\mathbb{R}} |F_n(x,\cdot) - \tilde{F}_\varphi(x)| \xrightarrow{a.s.} 0 \;\; as \;\; n \to \infty. \tag{2.10}$$

Remark 2.2.2. *Note that*

$$\tilde{F}_\varphi(x) = \frac{1}{\mu_T} E\left(\sum_{j=T_1}^{T_2-1} I_{(-\infty,x]}(\varphi(X_j))\right) \tag{2.11}$$

Roughly, (2.10) means that the empirical distribution of $\{\varphi(X_n)\}_{n\geq 1}$ when n is large is approximated uniformly by the expected value of the empirical empirical distribution of the i.i.d. blocks in which the sequence is divided.
On the other hand, let $\{X_n\}_{n\geq 0}$ be a positive recurrent irreducible Markov chain on countable space S with transition probability matrix $\boldsymbol{P}$ and limiting distribution $\tilde{\pi} = (\pi_j)_{j\in S}$. We can assume that X_0 has distribution $\tilde{\pi}$. Then $\tilde{F}_\varphi(x) = P(\varphi(X_0) \leq x)$ where φ is a real-valued function on S.

The following is a regenerative analogue of the classical CLT for sums of independent random variables.

Theorem 2.2.5. *[Glynn and Whitt (1993), Theorem 3] Let $\{X_n\}_{n\geq 1}$ be an aperiodic and positive recurrent regenerative sequence with regeneration times $\{T_n\}_{n\geq 1}$. Suppose that $\mu_T = E\{T_2 - T_1\} > 0$, $E\{(T_2-T_1)^2\} < \infty$ and $E\left\{\left(\sum_{k=T_1}^{T_2-1} \varphi(X_k)\right)^2\right\} < \infty$. Then*

$$\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}} \xrightarrow{d} Z_0 \stackrel{d}{=} N(0,1) \;\; as \;\; n \to \infty,$$

where $\tilde{\mu}_\varphi = E_{\tilde{\pi}}\{\varphi\}$ and $\tilde{\sigma}^2_\varphi := \dfrac{E\left\{\left(\sum_{k=T_1}^{T_2-1}(\varphi(X_k) - \tilde{\mu}_\varphi)\right)^2\right\}}{\mu_T}$.

In the next subsection we present an alternative demonstration of CLT for regenerative sequences. The proof makes use of some ideas from the proof of CLT for Markov chains with enumerable state space from Chung (1967).

2.3 CLT for Regenerative Sequences

Let $\{X_n\}_{n\geq 0}$ be a regenerative sequence on $(S, \mathcal{G})$ with regeneration times $\{T_n\}_{n\geq 0}$. For $\varphi : S \longrightarrow \mathbb{R}$ let $\tilde{\mu}_\varphi = \int \varphi d\tilde{\pi} = E_{\tilde{\pi}}\{\varphi\}$ and $S_n = \sum_{j=0}^{n} \varphi(X_j)$. We can write

$$S_n - n\tilde{\mu}_\varphi = \sum_{k=0}^{T_1-1}(\varphi(X_k) - \tilde{\mu}_\varphi) + \sum_{k=1}^{N_n-1} Y_k + \sum_{j=T_{N_n}}^{n} (\varphi(X_j) - \tilde{\mu}_\varphi), \tag{2.12}$$

where

$$Y_k = \sum_{j=T_k}^{T_{k+1}-1} (\varphi(X_j) - \tilde{\mu}_\varphi) \;\; \textit{and} \;\; N_n = \sup\{k : T_k \leq n < T_{k+1}\}.$$

Since $\{X_n\}_{n\geq 0}$ is a regenerative sequence, $\{Y_k\}_{k\geq i}$ are i.i.d. random variables with $E(Y_k) = 0$. In fact, $\mu_T = E(T_2 - T_1)$ and by (2.5) we have

$$E(Y_k) = E\left(\sum_{j=T_k}^{T_{k+1}-1} \varphi(X_j)\right) - \tilde{\mu}_\varphi E(T_2 - T_1) = \mu_T \int \varphi d\tilde{\pi} - \tilde{\mu}_\varphi \mu_T = 0.$$

Thus, to analyse the asymptotic normality of S_n when $0 < var(Y_k) < \infty$, we must guarantee the asymptotic normality of the central term in (2.12), since we can show that the other two terms are negligible. For the central term, in the same way as in Serfozo (2009) (Chapter 2, Theorem 65) we use a CLT for randomized sums and for other two terms in (2.12) we adapt to the case of regenerative sequences the arguments used in the proof of the CLT for Markov chains in Chung (1967) (see Theorem 8, Chapter 14).

Theorem 2.3.1. *[Gut (2013), Theorem 3.1. (A version of Anscombe's Theorem)] Let $Y_1, Y_2, \ldots$ i.i.d. random variables with mean μ and variance $\sigma^2 > 0$. Let $N(t)$ be an integer-valued process defined on the same probability space as the Y_n, where $N(t)$ may depend on the Y_n. If $\frac{N(t)}{t} \xrightarrow{p} c$, where c is a positive constant, then*

$$\frac{S_{N(t)}}{\sigma\sqrt{N(t)}} \xrightarrow{d} Z_0 \overset{d}{=} N(0,1) \;\; \textit{and} \;\; \frac{S_{N(t)}}{\sqrt{c}\sigma\sqrt{t}} \xrightarrow{d} Z_0 \overset{d}{=} N(0,1) \;\; \textit{as } n \to \infty,$$

where $S_{N(t)} = \sum_{k=1}^{N(t)}(Y_k - \mu)$.

Now we present an alternative proof of the central limit theorem for regenerative processes that provides conditions under which this limiting distribution is a normal distribution. We will see in Remark 2.3.1 that as compared to similar results such as CLT from Glynn and Whitt (1993) our hypotheses are somehow weaker. Special cases of this result are CLT's for renewal and Markovian processes.

Theorem 2.3.2. *[CLT for regenerative sequences] Let $\{X_n\}_{n\geq 1}$ be an aperiodic and positive recurrent regenerative sequence with regeneration times $\{T_n\}_{n\geq 1}$. Assume that*

$$\mu_T = E\{T_2 - T_1\} > 0 \ \ and \ \ 0 < Var(Y_k) = E\left\{\left(\sum_{j=T_1}^{T_2-1}(\varphi(X_j) - \tilde{\mu}_\varphi)\right)^2\right\} < \infty.$$

Then

$$\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}} \xrightarrow{d} Z_0 \stackrel{d}{=} N(0,1) \ \ as \ \ n \to \infty, \tag{2.13}$$

where $\tilde{\sigma}_\varphi^2 := \dfrac{Var(Y_k)}{\mu_T}$.

Proof. By decomposition (2.12), we have that

$$\frac{1}{\tilde{\sigma}_\varphi\sqrt{n}}\sum_{k=0}^{n}(\varphi(X_k) - \tilde{\mu}_\varphi) = A_n + \frac{1}{\tilde{\sigma}_\varphi\sqrt{n}}\sum_{k=1}^{N_n-1} Y_k + B_n, \tag{2.14}$$

where

$$A_n = \frac{1}{\tilde{\sigma}_\varphi\sqrt{n}}\sum_{k=0}^{T_1-1}(\varphi(X_k) - \tilde{\mu}_\varphi) \ \text{ and } \ B_n = \frac{1}{\tilde{\sigma}_\varphi\sqrt{n}}\sum_{k=T_{N_n}}^{n}(\varphi(X_k) - \tilde{\mu}_\varphi).$$

To obtain the convergence in (2.13) we will show that

$$\frac{1}{\tilde{\sigma}_\varphi\sqrt{n}}\sum_{k=1}^{N_n-1} Y_k \xrightarrow{d} Z \stackrel{d}{=} N(0,1), \ \ A_n \xrightarrow{a.s.} 0 \ \text{ and } \ B_n \xrightarrow{p} 0.$$

In fact, we know that the Y_k´s are i.i.d. with mean 0 and finite variance and from Theorem 1.4.1 we have $\dfrac{N_n}{n} \xrightarrow{a.s.} \dfrac{1}{\mu_T}$ as $n \to \infty$. Thus, by Theorem 2.3.1 as $n \to \infty$

$$\frac{1}{\tilde{\sigma}_\varphi\sqrt{n}}\sum_{k=1}^{N_n-1} Y_k \xrightarrow{d} Z \stackrel{d}{=} N(0,1) \ \ where \ \ \tilde{\sigma}_\varphi^2 = \frac{Var(Y_k)}{\mu_T}. \tag{2.15}$$

Since $\displaystyle\sum_{k=0}^{T_1-1}(\varphi(X_k) - \tilde{\mu}_\varphi)$ does not depend on n, we have

$$A_n \xrightarrow{a.s.} 0, \ \ n \to \infty. \tag{2.16}$$

On the other hand, to prove the convergence $B_n \xrightarrow{p} 0$ we makes use some ideas from the proof of Theorem 8 (chapter 14), from Chung (1967). First, we will show that $\sum_{j=T_{N_n}}^{n}(\varphi(X_j)-\tilde{\mu}_\varphi)$ is bounded in probability. In fact, for $u>0$ and fixed k , we have

$$\begin{aligned} P\left\{\left|\sum_{j=T_{N_n}}^{n}(\varphi(X_j)-\tilde{\mu}_\varphi)\right|>u\right\} &\leq P\left\{\max_{0\leq i\leq n}\left|\sum_{j=T_{N_n}}^{T_{N_n}+i}(\varphi(X_j)-\tilde{\mu}_\varphi)\right|>u\right\} \\ &\leq P\left\{\max_{0\leq i\leq n}\left|\sum_{j=T_{N_n}}^{T_{N_n}+i}(\varphi(X_j)-\tilde{\mu}_\varphi)\right|>u, n-T_{N_n}\leq k\right\} \\ &+ \left\{\max_{0\leq i\leq n}\left|\sum_{j=T_{N_n}}^{T_{N_n}+i}(\varphi(X_j)-\tilde{\mu}_\varphi)\right|>u, n-T_{N_n}> k\right\} \\ &\leq P\left\{\max_{0\leq i\leq k}\left|\sum_{j=T_{N_n}}^{T_{N_n}+i}(\varphi(X_j)-\tilde{\mu}_\varphi)\right|>u\right\}+P\left\{n-T_{N_n}\geq k\right\}. \end{aligned} \tag{2.17}$$

The second term in the last inequality tends to 0 as $k\to\infty$ uniformly with respect to n. Indeed, by hypothesis $\mu_T=\sum_{j=1}^{\infty}P(T_2-T_1\geq j)<\infty$ and by (1.6) we have

$$\lim_{n\to\infty}P(n-T_{N_n}\geq k)=\frac{1}{\mu_T}\sum_{j=k+1}^{\infty}P(T_2-T_1\geq j).$$

Thus given any $\epsilon>0$ there exists n_0 and k_0 such that $n>n_0$ and $k>k_0$ imply

$$P\left\{n-T_{N_n}\geq k\right\}<\epsilon.$$

Then $n-T_{N_n}$ is bounded in probability. On the other hand, since $\{X_n\}_{n\geq 1}$ is regenerative the first term in (2.17) does not depend on n and tends to 0 as $u\to\infty$, for each fixed k. This implies that $\sum_{j=T_{N_n}}^{n}(\varphi(X_j)-\mu_\varphi)$ is bounded in probability.
Therefore,

$$B_n \xrightarrow{p} 0. \tag{2.18}$$

Finally, from (2.15), (2.16) and (2.18) we obtain the convergence in (2.13). □

Corollary 2.3.3. *Let $\{X_n\}_{n\geq 0}$ be a positive recurrent irreducible Markov chain on countable space S with transition probability matrix $\boldsymbol{P}$ and limiting distribution $\tilde{\pi} = (\pi_j)_{j\in S}$. Let i be a fixed arbitrary state and denote by $T_r(i)$ the r-th time of visit to state i. Assume that $\tilde{\mu}_\varphi = \sum_{j\in S}\varphi(j)\pi_j < \infty$ and $0 < \sigma_\varphi(i) = E\left\{\left(\sum_{j=T_r(i)}^{T_{r+1}(i)-1}(\varphi(X_j)-\tilde{\mu}_\varphi)\right)^2\right\} < \infty$. Then S_n is asymptotically normally distributed with mean $n\tilde{\mu}_\varphi$ and variance $n\sigma_\varphi^2$ with $\sigma_\varphi^2 := \pi_i\sigma_\varphi^2(i)$, i.e.,*

$$\frac{S_n - n\tilde{\mu}_\varphi}{\sigma_\varphi\sqrt{n}} \xrightarrow{d} Z_0 \stackrel{d}{=} N(0,1) \text{ as } n\to\infty. \tag{2.19}$$

On the other hand, Theorem 2.3.2 also leads to CLT for Harris Chains, since the Harris chains are regenerative.

Corollary 2.3.4. *Let $\{X_n\}_{n\geq 0}$ be an aperiodic and positive recurrent Harris Markov chain with an accessible atom A. Let $\tilde{\mu}_\varphi = \{E_A(T_1(A))\}^{-1} E_A\left\{\sum_{j=0}^{T_1(A)-1}\varphi(X_j)\right\} < \infty$. Assume that $0 < E_A\left\{\left(\sum_{j=0}^{T_1(A)-1}(\varphi(X_j)-\tilde{\mu}_\varphi)\right)^2\right\} < \infty$. Then*

$$\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}} \xrightarrow{d} Z_0 \stackrel{d}{=} N(0,1) \text{ as } n\to\infty, \tag{2.20}$$

where $\tilde{\sigma}_\varphi^2 := \dfrac{E_A\left\{\left(\sum_{j=0}^{T_1(A)-1}(\varphi(X_j)-\tilde{\mu}_\varphi)\right)^2\right\}}{E_A(T_1(A))}$ and $T_1(A) = \inf\{n\geq 1, X_n\in A\}$ the hitting time on A.

It is worth pointing out the CLT for ergodic Markov chains has been under study for years and an extensive literature exists (e.g. see, Athreya and Ney (1978), Chen (1999), Chung (1967), Meyn and Tweedie (1993)).

Remark 2.3.1. *Note that if $E\{(T_2-T_1)^2\} < \infty$ and $E\left\{\left(\sum_{j=T_1}^{T_2-1}|\varphi(X_j)|\right)^2\right\} < \infty$ then $Var(Y_k)$ is finite. In fact, from Cauchy–Schwarz inequality follows that*

$$
\begin{aligned}
Var(Y_k) &= E\left\{\left(\sum_{j=T_1}^{T_2-1}(\varphi(X_j)-\mu_\varphi)\right)^2\right\} \\
&= E\left\{\left(\sum_{j=T_1}^{T_2-1}\varphi(X_j)-(T_2-T_1)\mu_\varphi\right)^2\right\} \\
&\leq E\left\{\left(\sum_{j=T_1}^{T_2-1}\varphi(X_j)\right)^2\right\}+2|\mu_\varphi|E\left\{(T_2-T_1)\left|\sum_{j=T_1}^{T_2-1}\varphi(X_j)\right|\right\}+\mu_\varphi^2E\left\{(T_2-T_1)^2\right\} \\
&\leq E\left\{\left(\sum_{j=T_1}^{T_2-1}|\varphi(X_j)|\right)^2\right\} \\
&+2\mu_T^{-1}E\left\{\sum_{j=T_1}^{T_2-1}|\varphi(X_j)|\right\}\left(E\left\{(T_2-T_1)^2\right\}\right)^{1/2}\left(E\left\{\left(\sum_{j=T_1}^{T_2-1}|\varphi(X_j)|\right)^2\right\}\right)^{1/2} \\
&+\mu_T^{-2}E\left\{(T_1-T_2)^2\right\}E\left\{\left(\sum_{j=T_1}^{T_2-1}|\varphi(X_j)|\right)^2\right\}<\infty.
\end{aligned}
$$

In the last inequality we have used that $\mu_\varphi = \mu_T^{-1}E\left\{\sum_{j=T_1}^{T_2-1}\varphi(X_j)\right\}$. *For instance, if* f *is bounded or the state space is finite* S *and* $E\left\{(T_2-T_1)^2\right\}<\infty$ *then* $Var(Y_k)<\infty$. *Thus, the CLT is valid for the regenerative sequence* $\left\{I_{(-\infty,x]}(X_n)\right\}_{n\geq 0}$ *for fixed* $x\in\mathbb{R}$ *whenever* $E\left\{(T_2-T_1)^2\right\}<\infty$. *In Chapter 3, we use this result to study the empirical process associated with a regenerative sequence.*

2.4 Asymptotic Behavior via Mallows Distance

Let a regenerative sequence $\{X_n\}_{n\geq 0}$ with regeneration times $\{T_n\}_{n\geq 0}$. As described in Section 2.2 the cycles,

$$\eta_0=\{X_n, 0\leq n\leq T_1-1\}, \eta_1=\{X_n, T_1\leq n\leq T_2-1\}, \eta_2=\{X_n, T_2\leq n\leq T_3-1\}, ...$$

are independent and, in addition, $\eta_1,\eta_2,...$ have the same distribution. Similarly as in the previous section we can write

$$S_n-n\mu_\varphi=\sum_{k=0}^{T_1-1}(\varphi(X_k)-\tilde{\mu}_\varphi)+\sum_{k=1}^{N_n-1}Y_k+\sum_{j=T_{N_n}}^{n}(\varphi(X_j)-\tilde{\mu}_\varphi), \tag{2.21}$$

where

$$Y_k = \sum_{j=T_k}^{T_{k+1}-1} (\varphi(X_j) - \tilde{\mu}_\varphi) \;\; are \;\; i.i.d. \;\; and \;\; N_n = \sup\{k : T_k \le n < T_{k+1}\}.$$

Note that if $d_r(Y_k, Z) < \infty$ where Z has normal distribution we have $E|Y_k|^r < \infty$. In fact, by Theorem 1.5.2

$$d_r^r(Y_k, Z) = E\{|Y_k - Z^*|\}^r,$$

where $Z^* \stackrel{d}{=} Z$ and $(Y_k, Z^*) \stackrel{d}{=} H$ with $H(x,y) = P(Y_k \le x, Z^* \le y)$. Since Z has normal distribution then $E|Z|^r < \infty$ and by Minkowski inequality

$$(E|Y_k|^r)^{1/r} \le d_r(Y_k, Z^*) + (E|Z^*|^r)^{1/r} < \infty. \tag{2.22}$$

The previous observation suggests the following condition.

Condition 2.4.1. *Let $\{X_n\}_{n\ge1}$ be an aperiodic and positive recurrent regenerative sequence with regeneration times $\{T_n\}_{n\ge1}$ satisfying $\sigma_\varphi^2 = Var(Y_k) > 0$. And assume that for some $r \ge 2$, $d_r(Y_k, Z) < \infty$ where Z has normal distribution.*

Theorem 2.4.2. *Assume that Condition 2.4.1 is satisfied and let $Z_0 \stackrel{d}{=} N(0,1)$. Then as $n \to \infty$,*

$$d_r\left(\frac{\sum_{k=1}^n Y_k}{\sigma_\varphi\sqrt{n}}, Z_0\right) \longrightarrow 0. \tag{2.23}$$

Moreover,

$$\frac{\sum_{k=1}^n Y_k}{\sigma_\varphi\sqrt{n}} \stackrel{d}{\longrightarrow} Z_0 \;\; and \;\; E\left\{\left|\frac{\sum_{k=1}^n Y_k}{\sigma_\varphi\sqrt{n}}\right|^r\right\} \longrightarrow E\{|Z_0|^r\}. \tag{2.24}$$

Proof. By hypothesis we have that $E|Y_k|^r < \infty$ for some $r \ge 2$ and then by Liapounov inequality $\sigma_\varphi^2 = E\{(Y_k)^2\} < \infty$. On the other hand the Y_k's are i.i.d. with $E(Y_k) = 0$. So (2.23) follows from Theorem 1.5.3. Now, to obtain the convergence (2.24) we will verify the conditions of Theorem 1.5.1. Since $Z_0 \stackrel{d}{=} N(0,1)$ then $E\{|Z|^r\} < \infty$. So we just need to show that

$$E\left\{\left|\frac{\sum_{k=1}^n Y_k}{\sigma_\varphi\sqrt{n}}\right|^r\right\} < \infty.$$

Since $\{\sum_{k=1}^{n} Y_k, \sigma(Y_1, Y_2, ..., Y_n)\}$ is a martingale by Lemma 1.6.1 there exists an constant $c_r > 0$ such that

$$\begin{aligned}
\frac{1}{n^{r/2}\sigma_\varphi^r} E\left\{\left|\sum_{k=1}^{n} Y_k\right|^r\right\} &\leq \frac{c_r}{n^{r/2}\sigma_\varphi^r} E\left\{\left|\sum_{k=1}^{n} Y_k^2\right|^{r/2}\right\} \\
&\leq \frac{c_r n^{r/2-1} n}{n^{r/2}\sigma_\varphi^r} E\left\{|Y_k|^r\right\} \\
&= c_r \sigma_\varphi^{-r} E\left\{|Y_k|^r\right\} < \infty.
\end{aligned}$$

Therefore (2.24) follows from Theorem 1.5.1. □

We can generalize the previous theorem taking N_n instead of n, where N_n is the random variable of the number of regenerations by time n, i.e, $N_n = \sup\{k : T_k \leq n < T_{k+1}\}, \ \ k, n = 1, 2, ...$ In this case it will be necessary to replace $\sigma_\varphi^2 = Var(Y_k)$ by $\tilde{\sigma}_\varphi^2 = \dfrac{\sigma_\varphi^2}{\mu_T}$.

Theorem 2.4.3. *Assume that Condition 2.4.1 is satisfied and let* $Z_0 \overset{d}{=} N(0, 1)$. *Then as* $n \to \infty$,

$$d_r\left(\frac{V_{N_n-1}}{\tilde{\sigma}_\varphi\sqrt{n}}, Z_0\right) \longrightarrow 0, \ V_n = \sum_{k=1}^{n} Y_k \ \textit{and} \ \tilde{\sigma}_\varphi^2 = \frac{\sigma_\varphi^2}{\mu_T}. \tag{2.25}$$

Moreover,

$$\frac{V_{N_n-1}}{\tilde{\sigma}_\varphi\sqrt{n}} \xrightarrow{d} Z_0 \ \textit{and} \ E\left\{\left|\frac{V_{N_n-1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r\right\} \longrightarrow E\left\{|Z_0|^r\right\}. \tag{2.26}$$

Proof. We will obtain the convergence in (2.26) and then the convergence in (2.25) will follow from Theorem 1.5.1. In this sense, first we will obtain the convergence of the left side in (2.26). We have that the Y_k's are i.i.d. with $E(Y_k) = 0$ and from Theorem 1.4.1 follows $\dfrac{N_n - 1}{n} \xrightarrow{a.s.} \dfrac{1}{\mu_T}$ as $n \to \infty$. So by Theorem 2.3.1 we have that

$$\frac{V_{N_n-1}}{\tilde{\sigma}_\varphi\sqrt{n}} \xrightarrow{d} Z_0 \ \text{with} \ \tilde{\sigma}_\varphi^2 = \frac{\sigma_\varphi^2}{\mu_T}. \tag{2.27}$$

Now we will obtain the convergence of the right side in (2.26). By convergence (2.27) and by Theorem 1.6.5 is sufficient to prove that

$$\left\{\left|\frac{V_{N_n-1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r, n \geq 1\right\} \ \text{is uniformly integrable.} \tag{2.28}$$

In fact, since $E|Y_k|^r < \infty$ given $\epsilon > 0$ we can choose $M > 0$ large such that

$$E\left(|Y_k|^r I\left\{|Y_k| > M\right\}\right) < \epsilon. \tag{2.29}$$

Let

$$V_n' = \sum_{k=1}^{n} Y_k' \text{ and } V_n'' = \sum_{k=1}^{n} Y_k'',$$

where

$$Y_k' = Y_k I\{|Y_k| \le M\} \text{ and } Y_k'' = Y_k I\{|Y_k| > M\}, k \ge 1. \tag{2.30}$$

Note that

$$E\left(\left|\frac{V'_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r I\left\{\left|\frac{V'_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r > \alpha\right\}\right) \le \alpha^{-r} E\left\{\left|\frac{V'_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^{2r}\right\}. \tag{2.31}$$

To see this, let V be a positive random variable. Then

$$E(V^r I\{V > \alpha\}) = \int_\alpha^\infty v^r dF_V(v) \le \alpha^{-r} \int_\alpha^\infty v^{2r} dF_V(v) \le \alpha^{-r} EV^{2r}.$$

On the other hand, the event

$$\{N(t) = n\} = \{T_n \le t, T_{n+1} > t\} = \{T_n \le t < T_{n+1}\}$$

is $\sigma(T_1, ..., T_{n+1})$-mensurable. So the event $N_n + 1 = n$ is $\sigma(T_1, ..., T_n)$-mensurable, i.e., $N_n + 1$ is stooping time with respect to $\sigma(T_1, ..., T_n)$. Thus by Theorem 1.6.4 (iii) there exist $c_r > 0$ such that

$$\begin{aligned}
\alpha^{-r} E\left\{\left|\frac{V'_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^{2r}\right\} &\le 2(\alpha\tilde{\sigma}_\varphi)^{-r} c_r \cdot E\left\{(Y_k')^{2r}\right\} \cdot E\left\{\left(\frac{N_n+1}{n}\right)^r\right\} \\
&\le 2^{r+1}(\alpha\tilde{\sigma}_\varphi)^{-r} c_r M^{2r}.
\end{aligned} \tag{2.32}$$

In the last inequality in (2.32) we use the definition of Y_k' given in (2.30) and the inequality $\dfrac{N_n+1}{n} \le 2$. So by (2.31) and (2.32) we obtain that

$$E\left(\left|\frac{V'_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r I\left\{\left|\frac{V'_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r > \alpha\right\}\right) \le 2^{r+1}(\alpha\tilde{\sigma}_\varphi)^{-r} c_r M^{2r}. \tag{2.33}$$

On the other hand, by Theorem 1.6.4 (iii) and by (2.29) for $\epsilon > 0$

$$\begin{aligned}
E\left\{\left|\frac{V''_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r\right\} &\le 2\tilde{\sigma}_\varphi^{-r} c_r \cdot E\left\{(Y_k'')^r\right\} \cdot E\left\{\left(\frac{N_n+1}{n}\right)^{r/2}\right\} \\
&\le 2^{r/2+1}\tilde{\sigma}_\varphi^{-r} c_r \epsilon.
\end{aligned} \tag{2.34}$$

From previous results we can show that $\left\{\left|\dfrac{V_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r, n \ge 1\right\}$ is uniformly integrable, i.e, for $\delta > 0$ given we have

$$E\left(\left|\frac{V_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r I\left\{\left|\frac{V_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right| > 2\alpha\right\}\right) < \delta. \tag{2.35}$$

To see this, let U and V be positive random variables, such that $EU^r < \infty$ and $EV^r < \infty$ for some $r > 0$. Then for $\alpha > 0$

$$\begin{aligned} E(U+V)^r I\{U+V>\alpha\} &\leq E(\max\{2U,2V\})^r I\{\max\{2U,2V\}>\alpha\} \\ &\leq 2^r EU^r I\{U>\alpha/2\} + 2^r EV^r I\{V>\alpha/2\} \end{aligned} \tag{2.36}$$

Thus by (2.34), (2.36),(2.33) and by the triangle inequality we obtain

$$\begin{aligned} E\left(\left|\frac{V_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r I\left\{\left|\frac{V_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|>2\alpha\right\}\right) &\leq E\left(\left|\frac{V'_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}+\frac{V''_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r I\left\{\left|\frac{V'_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|+\left|\frac{V''_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|>2\alpha\right\}\right) \\ &\leq 2^r E\left(\left|\frac{V'_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r I\left\{\left|\frac{V'_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|>\alpha\right\}\right) \\ &+ 2^r E\left(\left|\frac{V''_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r I\left\{\left|\frac{V''_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|>\alpha\right\}\right) \\ &\leq \frac{2^{2r+1}M^{2r}c_r}{(\alpha\tilde{\sigma}_\varphi)^r}+\frac{2^{3r/2+1}c_r\epsilon}{\tilde{\sigma}_\varphi^r}<\delta, \end{aligned} \tag{2.37}$$

provided we first choose M so large that ϵ is so small that $\dfrac{2^{3r/2+1}c_r\epsilon}{\tilde{\sigma}_\varphi^r} < \delta/2$ and α so large that $\dfrac{2^{2r+1}M^{2r}c_r}{(\alpha\tilde{\sigma}_\varphi)^r} < \delta/2$.

Now note that

$$\left|\frac{V_{N_n-1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r \leq 3^{r-1}\left(\left|\frac{V_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r+\left|\frac{Y_{N_n}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r+\left|\frac{Y_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r\right). \tag{2.38}$$

Since the sequence $\{Y_n\}_{n\geq 1}$ is i.i.d. with $E\{Y_n^r\} < \infty$ we have that the family of random variables $\left\{\left|\dfrac{Y_n}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r, n \geq 1\right\}$ is uniformly integrable. On the other hand, we proved above that the family $\left\{\left|\dfrac{V_{N_n+1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r, n \geq 1\right\}$ is also uniformly integrable . So, by Theorem 1.6.6 and by inequality (2.38) we obtain (2.28).

Finally, by (2.27) and (2.28) we obtain the two convergences in (2.26) and the convergence in (2.25) follows as a direct application of Theorem 1.5.1 . This complete the proof.

□

Let $V_n = \sum_{k=1}^{n} Y_k$ as before. The CLT for regenerative sequences (Theorem 2.3.2) states that if $0 < Var(Y_k) < \infty$ then as $n \to \infty$

$$\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}} \xrightarrow{d} Z_0 \stackrel{d}{=} N(0,1) \ where \ \tilde{\sigma}_\varphi^2 = \frac{\sigma_\varphi^2}{\mu_T}. \tag{2.39}$$

Moreover, we can write

$$\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}} = A_n + \frac{V_{N_n-1}}{\tilde{\sigma}_\varphi\sqrt{n}} + B_n \tag{2.40}$$

where

$$A_n = \frac{\sum_{k=0}^{T_1-1}(\varphi(X_k) - \tilde{\mu}_\varphi)}{\tilde{\sigma}_\varphi\sqrt{n}} \quad \textit{and} \quad B_n = \frac{\sum_{k=T_{N_n}}^{n}(\varphi(X_k) - \tilde{\mu}_\varphi)}{\tilde{\sigma}_\varphi\sqrt{n}}.$$

Note that if Condition 2.4.1 is valid for some $r \geq 2$, we can apply Theorem 2.4.3 to obtain the convergence

$$d_r\left(\frac{V_{N_n-1}}{\tilde{\sigma}_\varphi\sqrt{n}}, Z_0\right) \longrightarrow 0. \tag{2.41}$$

So, we would like to obtain convergence d_r in (2.39). For this, we will need a condition a little stronger than Condition 2.4.1.

Condition 2.4.4. *Let $\{X_n\}_{n\geq 1}$ be an aperiodic and positive recurrent regenerative sequence with regeneration times $\{T_n\}_{n\geq 1}$ satisfying $\sigma_\varphi^2 = Var(Y_k) > 0$. And assume that for some $r \geq 2$, $E\left(\sum_{k=T_1}^{T_2-1}|\varphi(X_k) - \tilde{\mu}_\varphi|\right)^r < \infty$.*

First we considerer the case $r = 2$. Thus, by considerations made above, we obtain the desired convergence in d_2.

Corollary 2.4.5. *Assume that Condition 2.4.4 is satisfied for $r = 2$ and let $Z_0 \stackrel{d}{=} N(0,1)$. Then as $n \to \infty$,*

$$d_2\left(\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}}, Z_0\right) \to 0. \tag{2.42}$$

Moreover,

$$\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}} \xrightarrow{d} Z_0 \quad \textit{and} \quad E\left\{\left(\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}}\right)^2\right\} \longrightarrow E\left\{Z_0^2\right\} = 1. \tag{2.43}$$

Proof. Since Condition 2.4.4 implies Condition 2.4.1 we can apply Theorem 2.4.3 to obtain the convergence

$$d_2\left(\frac{V_{N_n-1}}{\tilde{\sigma}_\varphi\sqrt{n}}, Z_0\right) \longrightarrow 0. \tag{2.44}$$

By decomposition (2.40) we have

$$A_n = \frac{\sum_{k=0}^{T_1-1}(\varphi(X_k) - \tilde{\mu}_\varphi)}{\tilde{\sigma}_\varphi\sqrt{n}} \quad \textit{and} \quad B_n = \frac{\sum_{k=T_{N_n}}^{n}(\varphi(X_k) - \tilde{\mu}_\varphi)}{\tilde{\sigma}_\varphi\sqrt{n}}.$$

Thus

$$E\left\{A_n^2\right\} \;= \frac{1}{\tilde{\sigma}_\varphi^2 n} E\left\{\left(\sum_{k=0}^{T_1-1}(\varphi(X_k)-\tilde{\mu}_\varphi)\right)^2\right\}.$$

Since $E\left(\sum_{k=0}^{T_1-1}|\varphi(X_k)-\tilde{\mu}_\varphi|\right)^2$ does not depend on n and is finite, we obtain

$$d_2(A_n,0) \leq E(A_n^2) \longrightarrow 0. \tag{2.45}$$

Now, let $U_k = \sum_{j=T_k}^{T_{k+1}-1} |\varphi(X_k)-\tilde{\mu}_\varphi|$ for $k \geq 1$. Since $\{X_n\}_{n\geq 1}$ is regenerative we have that the sequence $\{U_k\}_{k\geq 1}$ is i.i.d. and by Condition 2.4.4 for $r=2$ we have $E\left\{|U_k|^2\right\} < \infty,\ k \geq 1$. On the other hand, $T_{N_n} \leq n < T_{N_n+1}$ and $1 \leq N_n \leq n$. So

$$\begin{aligned}
E\left\{B_n^2\right\} &= E\left\{\frac{\left(\sum_{k=T_{N_n}}^{n}(\varphi(X_k)-\tilde{\mu}_\varphi)\right)^2}{n\tilde{\sigma}_\varphi^2}\right\} \\
&\leq E\left\{\frac{\left(\sum_{k=T_{N_n}}^{n}|\varphi(X_k)-\tilde{\mu}_\varphi|\right)^2}{n\tilde{\sigma}_\varphi^2}\right\} \\
&\leq \frac{1}{\tilde{\sigma}_\varphi^2 n} E\left\{U_{N_n}^2\right\} \\
&\leq \frac{1}{\tilde{\sigma}_\varphi^2 n} E\left\{\max_{1\leq m\leq n} U_m^2\right\} \longrightarrow 0 \ as\ n \to \infty.
\end{aligned}$$

In the last convergence we use the following result by Chung (1967). Let $U_1, U_2, ..., U_n$ i.i.d. random variables with finite mean, then

$$\lim_{n\to\infty} \frac{1}{n} E\left\{\max_{1\leq m\leq n} |U_m|\right\}. \tag{2.46}$$

To see this, let F to be the common distribution of the $|U_m|$. We have

$$P\left\{\max_{1\leq m\leq n} |U_m| > k\right\} = 1 - [F(k)]^n \leq n\,[1-F(k)]$$

It follows that

$$\frac{1}{n}E\left\{\max_{1\leq m\leq n}|U_m|\right\} = \frac{1}{n}\sum_{k=0}^{\infty} P\left\{\max_{1\leq m\leq n}|U_m| > k\right\} = \sum_{k=0}^{\infty}\int_{F(k)}^{1} y^{n-1}dy \leq \frac{1}{n}\sum_{k=0}^{\infty}[1-F(k)] < \infty.$$

Thus the series in the middle converges uniformly in n. Upon letting $n \to \infty$ each integral tends to zero and (2.46) is proved. So

$$d_2(B_n,0) \leq E(B_n^2) \longrightarrow 0. \tag{2.47}$$

On the other hand, by Minkowski Inequality and by decomposition (2.40) we obtain the convergence in (2.42). Instead,

$$d_2\left(\frac{S_n - n\tilde{\mu}_\varphi}{\sigma\sqrt{n}}, Z_0\right) \le d_2(A_n, 0) + d_2\left(\frac{V_{N_n-1}}{\tilde{\sigma}_\varphi\sqrt{n}}, Z_0\right) + d_2(B_n, 0) \longrightarrow 0 \ as \ n \to \infty.$$

Since $E\left\{\left(\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}}\right)^2\right\} < \infty$ by Theorem (1.5.1) we have the two convergences in (2.43). This concludes the proof. □

The following theorem generalizes the previous result for $r > 2$.

Theorem 2.4.6. *Assume that Condition 2.4.4 is satisfied and let* $Z_0 \stackrel{d}{=} N(0,1)$. *Then as* $n \to \infty$,

$$d_r\left(\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}}, Z_0\right) \to 0. \tag{2.48}$$

Moreover,

$$\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}} \xrightarrow{d} Z_0 \ \ and \ \ E\left\{\left|\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r\right\} \longrightarrow E\left\{|Z_0|^r\right\}. \tag{2.49}$$

Proof. To obtain the two convergences in (2.49) we will verify the conditions of Theorem (1.5.1), i.e., we will show that (2.48) is valid and

$$E\left\{\left|\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r\right\} < \infty. \tag{2.50}$$

In fact, by decomposition (2.40) and Minkowsky inequality we have

$$\left\|\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}}\right\|_r \le \|A_n\|_r + \left\|\frac{V_{N_n-1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right\|_r + \|B_n\|_r. \tag{2.51}$$

In (2.28) we proved that $\left\{\left|\frac{V_{N_n-1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r, n \ge 1\right\}$ is uniformly integrable. Then

$$\left\|\frac{V_{N_n-1}}{\tilde{\sigma}_\varphi\sqrt{n}}\right\|_r < \infty. \tag{2.52}$$

On the other hand

$$E\left\{|A_n|^r\right\} = \frac{1}{\tilde{\sigma}_\varphi^r n^{r/2}} E\left\{\left|\sum_{k=0}^{T_1-1}(\varphi(X_k) - \tilde{\mu}_\varphi)\right|^r\right\}$$

Since $E\left|\sum_{k=0}^{T_1-1}(\varphi(X_k) - \tilde{\mu}_\varphi)\right|^r$ does not depend on n and is finite follows that

$$d_r^r(A_n, 0) \leq E(|A_n|^r) \longrightarrow 0. \tag{2.53}$$

As in the previous proof, consider the i.i.i random variables $U_k = \sum_{j=T_k}^{T_{k+1}-1} |\varphi(X_k) - \tilde{\mu}_\varphi|$ for $k \geq 1$. By Condition 2.4.4 we have $E\{|U_k|^r\} < \infty \; for \; k \geq 1$. Since $T_{N_n} \leq n < T_{N_n+1}$ and $1 \leq N_n \leq n$ and by (2.46) we have

$$\begin{aligned}
E\{|B_n|^r\} &= E\left\{ \frac{\left|\sum_{k=T_{N_n}}^{n} (\varphi(X_k) - \tilde{\mu}_\varphi)\right|^r}{\tilde{\sigma}_\varphi^r n^{r/2}} \right\} \\
&\leq E\left\{ \frac{\left(\sum_{k=T_{N_n}}^{n} |\varphi(X_k) - \tilde{\mu}_\varphi|\right)^r}{\tilde{\sigma}_\varphi^r n^{r/2}} \right\} . \\
&\leq \frac{1}{\tilde{\sigma}_\varphi^r n^{r/2}} E\left\{U_{N_n}^r\right\} \\
&\leq \frac{1}{\tilde{\sigma}_\varphi^r n^{r/2-1}} \frac{1}{n} E\left\{ \max_{1\leq m\leq n} U_m^r \right\} \longrightarrow 0.
\end{aligned}$$

Since $r \geq 2$ and $E\{|U_k|^r\} < \infty$ the last convergence follows by (2.46). Thus

$$d_r^r(B_n, 0) \leq E(|B_n|^r) \longrightarrow 0. \tag{2.54}$$

Thus, the finitude of the r-th moment in (2.50) follows from (2.52),(2.53) and (2.54). To obtain convergence in (2.48) we can use the same argument used for case $r = 2$ and so convergence in (2.49) follows from Theorem (1.5.1). □

If φ is bounded and $E\{(T_2 - T_1)^r\} < \infty$ for some $r \geq 2$ then the condition 2.4.4 holds. So, from last Theorem we obtain the following result.

Corollary 2.4.7. *If φ is bounded and $E\{(T_2 - T_1)^r\} < \infty$, $r \geq 2$ then as $n \to \infty$,*

$$d_r\left(\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}}, Z_0\right) \to 0.$$

Moreover,

$$\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}} \xrightarrow{d} Z_0 \;\; and \;\; E\left\{\left|\frac{S_n - n\tilde{\mu}_\varphi}{\tilde{\sigma}_\varphi\sqrt{n}}\right|^r\right\} \longrightarrow E\{|Z_0|^r\}.$$

2.5 Strong Approximation

Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. centred real-valued random variables with a finite moment generating function in a neighbourhood of 0 and let $\sigma^2 = VarX_1$ and $S_n = X_1 + X_2 + \cdots + X_n$. Komlós−Major−Tusnády Theorem (1975 and 1976) proved that one can construct a standard Wiener process $\{W(t)\}_{t\geq 0}$ in such a way that

$$P\left(\sup_{k\leq n} |S_k - \sigma W(k)| > c\log n + x\right) \leq a\exp(-bx), \tag{2.55}$$

where a, b and c are positive constants depending only on the distribution of X_1. From this result, the almost sure approximation of the partial sum process by a Wiener process holds with the rate $O(\log n)$.

The Komlós−Major−Tusnády Theorem is one of the most important in probability approximations because many well known probability theorems can be considered as consequences of results about strong approximation of sequences of sums by corresponding Gaussian sequences. Due to the powerful consequences of KMT approximation (see, e.g., Csorgo and Hall (1984) or the books of Csorgo and Révész (1981) and Shorack and Wellner (1986) for its applications), extending these results for dependent random variables would have a great importance but the dyadic construction of Komlós, Major and Tusnády is highly technical and utilizes conditional large deviation techniques, which makes it very difficult to extend to dependent processes.

In this section, we present a version for regenerative sequences of KMT approximation obtained in the paper "Strong approximation for additive functionals of geometrically ergodic Markov chains" by Merlevede and Rio (2015). This adaptation was possible because the authors used the fact that an irreducible and aperiodic Harris recurrent Markov chain is regenerative. Thus, the chain can be divided into i.i.d blocks. and then it is possible to apply known approximations.

As in the previous sections, consider $\{X_n\}_{n\geq 0}$ be a regenerative sequence on $(S, \mathcal{G})$ with regeneration times $\{T_n\}_{n\geq 0}$ and $\varphi : S \longrightarrow \mathbb{R}$ a measurable function. Let $\tilde{\mu}_\varphi = E_{\tilde{\pi}}\{\varphi\}$, $S_n = \sum_{j=1}^{n}(\varphi(X_j) - \tilde{\mu}_\varphi)$, $Y_k = \sum_{j=T_k}^{T_{k+1}-1}(\varphi(X_j) - \tilde{\mu}_\varphi)$ *and* $N_n = \sup\{k : T_k \leq n < T_{k+1}\}$.

Theorem 2.5.1 (KMT approximation for Regenerative Sequences)**.** *Let* $\{X_n\}_{n\geq 0}$ *be an aperiodic and positive recurrent regenerative sequence with regeneration times* $\{T_n\}_{n\geq 0}$ *such that* $E\{(T_2 - T_1)\}^2 < \infty$. *Assume that* $Var(Y_k) = E\left\{\left(\sum_{j=T_1}^{T_2-1}(\varphi(X_j) - \tilde{\mu}_\varphi)\right)^2\right\} > 0$ *and* $\varphi : S \longrightarrow \mathbb{R}$ *is bounded. Suppose there is* $\delta > 0$ *such that* $E(e^{t(T_2-T_1)}) < \infty$ *for any* $|t| < \delta$. *Then, there exists a standard Wiener process* $(W(t))_{t\geq 0}$ *and positive constants* a, b *and* c *depending of* φ *such that, for any* $x > 0$ *and any integer* $n \geq 2$,

$$P\left(\sup_{1\leq k\leq n}|S_k - \tilde{\sigma}_\varphi W(k)| \geq c\log n + x\right) \leq a\exp(-bx). \tag{2.56}$$

where $\tilde{\sigma}_\varphi^2 = \dfrac{Var(Y_k)}{\mu_T}$.

The following corollary is an immediate consequence of this theorem.

Corollary 2.5.2. *For* S_n *and* $W(t)$ *given in Theorem 2.5.1*

$$S_n - \tilde{\sigma}_\varphi W(n) = O(\log n) \;\; a.s. \tag{2.57}$$

Proof. If $x = \dfrac{2\log n}{b}$ in (2.56) we have for $C = c + \dfrac{2}{b}$

$$\begin{aligned}\sum_{n=1}^{\infty} P\left(\frac{|S_n - \tilde{\sigma}_\varphi W(n)|}{\log n} > C\right) &\leq \sum_{n=1}^{\infty} P\left(\sup_{1\leq k\leq n}|S_n - \tilde{\sigma}_\varphi W(n)| > c\log n + \frac{2\log n}{b}\right)\\ &\leq a\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty\end{aligned}$$

Thus by Borel-Cantelli lemma we obtain

$$P\left(\limsup_{n\to\infty}\frac{|S_n - \tilde{\sigma}_\varphi W(n)|}{\log n} \leq C\right) = 1.$$

□

Proof Theorem 2.5.1 . First, note that is sufficient to show (2.56) for any real positive x such that $x \leq 4n\|\varphi\|_\infty$. Indeed, consider $x > 4n\|\varphi\|_\infty$. Since φ is bounded we have that

$|S_k| \leq \sum_{j=1}^{k} |(\varphi(X_j)| + |\tilde{\mu}_\varphi|) \leq 2k \|\varphi\|_\infty$ for any integer $k \geq 0$. On the other hand, for any standard Wiener process $(W(t))_{t\geq 0}$ we have $P\{W(1) > x\} \leq \dfrac{1}{\sqrt{2\pi}} \dfrac{1}{x} e^{-x^2/2}$ and by Lévi's inequality

$$P\left(\max_{1\leq k\leq n} |W(k)| \geq x\right) \leq 2P\left(|W(n)| \geq x\right).$$

This implies that

$$\begin{aligned}
P\left(\sup_{1\leq k\leq n} |S_k - \tilde{\sigma}_\varphi W(k)| \geq c\log n + x\right) &\leq P\left(\sup_{1\leq k\leq n} \{|S_k| + |\tilde{\sigma}_\varphi W(k)|\} \geq c\log n + x\right) \\
&\leq P\left(\sup_{1\leq k\leq n} |\tilde{\sigma}_\varphi W(k)| \geq c\log n + x - 2n\|\varphi\|_\infty\right) \\
&\leq P\left(\sup_{1\leq k\leq n} |\tilde{\sigma}_\varphi W(k)| \geq c\log n + x - \frac{x}{2}\right) \\
&\leq P\left(\sup_{1\leq k\leq n} |\tilde{\sigma}_\varphi W(k)| \geq \frac{x}{2}\right) \\
&\leq 2P\left(|\tilde{\sigma}_\varphi W(n)| \geq \frac{x}{2}\right) \\
&= 2P\left(|W(1)| \geq \frac{x}{2\tilde{\sigma}_\varphi \sqrt{n}}\right) \\
&\leq \frac{2\sqrt{2}}{\sqrt{\pi}} \times \frac{\tilde{\sigma}_\varphi \sqrt{n}}{x} \exp\left\{-\frac{x^2}{8\sigma^2 n}\right\} \\
&\leq \frac{\sqrt{2}\tilde{\sigma}_\varphi}{\|\varphi\|_\infty \sqrt{\pi n}} \exp\left\{-\frac{x\|\varphi\|_\infty}{2\tilde{\sigma}_\varphi^2}\right\} \\
&\leq \frac{2\sqrt{2}\tilde{\sigma}_\varphi}{\|\varphi\|_\infty \sqrt{\pi}} \exp\left\{-\frac{x\|\varphi\|_\infty}{2\tilde{\sigma}^2}\right\}.
\end{aligned}$$

Therefore, we will consider $x \leq 4n\|\varphi\|_\infty$. Recall that

$$Y_k = \sum_{j=T_{k-1}}^{T_k - 1} (\varphi(X_j) - \tilde{\mu}_\varphi) \ \textit{and} \ \tau_k = T_k - T_{k-1},$$

are i.i.d. sequences and from this notation we have that $\sum_{j=1}^{k} Y_j = S_{T_k - 1} + (\varphi(X_0) - \tilde{\mu}_\varphi)$. On the other hand, $Var(\tau_1) > 0$ because we are assuming that $\{X_n\}_{n\geq 0}$ is a aperiodic regenerative sequence. Thus, $\{(\tau_k, Y_k - \alpha(\tau_k - E(\tau_k))\}_{k\geq 1}$ with $\alpha = \dfrac{Cov(\tau_1, Y_1)}{Var(\tau_1)}$ is a sequence of i.i.d. random vectors in $\mathbb{R}^2$ with $E(Y_k) = 0$ and $Cov(\tau_k, Y_k - \alpha(\tau_k - E(\tau_k))) = 0$.
By hypotheses $E(e^{t\tau_1}) < \infty$ for $|t| \leq \delta$, so

$$E\left(e^{t(Y_1 - \alpha(\tau_1 - E(\tau_1)))}\right) \leq e^{t\alpha E(\tau_1)} E\left(e^{t(2\|\varphi\|_\infty + |\alpha|)\tau_1}\right) < \infty \ \text{ for } \ |t| \leq \delta(2\|\varphi\|_\infty + |\alpha|)^{-1}.$$

Taking into account all the considerations above mentioned, we can apply Theorem 1.3 in Zaitsev (1998) (which is the multidimensional extension of the results of Komlós, Major and

Tusnády (1976)) to the multivariate sequence of independent and identically distributed random variables $\{\tau_k, Y_k - \alpha(\tau_k - E(\tau_k))\}_{k\geq 1}$. Therefore, there exists a sequence $(\tilde{Y}_i, Z_i)_{i\geq 1}$ in $\mathbb{R}^2$ of independent random variables such that $(\tilde{Y}_i)_{i\geq 1}$ is independent of $(Z_i)_{i\geq 1}$ and

$$\tilde{Y}_i \overset{D}{=} N(0, v^2), \;\; Z_i \overset{D}{=} N(0, Var(\tau_1)) \;\; where \;\; v^2 = Var(Y_1 - \alpha(\tau_1 - E(\tau_1))),$$

and satisfying, for some positive constants C_1, A_1 e B_1 depending on φ, the following inequalities: for any integer $n \geq 2$.

$$\begin{aligned} & P\left(\sup_{1\leq k\leq n}\left|\sum_{i=1}^{k}[Y_i - \alpha(\tau_i - E(\tau_i))] - \sum_{i=1}^{k}\tilde{Y}_i\right| \geq C_1 \log n + x\right) \\ = \;& P\left(\sup_{1\leq k\leq n}\left|S_{T_k-1} + (\varphi(X_0) - \tilde{\mu}_\varphi) - \alpha(T_k - kE(\tau_1)) - \sum_{i=1}^{k}\tilde{Y}_i\right| \geq C_1 \log n + x\right) \\ \leq \;& A_1 \exp(-B_1 x) \end{aligned} \tag{2.58}$$

and

$$\begin{aligned} & P\left(\sup_{1\leq k\leq n}\left|\sum_{i=1}^{k}[\tau_i - E(\tau_i)] - \sum_{i=1}^{k} Z_i\right| \geq C_1 \log n + x\right) \\ = \;& P\left(\sup_{1\leq k\leq n}\left|T_k - kE(\tau_1)) - \sum_{i=1}^{k} Z_i\right| \geq C_1 \log n + x\right) \\ \leq \;& A_1 \exp(-B_1 x). \end{aligned} \tag{2.59}$$

Using the Skorohod embedding theorem, we can then construct two independent standard Wiener processes $\{B(t)\}_{\geq 0}$ and $\left\{\tilde{B}(t)\right\}_{t\geq 0}$ such that for any positive integer k,

$$vB(k) = \sum_{i=1}^{k}\tilde{Y}_i \;\; and \;\; \sqrt{Var(\tau_1)}\tilde{B}(k) = \sum_{i=1}^{k} Z_i.$$

Thus, by (2.58) and (2.59) and using the same argument in the proof of Corollary 2.5.2 we obtain

$$S_{T_{n-1}} + \varphi(X_0) - \tilde{\mu}_\varphi - \alpha(T_n - nE(\tau_1)) - vB(n) = O(\log n) \; a.s.$$

and

$$T_n - nE(\tau_1 - \sqrt{Var(\tau_1)}\tilde{B}(n) = O(\log n) \; a.s.$$

Next, since a Poisson process is a partial sum process associated to i.i.d. random variables with exponential law, using the Komlós, Major and Tusnády strong approximation Theorem, we can construct a Poisson process $N(t)$ with parameter $\lambda = \dfrac{(E(\tau_1))^2}{Var(\tau_1)}$ from $\tilde{B}(t)$ in such a way that

$$nE(\tau_1) + \sqrt{Var(\tau_1)}\tilde{B}(n) - \gamma N(n) = O(\log n) \ \ a.s. \tag{2.60}$$

and the processes $B(t)$ e $N(t)$ are independent. From previous strong approximations we can show that there are two independent standard Wiener processes $W^*(t)$ and $\tilde{W}(t)$ such that

$$S_n = W^*(n) + \tilde{W}(n) + O(\log n) \ \ a.s. \tag{2.61}$$

Let $W(t) = W^*(n) + \tilde{W}(n)$. Since $W(t)$ is a Wiener process, (2.61) implies the strong approximation in (2.56). The proofs of approximations (2.60) and (2.61) are too technical and too extensive. These proofs are made in detail in the proof of Theorem 1.1. in Merlevede and Rio (2015). □

Remark 2.5.1. *If there exists $\delta > 0$ such that $E(e^{tT_0}) < \infty$ for any $|t| < \delta$, then the Theorem 2.5.1 is also valid for delayed regenerative sequences. See Theorem 1.1. in Merlevede and Rio (2015).*

Chapter 3

Weak Convergence

3.1 Introduction

Let $\{X_n\}_{n\geq 0}$ be an aperiodic and a positive recurrent regenerative sequence with values in $\mathbb{R}$ and regeneration times $\{T_n\}_{n\geq 0}$ such that $\mu_T = E(T_2 - T_1) > 0$. Consider the canonical measure

$$\tilde{F}(x) = \frac{1}{\mu_T} E\left(\sum_{j=T_1}^{T_2-1} I_{(-\infty,x]}(X_j)\right). \tag{3.1}$$

Define the empirical distribution function $F_n(x), x \in \mathbb{R}$ and the empirical process $\beta_n(x), x \in \mathbb{R}$ by

$$F_n(x) = \frac{1}{n}\sum_{j=1}^{n} I_{(-\infty,x]}(X_j) \;\; , x \in \mathbb{R},\; n \geq 1, \tag{3.2}$$

$$\beta_n(x) = \sqrt{n}(F_n(x) - \tilde{F}(x)), x \in \mathbb{R}. \tag{3.3}$$

The empirical process plays a prominent role in non-parametric statistical inference. In all statical applications, information about the distribution of the empirical processes is needed. One says that a process $\beta_n(x)$ satisfies an invariance principle if it converges weakly to a mean-zero Gaussian process.

For the case of i.i.d. observations, Donsker proved in 1952 that empirical process converges in distribution to a Brownian bridge process but this is not always the case for dependent variables. Donsker's result has been extended to sequences of weakly dependent random variables by many authors. Among others, it shall be remarked that Billingsley (1968) gave a result for functionals of $\phi-$mixing process, Berkes and Philipp (1977/78) under strong mixing assumptions, Doukhan, Massart and Rio (1995) for absolutely regular sequences, Borovkova, Burton and Dehling (2001) for functionals of absolutely regular processes, Dedecker and Prieurd (2007) for new dependence coefficients, Shao and Yu (1996) for mixing and associated processes, Dehling, Durieu and Volny (2009) for Markov chains and dynamical systems.

In this chapter, we study weak convergence in the Skorokhod space D of the empirical and empirical quantile processes associated to an aperiodic and a positive recurrent regenerative sequence $\{X_n\}_{n\geq 0}$ with regeneration times $\{T_n\}_{n\geq 0}$. More explicitly, we obtain an invariance principle for regenerative processes using alternative techniques such as the Mallows distance for the empirical case, and Skorokhod's Representation Theorem and properties of locally uniformly aproximation of monotone functions (Lemma 1.8.6), for the empirical quantile case.

In section 3.3, under certain regularity conditions, our Theorem 3.3.5 shows that the empirical process $\beta_n(x)$ converges weakly to a zero-mean Gaussian process $\tilde{B}_{\tilde{F}}(x) = \left\{\tilde{B}(\tilde{F}(x)) : x \in \mathbb{R}\right\}$ with covariance function given by

$$\begin{aligned} E(\tilde{B}_{\tilde{F}}(x), \tilde{B}_{\tilde{F}}(y)) &= \tilde{F}(x \wedge y) - \tilde{F}(x)\tilde{F}(y) \\ &+ \sum_{j=1}^{\infty} E\left\{I_{(-\infty,x]}(X_0) - \tilde{F}(x), I_{(-\infty,y]}(X_j) - \tilde{F}(y)\right\} \\ &+ \sum_{j=1}^{\infty} E\left\{I_{(-\infty,y]}(X_0) - \tilde{F}(y), I_{(-\infty,x]}(X_j) - \tilde{F}(x)\right\}. \end{aligned} \tag{3.4}$$

Proofs of invariance principles usually consist of two parts, establishing finite dimensional convergence and tightness of the empirical process. Since we can write

$$\beta_n(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left\{I_{(-\infty,x]}(X_j) - \tilde{F}(x)\right\}.$$

then we apply the results obtained in Chapter 2 to study the process $\beta_n(x)$. In our Theorem 3.3.3 we use convergence in Mallows distance to obtain the finite dimensional convergence, i.e. convergence in distribution of the sequence of vectors $(\beta_n(x_1), ..., \beta_n(x_k))_{n\geq 1}$ to $\Big(\tilde{B}_{\tilde{F}}(x_1), \tilde{B}_{\tilde{F}}(x_2), \cdots, \tilde{B}_{\tilde{F}}(x_k)\Big)$,

Tightness is far more difficult to establish. One ingredient is usually a probability bound on the increments of the empirical process $\beta_n(t)$. In this sense, by Theorem 3.2.9 we have that a regenerative sequence is $\alpha-$mixing (or strong mixing) then we can use Rosenthal-type inequality for $\alpha-$mixing (Theorem 1.6.3) to obtain an estimate for the moments of the increments of the empirical process. So, tightness follows from Shao and Yu's tightness criterion (Theorem 3.2.7) in the same way as in the proof of Theorem 2.2. in Shao and Yu (1996).

At the end of section 3.3, our Theorem 3.3.8 present an alternative invariance principle for regenerative sequences substituting mixing conditions for the condition that inter-regeneration times (or length of the cycles) of the regenerative sequence be bounded. This result is interesting because we show tightness of $\beta_n(\cdot)$ without to use estimates of the mixing theory, we obtain moment bounds for partial sums of regenerative sequences using definitions and properties we studied in Chapter 2.

Once the weak convergence of the empirical process is obtained, the next logical step is to prove the weak convergence for the empirical quantile process associated to regenerative process. Theory and important results related to the empirical quantile process in the i.i.d. case can be studied in Shorack and Wellner (1986) and Csörgő and Révész (1981), among other references. In this sense, we extend some known results for i.i.d. data.

In section 3.4, we study weak convergence of the empirical quantile process $q_n(t) = \sqrt{n}(F_n^{-1}(t) - \tilde{F}^{-1}(t)), t \in (0,1)$, associated to an aperiodic and positive recurrent regenerative sequence $\{X_n\}_{n\geq 0}$. First, for fixed $t \in (0,1)$ in our Theorem 3.4.1 we obtain the convergence in distribution of the uniform quantile process and then in our Theorem 3.4.3 we prove that for t fixed the quantile process $q_n(t)$ converges in distribution to the random variable $-\dfrac{\tilde{B}_{\tilde{F}}(t)}{\tilde{f}(\tilde{F}^{-1}(t))}$, where $\tilde{f}(t) = \dfrac{d\tilde{F}(t)}{dt}$. Next, in our Theorems 3.4.4 and 3.4.5 we establish the weak convergence of the uniform quantile process and of the process $q_n(t)$ in the Skorokhod space D.

In the i.i.d. case these results can be shown using the Delta method which is inconsistent with our theory. Thus, our arguments are based on properties of locally uniformly aproximation of monotone functions (Lemma 1.8.6) together with the Skorokhod Theorem (Theorem 1.8.5) with the same approach adopted in Vervaat (1971), Haan and Ferreira (2006) and Resnick (2007). We also use the Bahadur representation for quantiles of $\alpha-$mixing samples obtained by Xing, Yang, Liu et al. (2012).

Finally, by Example 2.2.3 we have that any Harris chains $\{X_n\}_{n\geq 1}$ on a general state space that possess an atom is regenerative and by Remark 2.2.1 if $\{X_n\}_{n\geq 1}$ is aperiodic and recurrent positive with limiting distribution F_{lim} then $F_{lim} = \tilde{F}$. So, it is worth pointing out that all the results obtained in this chapter can be applied for this type of Markov chains.

3.2 Auxiliary Results

We mentioned in the introduction of this chapter that some authors have studied principles of invariance for samples with some type of dependence. In this subsection we present an invariance principle for stationary processes obtained by Dehling, Durieu and Volny in 2009, which can be apply to a large class of Markov chains under some assumptions on the Markov transition function , namely geometrically ergodic Markov chains. We also present a result about weak convergence for empirical processes of strong mixing sequences by Shao and Yu (1996). Finally, we study mixing conditions of regenerative processes which will allow us to establish conditions for an invariance principle for this type of process.

3.2.1 Invariance principle for stationary processes

Dehling, Durieu and Volny (2009) proposed a new technique to obtain an invariance principle for stationary processes. They developed an approach that is strictly based on properties of Lipschitz functions $\varphi(X_i)$ the original data $\{X_i\}_{i\geq 0}$. More precisely, they made two assumptions, namely that the partial sums of Lipschitz functions satisfy the CLT and that a suitable fourth-moment bound is satisfied. Thus, to prove a principle of invariance over these conditions they replaced the usual finite dimensional convergence plus tightness approach by a method of approximation by a sequence of finite dimensional processes. This method is different from the traditional methods and requires the following assumptions.

Let $\{X_n\}_{n\geq 0}$ be a stationary ergodic process of $\mathbb{R}$-valued random variables with marginal distribution function $F(x) = P(X_0 \leq x)$ satisfying the following condition

Condition 3.2.1. *i) For any Lipschitz function φ, the CLT holds, i.e.*

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\{\varphi(X_j) - E\varphi(X_j)\} \xrightarrow{d} Z \overset{d}{=} N(0,\sigma^2) \text{ as } n \to \infty, \tag{3.5}$$

where

$$\sigma^2 = E(\varphi(X_0) - E\varphi(X_0))^2 + 2\sum_{j=1}^{\infty} Cov(\varphi(X_0), \varphi(X_j)).$$

ii) A bound on the fourth central moments of partial sums of $\{\varphi(X_n)\}_{n\geq 0}$, φ bounded Lipschitz with $E(\varphi(X_0)) = 0$, of the type

$$\begin{aligned} E\left\{\left(\sum_{j=1}^{n}\varphi(X_j)\right)^4\right\} &\leq Cm_\varphi^3\big(n\,\|\varphi(X_0)\|_1 \log^\alpha(1+\|\varphi\|) \\ &+ n^2\,\|\varphi(X_0)\|_1^2 \log^\beta(1+\|\varphi\|)\big), \end{aligned} \tag{3.6}$$

where C is some universal constant,α and β are some nonnegative integers,

$$\|\varphi\| = \sup_x |\varphi(x)| + \sup_{x\neq y}\frac{|\varphi(x) - \varphi(y)|}{|x-y|} \tag{3.7}$$

and

$$m_\varphi = \max\left\{1, \sup_x |\varphi(x)|\right\}.$$

As before, define the empirical distribution function $F_n(x)$ and the empirical process $\beta_n(x)$ by

$$F_n(x) = \frac{1}{n}\sum_{j=1}^{n} I_{(-\infty,x]}(X_j),\ x\in\mathbb{R},$$

$$\beta_n(x) = \sqrt{n}(F_n(x) - F(x)),\ x \in \mathbb{R}.$$

Consider the modulus of continuity of a function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$

$$\omega_\varphi(\delta) = \sup\left\{|\varphi(x) - \varphi(y)| : x, s \in \mathbb{R}, |x - y| < \delta\right\}. \tag{3.8}$$

Theorem 3.2.2. *[Dehling, Durieu and Volny (2009)] Let $\{X_n\}_{n\geq 0}$ be an $\mathbb{R}$-valued stationary ergodic random process such that the condition 3.2.1 holds. Assume that X_0 has a distribution function F satisfying the following condition:*

$$\omega_F(\delta) \leq D|\log(\delta)|^{-\gamma} \ \textit{for some}\ D > 0 \ \textit{and}\ \gamma > \max\left\{\frac{\alpha}{2}, \beta\right\}. \tag{3.9}$$

Then

$$\beta_n(x) \Rightarrow B^*(x) \ \textit{in}\ D(-\infty, \infty), \tag{3.10}$$

where $B^*(x)$ is a mean-zero Gaussian process with covariances

$$\begin{aligned} E(B^*(x), B^*(y)) &= F(x \wedge y) - F(x)F(y) \\ &+ \sum_{j=1}^{\infty} E\left\{I_{(-\infty,x]}(X_0) - F(x), I_{(-\infty,y]}(X_j) - F(y)\right\} \\ &+ \sum_{j=1}^{\infty} E\left\{I_{(-\infty,y]}(X_0) - F(y), I_{(-\infty,x]}(X_j) - F(x)\right\}. \end{aligned} \tag{3.11}$$

The assumptions of last theorem can be verified for a large class of Markov chains under some assumptions on the Markov transition operator. Let (S, d) be a separable metric space and $\{X_n\}_{n\geq 0}$ be a homogeneous and S-valued Markov chain with stationary measure ν and transition function P. Denote by $\mathcal{L}$ the space of all bounded Lipschitz continuous functions from S to $\mathbb{R}$ equipped with the norm defined in (3.7).

Definition 3.2.1. *The Markov chain $\{X_n\}_{n\geq 0}$ is $\mathcal{L}$ -geometrically ergodic or strongly ergodic if there exist $C > 0$ and $0 < \theta < 1$ such that for all $\varphi \in \mathcal{L}$,*

$$\|P^n\varphi - \Pi\varphi\| \leq C\theta^n \|\varphi\|$$

where $\Pi\varphi = E_\nu\varphi(X_0)$.

The invariance principle for $\mathcal{L}$-geometrically ergodic Markov chains is a consequence of the following statements in Durieu (2008) and of Theorem 3.2.2.

Proposition 3.2.3. *[Durieu (2008), Corollary 2] If $\{X_n\}_{n\geq 0}$ is $\mathcal{L}$-geometrically ergodic Markov chain then (3.6) holds for all $\varphi \in \mathcal{L}$ such that $E\varphi(X_0) = 0$, with $\alpha = 3$ and $\beta = 2$.*

Proposition 3.2.4. *[Durieu (2008), Proposition 3] If $\{X_n\}_{n\geq 0}$ is ergodic and $\mathcal{L}$-geometrically ergodic Markov chain, then the CLT given by (3.5) holds for all $\varphi \in \mathcal{L}$.*

Corollary 3.2.5. *[Corollary of Theorem 3.2.2] Let $\{X_n\}_{n\geq 0}$ be an $\mathcal{L}$-geometrically ergodic Markov chain with values in $\mathbb{R}$. Assume that the distribution function F of X_0 satisfies*

$$\omega_F(\delta) \leq D|\log(\delta)|^{-\gamma} \ for \ some \ D > 0 \ and \ \gamma > 2. \tag{3.12}$$

Then the empirical process associated with the Markov chain $\{X_n\}_{n\geq 0}$ satisfies the invariance principle of Theorem 3.2.2.

For more details and concrete examples of the results above see Section 4 in Dehling, Durieu and Volny (2009).

On the other hand, Shao and Yu (1996) established weak convergence theorems for empirical processes of strong mixing, ρ- mixing and associated sequences. Below we present this results for stationary strong mixing sequences because regenerative processes satisfy this conditions. First we recall definition of strong mixing or $\alpha-$mixing sequence.
Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_1$ and $\mathcal{F}_2$ be two σ-algebras contained in $\mathcal{F}$. Define the following measures of dependence between $\mathcal{F}_1$ and $\mathcal{F}_2$:

$$\alpha(\mathcal{F}_1, \mathcal{F}_2) = \sup_{A\in\mathcal{F}_1, B\in\mathcal{F}_2} |P(A\cap B) - P(A)P(B)|.$$

Let $\{X_n\}_{n\geq 1}$ be a sequence of real-valued random variables on $(\Omega, \mathcal{F}, P)$ and let $\mathcal{F}_n^m = \sigma(X_i, n \leq i \leq m)$ be σ-algebras generated by the indicated random variables and put

$$\alpha(n) = \sup_{k\geq 1} \alpha(\mathcal{F}_1^k, \mathcal{F}_{n+k}^\infty).$$

The sequence $\{X_n\}_{n\geq 1}$ is said to be $\alpha-$mixing (or strong mixing), according as $\alpha(n) \to \infty$.

Theorem 3.2.6. *[Shao and Yu (1996), Theorem 2.2.] Let $\{U_n\}_{n\geq 1}$ be a stationary $\alpha-$mixing sequence of uniform $[0,1]$ random variables. If*

$$\alpha(n) = O(n^{-\theta-\epsilon}) \ for \ some \ \theta \geq 1 + \sqrt{2} \ and \ \epsilon > 0,$$

then we have

$$u_n(t) \Rightarrow B^*(t) \text{ in } D[0,1].$$

where $u_n(t)$ *is the uniform empirical process of* $U_1, ..., U_n$ *and* $B^*(t)$ *is a zero-mean Gaussian process specified by* $B^*(0) = B^*(1) = 1$ *and*

$$\begin{aligned} Cov(B^*(s), B^*(t)) &= s \wedge t - st \\ &+ \sum_{j=2}^{\infty} Cov\left\{I_{[0,s]}(U_1), I_{[0,t]}(U_j)\right\} \\ &+ \sum_{j=2}^{\infty} Cov\left\{I_{[0,t]}(U_1), I_{[0,s]}(U_j)\right\}. \end{aligned} \tag{3.13}$$

The key point to establishing the last weak convergence was the Rosenthal-type inequality for $\alpha-$mixing sequences (1.16) because it allows to obtain an estimate of type (3.14) and then to use the following Shao and Yu's tightness criterion for the empirical process in the space $D[0,1]$.

Theorem 3.2.7. *[Shao and Yu (1996)] Let* $\{U_n\}_{n\geq 1}$ *be a stationary sequence of uniform* $[0,1]$ *random variables and let* $u_n(t)$ *is the uniform empirical process of* $U_1, ..., U_n$*. If there exist constants* $C > 0, p > 2, p_1 > 1, 0 \leq p_3 \leq 1, p_2 > 1 - p_3$ *such that for any* $s, t \in [0,1]$ *and* $n \geq 1$ *the following inequality holds*

$$|u_n(t) - u_n(s)|^p \leq C\left(|t-s|^{p_1} + n^{-p_2/2}|t-s|^{p_3}\right) \tag{3.14}$$

then the process $u_n(t)$ *is tight in* $D[0,1]$.

3.2.2 Mixing conditions for regenerative processes

Let $\{X_n\}_{n\geq 0}$ be an aperiodic positive recurrent regenerative sequence on $(S, \mathcal{G})$ with regeneration times $\{T_n\}_{n\geq 0}$ such that $\mu_T = E(T_2 - T_1) > 0$ and let $\varphi : S \times S \times S \cdots \longrightarrow \mathbb{R}$ be a bounded function. By definition of regenerative process we have that $\{T_n\}_{n\geq 0}$ is a renewal process i.e.,

$$\tau_1 = T_1 - T_0,\ \tau_2 = T_2 - T_1,\ \tau_3 = T_3 - T_2,\ \cdots,$$

are i.i.d. random variables and $T_n = \tau_1 + \tau_1 + \cdots \tau_n$. As before, let N_n be the random variable of the number of regenerations by time n, i.e., $N_n = k$ if $T_k \leq n < T_{k+1},\ \ k = 0, 1, 2, ...$

In this subsection, for a better understanding we will rewrite some results shown in technical report "Some New Results in Regenerative Process Theory" by Glynn (1982). This results are based on the renewal theory applied to the sequence $\{T_n\}_{n\geq 0}$ and guarantee under some conditions that $\{X_n\}_{n\geq 0}$ is strong mixing.

Lemma 3.2.8. *Let $\{X_n\}_{n\geq 0}$ be an aperiodic positive recurrent regenerative with regeneration times $\{T_n\}_{n\geq 0}$. Then*

i) There exist constants $\gamma_n \longrightarrow 0$ as $n \to \infty$ such that

$$\sup_{\{\varphi:\|\varphi\|_\infty \leq 1\}} |E\{\varphi(X_{T_1+n}, X_{T_1+n+1}, ...)\} - E_{\pi^*}\{\varphi\}| = \gamma_n. \tag{3.15}$$

Moreover, if $E(T_2 - T_1)^k < \infty$ for some $k > 1$ then $\gamma_n = o(n^{1-k})$.

ii) Let $\mathcal{F}_k = \sigma(X_0, X_1, ..., X_k)$. For $\gamma(n) = \sup_{j\geq n} \gamma_j,\ n \geq 0$ we have

$$\begin{aligned} |\{\varphi(X_{k+n}, X_{k+n+1}, ...)|\mathcal{F}_k\} \quad &- \quad E_{\pi^*}\{\varphi\}| \\ &\leq \quad \|\varphi\|_\infty (1+\gamma(0))P(T_{N_k+1} - k > n/2|\mathcal{F}_k) + \|\varphi\|_\infty \gamma(n/2) \end{aligned} \tag{3.16}$$

where E^* denotes expectation with respect to the probability measure π^* given by

$$\pi^*(A) = \frac{1}{\mu_T} E\left\{\sum_{j=T_1}^{T_2-1} I_A(X_j, X_{j+1}, ...)\right\},\ A \in \mathcal{G}\times\mathcal{G}\times\mathcal{G}\cdots.$$

Proof. i) Let

$$\begin{aligned} a_n &= E\{\varphi(X_{T_1+n}, X_{T_1+n+1}, ...)\} \\ b_n &= E\{\varphi(X_{T_1+n}, X_{T_1+n+1}, ...)I(T_2 - T_1 > n)\}. \end{aligned} \tag{3.17}$$

By the regenerative property, the sequence $\{a_n\}_{n\geq 1}$ satisfies the renewal equation

$$a_n = b_n + \sum_{j=0}^{n} a_{n-j} P(T_2 - T_1 = j).$$

In fact,

$$\begin{aligned} a_n &= b_n + E\{\varphi(X_{T_1+n}, X_{T_1+n+1}, ...)I(T_2 - T_1 \leq n)\} \\ &= b_n + \sum_{j=0}^{n} E\{\varphi(X_{T_1+n}, X_{T_1+n+1}, ...)I(T_2 - T_1 = j)\} \\ &= b_n + \sum_{j=0}^{n} E\{\varphi(X_{T_1+n-j}, X_{T_1+n-j+1}, ...)\} P(T_2 - T_1 = j). \end{aligned}$$

Hence, since $T_k = \tau_1 + \tau_1 + \cdots \tau_n$ where $\tau_j = T_j - T_{j-1}$ from Theorem 1.4.3 follows

$$a_n = \sum_{j=0}^{n} b_j u_j \text{ where } u_j = P(T_k = j).$$

On the other hand, $|b_n| \leq \|\varphi\|_\infty P(T_2 - T_1 > n) \leq \|\varphi\|_\infty$, so, by Fubini's theorem and by (2.5),

$$\begin{aligned}
\sum_{j=0}^{\infty} b_j &= E\left\{\sum_{j=0}^{\infty} \varphi(X_{T_1+n}, X_{T_1+n+1}, ...) I(T_2 - T_1 > j)\right\} \\
&= E\left\{\sum_{j=T_1}^{T_2-1} \varphi(X_{T_1+n}, X_{T_1+n+1}, ...)\right\} \\
&= \mu_T E_{\pi^*}\{\varphi\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|E\{\varphi(X_{T_1+n}, X_{T_1+n+1}, ...)\} - E_{\pi^*}\{\varphi\}| &= \left| a_n - \frac{1}{\mu_T}\sum_{j=0}^{\infty} b_j \right| \\
&= \left| \sum_{j=0}^{n} b_j u_{n-j} - \frac{1}{\mu_T}\sum_{j=0}^{\infty} b_j \right| \\
&\leq \sum_{j=0}^{n} |b_j| \left(u_{n-j} - \frac{1}{\mu_T} \right) + \frac{1}{\mu_T} \sum_{j=n+1}^{\infty} |b_j| \\
&\leq \|\varphi\|_\infty \sum_{j=0}^{n} P(T_2 - T_1 > j) \left(u_{n-j} - \frac{1}{\mu_T} \right) \\
&+ \|\varphi\|_\infty \sum_{j=n+1}^{\infty} P(T_2 - T_1 > j). \qquad (3.18)
\end{aligned}$$

Let $\gamma_n = \sum_{j=0}^{n} P(T_2 - T_1 > j) \left(u_{n-j} - \frac{1}{\mu_T} \right) + \sum_{j=n+1}^{\infty} P(T_2 - T_1 > j)$ then $\gamma_n \to 0$ as $n \to \infty$. Indeed, by Theorem 1.4.2 i) $u_n \longrightarrow \frac{1}{\mu_T}$ as $n \to \infty$. Since $\mu_T = E(T_2 - T_1) < \infty$, given $\epsilon > 0$ there exists n such that $\sum_{j=n+1}^{\infty} P(T_2 - T_1 > j) < \frac{\epsilon}{2}$. Thus, by (3.18)

$$\sup_{\{\varphi: \|\varphi\|_\infty \leq 1\}} |E\{\varphi(X_{T_1+n}, X_{T_1+n+1}, ...)\} - E_{\pi^*}\{\varphi\}| = \gamma_n.$$

Now, if $E\left\{(T_2 - T_1)^k\right\} < \infty$ for some $k > 1$ from Theorem 1.4.2 ii) we have that

$$u_n = \frac{1}{\mu_T} + o(n^{1-k}).$$

This implies that

$$\sup_{j\geq n}\left|u_j-\frac{1}{\mu_T}\right|=o(n^{1-k}). \tag{3.19}$$

Substituting this relation in (3.18) we obtain

$$\begin{aligned}
|E\{\varphi(X_{T_1+n},X_{T_1+n+1},...)\}-E_{\pi^*}\{\varphi\}| &\leq \|\varphi\|_\infty\sum_{j=0}^{n}P(T_2-T_1>j)\sup_{k\geq n/2}\left(u_k-\frac{1}{\mu_T}\right)\\
&+ \|\varphi\|_\infty\sum_{j=n+1}^{\infty}P(T_2-T_1>j).\\
&\leq \|\varphi\|_\infty\sup_{k\geq n/2}\left(u_k-\frac{1}{\mu_T}\right)+\|\varphi\|_\infty\sum_{j=n/2}^{\infty}P(T_2-T_1>j)\\
&\leq \|\varphi\|_\infty\sup_{k\geq n/2}\left(u_k-\frac{1}{\mu_T}\right)\\
&+ \|\varphi\|_\infty n^{1-k}\sum_{j=n/2}^{\infty}j^{k-1}P(T_2-T_1>j)\\
&= \|\varphi\|_\infty(n^{1-k}o(1)+o(n^{1-k}))=\|\varphi\|_\infty o(n^{1-k}), \qquad (3.20)
\end{aligned}$$

the second-last equality is valid because $E\left\{(T_2-T_1)^k\right\}<\infty$. This completes the proof of i).

ii) Let $\hat{\varphi}=\varphi-E_{\pi^*}\{\varphi\}$. Then

$$\begin{aligned}
|E\{\hat{\varphi}(X_{k+n},X_{k+n+1},...)|\mathcal{F}_k\}| &\leq |E\{\hat{\varphi}(X_{k+n},X_{k+n+1},...)I(T_{N_k+1}\leq k+n)|\mathcal{F}_k\}|\\
&+ |E\{\hat{\varphi}(X_{k+n},X_{k+n+1},...)I(T_{N_k+1}>k+n)|\mathcal{F}_k\}|\\
&\leq |E\{\hat{\varphi}(X_{k+n},X_{k+n+1},...)I(T_{N_k+1}\leq k+n)|\mathcal{F}_k\}|\\
&+ \|\varphi\|_\infty P(T_{N_k+1}-k>n|\mathcal{F}_k)\\
&\leq |E\{\hat{\varphi}(X_{k+n},X_{k+n+1},...)I(T_{N_k+1}\leq k+n)|\mathcal{F}_k\}|\\
&+ \|\varphi\|_\infty P(T_{N_k+1}-k>n/2|\mathcal{F}_k). \qquad (3.21)
\end{aligned}$$

For the first term in the last inequality, we use i) and the regenerative property to obtain

$$
\begin{aligned}
&|E\left\{\hat{\varphi}(X_{k+n}, X_{k+n+1}, ...)I(T_{N_k+1} \leq k+n)|\mathcal{F}_k\right\}| \\
&= |E\left\{\hat{\varphi}(X_{k+n-T_{N_k+1}}, X_{k+n-T_{N_k+1}+1}, ...)\right\} E\left\{I(T_{N_k+1} - k \leq n)|\mathcal{F}_k\right\}| \\
&\leq \|\varphi\|_\infty \gamma(n+k-T_{N_k+1})P(T_{N_k+1} - k \leq n/2|\mathcal{F}_k) \\
&+ \|\varphi\|_\infty \gamma(0)P(T_{N_k+1} - k > n/2|\mathcal{F}_k) \\
&\leq \|\varphi\|_\infty \left\{\gamma(n/2) + \gamma(0)P(T_{N_k+1} - k > n/2|\mathcal{F}_k)\right\}. \qquad (3.22)
\end{aligned}
$$

So from (3.21) and (3.22), ii) follows. □

Theorem 3.2.9. *Let $\{X_n\}_{n\geq 0}$ be an aperiodic positive recurrent regenerative with regeneration times $\{T_n\}_{n\geq 0}$. Then*

i) $\{X_n\}_{n\geq 0}$ is $\alpha-$mixing.

ii) If, In addition to the above hypotheses, $\{X_n\}_{n\geq 0}$ is stationary and $E\left\{(T_2 - T_1)^k\right\} < \infty$ for some $k > 1$ then $\{X_n\}_{n\geq 0}$ is $\alpha-$mixing with constants $\alpha(n) = o(n^{1-k})$.

Proof. i) Let W be a bounded $\mathcal{F}_k$-measurable random variable and let $g : S\times S\times S\cdots \longrightarrow \mathbb{R}$ be a bounded and measurable function. We will show that

$$
|E\left\{W\varphi(X_{k+n}, X_{k+n+1}, ...)\right\} - E\left\{W\right\} E\left\{\varphi(X_{k+n}, X_{k+n+1}, ...)\right\}| \leq \alpha(n) \longrightarrow 0 \text{ as } n \to \infty, \qquad (3.23)
$$

uniformly in k. Clearly, (3.23) is equivalent with the definition of $\alpha-$mixing.
First, note that from Lemma 3.2.8 ii) follows there are constants $a_n \to 0$ such that

$$
\sup_{\{\varphi:\|\varphi\|_\infty \leq 1\}} |E\left\{\varphi(X_n, X_{n+1}, ...)\right\} - E_{\pi^*}\left\{\varphi\right\}| = a_n. \qquad (3.24)
$$

On the other hand, using properties of expectation conditional, Lemma 3.2.8 ii) and (3.24) we obtain

$$
\begin{aligned}
&|E\left\{W\varphi(X_{k+n}, X_{k+n+1}, ...)\right\} - E\left\{W\right\} E\left\{\varphi(X_{k+n}, X_{k+n+1}, ...)\right\}| \\
&\leq |E\left\{W\varphi(X_{k+n}, X_{k+n+1}, ...)\right\} - E_{\pi^*}\left\{\varphi\right\}| + |E\left\{W\right\} E\left\{\varphi(X_{k+n}, X_{k+n+1}, ...)\right\} - E_{\pi^*}\left\{\varphi\right\}| \\
&\leq |E\left\{W E\left\{\varphi(X_{k+n}, X_{k+n+1}, ...)|\mathcal{F}_k\right\} - E_{\pi^*}\left\{\varphi\right\}\right\}| + \|W\|_\infty |E\left\{\varphi(X_{k+n}, X_{k+n+1}, ...)\right\} - E_{\pi^*}\left\{\varphi\right\}| \\
&\leq \|W\|_\infty \|\varphi\|_\infty \left\{(1+\gamma(0))P(T_{N_k+1} - k > n/2) + \gamma(n/2) + a_n\right\} \to 0 \text{ as } n \to \infty,
\end{aligned}
$$

uniformly in k. The last convergence is valid because $\mu_T = \sum_{j=1}^{\infty} P(T_2 - T_1 \geq j) < \infty$ by hypothesis and by (1.6) we have

$$\lim_{k\to\infty} P(T_{N_k+1} - k > n) = \frac{1}{\mu_T} \sum_{j=n+1}^{\infty} P(T_2 - T_1 \geq j).$$

Thus $P(T_{N_k+1} - k > n)$ tends to 0 as $n \to \infty$ uniformly with respect to k. Since $\gamma(n/2) \to 0$ and $a_n \to 0$ as $n \to \infty$, uniformly in k, the proof of i) is complete.

ii) Since we are assuming that $\{X_n\}_{n\geq 1}$ is stationary then

$$E\{\varphi(X_k, X_{k+1}, ...)\} = \{\varphi(X_0, X_1, ...)\} = E_{\pi^*}\{\varphi\}.$$

Thus, using properties of expectation conditional and Lemma 3.2.8 ii) we obtain

$$\begin{aligned}
&|E\{W\varphi(X_{k+n}, X_{k+n+1}, ...)\} - E\{W\} E\{\varphi(X_{k+n}, X_{k+n+1}, ...)\}| \\
&= |E\{E\{W\varphi(X_{k+n}, X_{k+n+1}, ...)|\mathcal{F}_k\}\} - E\{W\} E_{\pi^*}\{\varphi\}| \\
&= |E\{WE\{\varphi(X_{k+n}, X_{k+n+1}, ...)|\mathcal{F}_k\} - WE_{\pi^*}\{\varphi\}\}| \\
&= |E\{W(E\{\varphi(X_{k+n}, X_{k+n+1}, ...)|\mathcal{F}_k\} - E_{\pi^*}\{\varphi\}\})| \\
&\leq \|W\|_\infty \|\varphi\|_\infty (1 + \gamma(0)) P(T_{N_k+1} - k > n/2) + \gamma(n/2) \qquad (3.25)
\end{aligned}$$

Since $E\left\{(T_2 - T_1)^k\right\} < \infty$ from Lemma 3.2.8 i) follows $\gamma(n/2) = o(n^{1-k})$. On the other hand, by (1.6) we have

$$P(T_{N_k+1} - k > n) \leq \frac{1}{\mu_T} \sum_{j=n+1}^{\infty} P(T_2 - T_1 \geq j) \leq \frac{n^{1-k}}{\mu_T} \sum_{j=n+1}^{\infty} j^{k-1} P(T_2 - T_1 > j) = \frac{n^{1-k}}{\mu_T} o(1),$$

the last equality is valid because $E\left\{(T_2 - T_1)^k\right\} < \infty$. Thus, $P(T_{N_k+1} - k > n/2) = o(n^{1-k})$. Finally, from last considerations and by (3.25) we obtain

$$|E\{W\varphi(X_{k+n}, X_{k+n+1}, ...)\} - E\{W\} E\{\varphi(X_{k+n}, X_{k+n+1}, ...)\}| = o(n^{1-k}).$$

This completes the proof of ii). □

3.3 Weak Convergence of the Empirical Process

In this section, we show that $\beta_n(x) \Rightarrow \tilde{B}_{\tilde{F}}(x)$ in the Skorokhod space D with $\beta_n(x)$ defined by (3.3) and $\tilde{B}_{\tilde{F}}(x)$ given by (3.4). We prove this weak convergence under $\alpha-$mixing conditions on $\{X_n\}_{n\geq 0}$. As in the classical approach, our invariance principle consist of two parts, establishing finite dimensional convergence and tightness of the empirical process $\beta_n(x)$. We first obtain the finite-dimensional convergence using Mallows distance, i.e., we will prove for fixed $x_1, ..., x_k \in \mathbb{R}$ and $\forall k \in \mathbb{N}$, $(\beta_n(x_1), \beta_n(x_2), \cdots, \beta_n(x_k)) \xrightarrow{d} \left(\tilde{B}_{\tilde{F}}(x_1), \tilde{B}_{\tilde{F}}(x_2), \cdots, \tilde{B}_{\tilde{F}}(x_k)\right)$. For this, note that we can write the empirical process $\beta_n(x)$ as the sum

$$\beta_n(x) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left\{ I_{(-\infty,x]}(X_j) - \tilde{F}(x) \right\}.$$

Since $\{X_n\}_{n\geq 1}$ is regenerative, by Proposition 2.2.1 we have the sequence $\left\{I_{(-\infty,x]}(X_n)\right\}_{n\geq 0}$ is also regenerative. So, we can apply the results obtained in section 2.4 to the empirical process $\beta_n(x)$. In this sense, from Corollary (2.4.5) we obtain the following results.

Lemma 3.3.1. *Let $\{X_n\}_{n\geq 0}$ be a regenerative sequence with regeneration times $\{T_n\}_{n\geq 0}$ such that $E\left\{(T_2 - T_1)^2\right\} < \infty$. Then, for x fixed*

$$d_2(\beta_n(x), \tilde{B}_{\tilde{F}}(x)) \to 0 \ \ as \ \ n \to \infty, \tag{3.26}$$

where $\tilde{B}_{\tilde{F}}(x)$ is a zero-mean Gaussian process defined by (3.4).

Proof. Let $\phi(X_j) = I_{(-\infty,x]}(X_j)$, $\tilde{\mu}_\phi = \int \phi d\tilde{\pi} = \tilde{F}(x)$ and

$$\tilde{\sigma}_\phi^2 = \frac{1}{\mu_T} E\left\{ \left(\sum_{j=T_1}^{T_2-1} I_{(-\infty,x]}(X_j) - \tilde{F}(x) \right)^2 \right\} < \infty.$$

Since $|\phi| \leq 1$ and $E\left\{(T_2 - T_1)^2\right\} < \infty$ from Corollary (2.4.5) follows

$$d_2\left(\frac{\beta_n(x)}{\tilde{\sigma}_\phi}, Z_0\right) \to 0 \ \ as \ n \to \infty,$$

where $Z_0 \stackrel{d}{=} N(0,1)$. The last expression is equivalent with (3.26). □

In the same way, we obtain the following result:

Lemma 3.3.2. *Let $\{X_n\}_{n\geq 0}$ be a regenerative sequence with regeneration times $\{T_n\}_{n\geq 0}$ such that $E\left\{(T_2 - T_1)^2\right\} < \infty$. Then, for x fixed*

$$\beta_n(x) \xrightarrow{d} \tilde{B}_{\tilde{F}}(x) \ \ as \ \ n \to \infty, \tag{3.27}$$

where $\tilde{B}_{\tilde{F}}(x)$ is a zero-mean Gaussian process defined by (3.4).

Next we show the finite dimensional convergence of the empirical process $\beta_n(x)$ to the zero-mean Gaussian process $\tilde{B}_{\tilde{F}}(x)$.

Theorem 3.3.3. *The empirical process $\beta_n(x)$ converges to a zero-mean Gaussian process $\tilde{B}_{\tilde{F}}(x)$ in the sense of finite-dimensional distributions, i.e., for fixed $x_1, ..., x_k \in \mathbb{R}$ and $\forall k \in \mathbb{N}$*

$$(\beta_n(x_1), \beta_n(x_2), \cdots, \beta_n(x_k)) \xrightarrow{d} \left(\tilde{B}_{\tilde{F}}(x_1), \tilde{B}_{\tilde{F}}(x_2), \cdots, \tilde{B}_{\tilde{F}}(x_k)\right) \tag{3.28}$$

Proof. Case $k = 1$ follows from Lemma 3.3.2. For $k = 2$, let $a, b \in \mathbb{R}$ and $F_{n,2}$ and G_2 be the distribution functions of the random variables

$$a\beta_n(x_1) + b\beta_n(x_2) \ \ and \ \ a\tilde{B}_{\tilde{F}}(x_1) + b\tilde{B}_{\tilde{F}}(x_2),$$

respectively.

By definition of Mallows distance and the classic inequality

$$|x + y|^p \leq 2^{p-1}(|x|^p + |y|^p) \ \ x, y \in \mathbb{R} \ \ and \ \ p \geq 1,$$

we obtain

$$\begin{aligned}
d_2^2(F_{n,2}, G_2) &\leq E|a\beta_n(x_1) + b\beta_n(x_2) - (a\tilde{B}_{\tilde{F}}(x_1) + b\tilde{B}_{\tilde{F}}(x_2))|^2 \\
&= E|a(\beta_n(x_1) - \tilde{B}_{\tilde{F}}(x_1)) + b(\beta_n(x_2) - \tilde{B}_{\tilde{F}}(x_2))|^2 \\
&\leq 2\left\{|a|^2 E|\beta_n(x_1) - \tilde{B}_{\tilde{F}}(x_1)|^2 + |b|^2 E|\beta_n(x_2) - \tilde{B}_{\tilde{F}}(x_2)|^2\right\} \\
&= 2\left\{|a|^2 d_2^2(\beta_n(x_1), \tilde{B}_{\tilde{F}}(x_1)) + |b|^2 d_2^2(\beta_n(x_2), \tilde{B}_{\tilde{F}}(x_2))\right\}
\end{aligned}$$

In the last equality we have used Theorem 1.5.2 with $(\beta_n(x_i), \tilde{B}_{\tilde{F}}(x_i)) \stackrel{d}{=} F_{\beta_n(x_i)} \wedge F_{\tilde{B}_{\tilde{F}}(x_i)}$, $i = 1, 2$. From Lemma 3.3.1 follows

$$d_2^2(\beta_n(x_i), \tilde{B}_{\tilde{F}}(x_i)) \stackrel{n\to\infty}{\longrightarrow} 0, \ \ i = 1, 2,$$

Thus,

$$d_2(F_{n,2}, G_2) \stackrel{n\to\infty}{\longrightarrow} 0.$$

i.e., for any $a, b \in \mathbb{R}$,

$$d_2^2\left(a\beta_n(x_1) + b\beta_n(x_2), a\tilde{B}_{\tilde{F}}(x_1) + b\tilde{B}_{\tilde{F}}(x_2)\right) \stackrel{n\to\infty}{\longrightarrow} 0.$$

Then by Theorem 1.5.1

$$a\beta_n(x_1) + b\beta_n(x_2) \stackrel{d}{\longrightarrow} a\tilde{B}_{\tilde{F}}(x_1) + b\tilde{B}_{\tilde{F}}(x_2),$$

and by Crámer- Wold Theorem follow that

$$(\beta_n(x_1), \beta_n(x_2)) \stackrel{d}{\longrightarrow} (\tilde{B}_{\tilde{F}}(x_1), \tilde{B}_{\tilde{F}}(x_2)).$$

For the general case we use mathematical induction. If $p \geq 1$, we have the inequality

$$\left|\sum_{j=1}^{k} a_j\right|^p \leq 2^{(k-1)(p-1)}|a_1|^p + \sum_{j=2}^{k} 2^{(k-j+1)(p-1)}|a_j|^p, \;\; a_j \in \mathbb{R},\; j = 1, ..., k. \tag{3.29}$$

Let $a_1, a_2, ..., a_k \in \mathbb{R}$ and let $F_{n,k}$ and G_k be the distribution functions of the random variables

$$\sum_{j=1}^{k} a_j\beta_n(x_j) \;\; and \;\; \sum_{j=1}^{k} a_j\tilde{B}_{\tilde{F}}(x_j),$$

respectively.

By definition of Mallows distance and Inequality (3.29) we have

$$\begin{aligned}
d_2^2(F_{n_k}, G_k) &\leq E\left|\sum_{j=1}^{k} a_j\beta_n(x_j) - \sum_{j=1}^{k} a_j\tilde{B}_{\tilde{F}}(x_j)\right|^2 \\
&= E\left|\sum_{j=1}^{k} a_j(\beta_n(x_j) - \tilde{B}_{\tilde{F}}(x_j))\right|^2 \\
&\leq 2^{(k-1)}|a_1|^2 E|\beta_n(x_1) - \tilde{B}_{\tilde{F}}(x_1)|^2 + \sum_{j=2}^{k} 2^{(k-j+1)}|a_j|^2 E|\beta_n(x_j) - \tilde{B}_{\tilde{F}}(x_j)|^2 \\
&= 2^{(k-1)}|a_1|^2 d_2^2(\beta_n(x_1), \tilde{B}_{\tilde{F}}(x_1)) + \sum_{j=2}^{k} 2^{(k-j+1)}|a_j|^2 d_2^2(\beta_n(x_j), \tilde{B}_{\tilde{F}}(x_j)).
\end{aligned}$$

In the last equality we have used Theorem 1.5.2 with $(\beta_n(x_i), \tilde{B}_{\tilde{F}}(x_i)) \stackrel{d}{=} F_{\beta_n(x_i)} \wedge F_{B(x_j)}$, $i = 1, 2, ..., k$. Again, from Lemma 3.3.1 follows

$$d_2^2(\beta_n(x_i), \tilde{B}_{\tilde{F}}(x_j)) \stackrel{n\to\infty}{\longrightarrow} 0, \;\; i = 1, 2, ..., k.$$

Thus

$$d_2(F_{n,k}, G_k) \stackrel{n\to\infty}{\longrightarrow} 0.$$

Then, we have for any $a_1, a_2, ..., a_k \in \mathbb{R}$

$$d_2\left(\sum_{j=1}^{k} a_j\beta_n(x_j), \sum_{j=1}^{k} a\tilde{B}_{\tilde{F}}(x_j)\right) \stackrel{n\to\infty}{\longrightarrow} 0.$$

By Theorem 1.5.1, the last convergence implies that

$$\sum_{j=1}^{k} a_j \beta_n(x_j) \xrightarrow{d} \sum_{j=1}^{k} a_j \tilde{B}_{\tilde{F}}(x_j) \ as \ n \to \infty,$$

and from Crámer- Wold Theorem follows that

$$(\beta_n(x_1), \beta_n(x_2), \cdots, \beta_n(x_k)) \xrightarrow{d} \left(\tilde{B}_{\tilde{F}}(x_1), \tilde{B}_{\tilde{F}}(x_2), \cdots, \tilde{B}_{\tilde{F}}(x_k)\right).$$

This completes the proof. □

Observe that from Theorem 2.3.2 we have that CLT is valid to regenerative sequences $\{\varphi(X_n)\}_{n\geq 0}$ where φ can be a Lipschitz functions. In the case that we have not an suitable moment bound of the type (3.6) that would allow us to use Theorem 3.2.2 obtained by Dehling, Durieu and Volny (2009), we establish an alternative invariance principle under strong mixing conditions. We already obtained the finite dimensional convergence, then it remains to show that $\beta_n(x)$ is tight. For this, from Theorem 3.2.9 we have that a regenerative sequence is $\alpha-$mixing. So, tightness follows from Shao and Yu's tightness criterion (Theorem 3.2.7) in the same way as in the proof of Theorem 2.2. in Shao and Yu (1996). In this sense, the previous observations suggest the following condition:

Condition 3.3.4. *Let $\{X_n\}_{n\geq 1}$ be an aperiodic, positive recurrent and stationary regenerative sequence with regeneration times $\{T_n\}_{n\geq 1}$ satisfying $E(T_2 - T_1)^2 < \infty$. Suppose that the canonical distribution $\tilde{F}$ is continuous. And, assume the following conditions*

i) $\alpha(n) = O(n^{-\theta-\epsilon})$ *for some* $\theta \geq 1 + \sqrt{2}$ *and* $\epsilon > 0$ *or*

ii) $\tilde{F}$ *satisfies (3.9) and the partial sums of* $\{\varphi(X_n)\}_{n\geq 0}$ *with* φ *bounded Lipschitz has a bound on the fourth moments of the type (3.6).*

Theorem 3.3.5. *Assume that Condition 3.3.4 is satisfied. Then*

$$\beta_n(x) \Rightarrow \tilde{B}_{\tilde{F}}(x) \ in \ D(-\infty, \infty), \tag{3.30}$$

where $\tilde{B}_{\tilde{F}}(x)$ *is a zero-mean Gaussian process defined by (3.4).*

Proof. If Condition 3.3.4, ii) is valid then the weak convergence (3.30) follows from Theorem 3.2.2. So, we assume Condition 3.3.4, i). Note that we may confine our attention to the case in which $\{X_n\}_{n\geq 1}$ is uniformly distributed over $[0,1]$. Indeed, $\left\{\tilde{F}(X_n)\right\}_{n\geq 1}$ is $\alpha-$mixing and regenerative with regeneration times $\{T_n\}_{n\geq 1}$, and, since $\tilde{F}$ is continuous, $\tilde{F}(X_n)$ is uniformly distributed. If $U_n(t)$ is the empirical distribution function for $\tilde{U}_1 = \tilde{F}(X_1), \tilde{U}_2 = \tilde{F}(X_2), ..., \tilde{U}_n = \tilde{F}(X_n)$, and if

$$u_n(t) = \sqrt{n}(U_n(t) - t) \tag{3.31}$$

then, with probability 1,

$$u_n(\tilde{F}(x)) = \beta_n(x)$$

for all x. If the theorem is true in the uniform case, then $u_n(\cdot) \Rightarrow \tilde{B}(\cdot)$ in $D[0,1]$, where $\tilde{B}$ is a zero-mean Gaussian process and

$$\begin{aligned} Cov(\tilde{B}(s), \tilde{B}(t)) &= s \wedge t - st \\ &+ \sum_{j=1}^{\infty} Cov\left\{I_{[0,s]}(\tilde{F}(X_0)), I_{[0,t]}(\tilde{F}(X_j))\right\} \\ &+ \sum_{j=1}^{\infty} Cov\left\{I_{[0,t]}(\tilde{F}(X_0)), I_{[0,s]}(\tilde{F}(X_j))\right\}. \end{aligned} \tag{3.32}$$

Define the mapping $z \longrightarrow z \circ \tilde{F}$ from the Skorokhod space D in D . Since z is continuous follows by mapping Theorem 1.8.1 that $u_n(\cdot) \Rightarrow \tilde{B}(\cdot)$ implies $\beta_n(\cdot) \Rightarrow \tilde{B}_{\tilde{F}}(\cdot)$. Thus we need only treat the uniform case. By Theorem 3.3.3, the finite dimensional distributions of $u_n(t)$ converge to the corresponding finite dimensional distributions of $\tilde{B}(t)$. The tightness of $u_n(t)$ follows from Shao and Yu's tightness criterion (Theorem 3.2.7) in the same way as in the proof of Theorem 2.2. in Shao and Yu (1996).

□

Corollary 3.3.6. *Let $\{X_n\}_{n\geq 1}$ be an aperiodic, positive recurrent and stationary regenerative sequence with regeneration times $\{T_n\}_{n\geq 1}$ satisfying $E\left\{(T_2 - T_2)^{\theta+\epsilon}\right\} < \infty$ for some $\theta \geq 2 + \sqrt{2}$ and $\epsilon > 0$. Then,*

$$\beta_n(x) \Rightarrow \tilde{B}_{\tilde{F}}(x) \text{ in } D(-\infty, \infty),$$

where $\tilde{B}_{\tilde{F}}(x)$ is a zero-mean Gaussian process defined by (3.4).

Proof. By Theorem 3.2.9 we have that $\{X_n\}_{n\geq 0}$ is $\alpha(n)-$mixing with constants $\alpha(n) = o(n^{1-(\theta+\epsilon)})$. Then Condition 3.3.4, i) is satisfied. So, the result follows from the previous theorem. □

Remark 3.3.1. *Our condition 3.3.4 requires stationarity of the regenerative sequence $\{X_n\}_{n\geq 1}$. In many cases $\{X_n\}_{n\geq 1}$ is not stationary, but is possible to make the regenerative sequence a stationary and regenerative sequence with marginal distribution $\tilde{\pi}$ and with the same asymptotic behavior. There are different works about the construction of the stationary version of $\{X_n\}_{n\geq 0}$ which is quite technical and for which we will omit the details, see, for example, [61] for a construction. When $\{X_n\}_{n\geq 1}$ is not stationary,, the covariance function of the limit process $\tilde{B}_{\tilde{F}}$ is given by*

$$\begin{aligned} E(\tilde{B}_{\tilde{F}}(x), \tilde{B}_{\tilde{F}}(y)) &= \tilde{F}(x \wedge y) - \tilde{F}(x)\tilde{F}(y) \\ &+ \sum_{j=2}^{\infty} E\left\{ I_{(-\infty,x]}(X_1^*) - \tilde{F}(x), I_{(-\infty,y]}(X_j^*) - \tilde{F}(y) \right\} \\ &+ \sum_{j=2}^{\infty} E\left\{ I_{(-\infty,y]}(X_1^*) - \tilde{F}(y), I_{(-\infty,x]}(X_j^*) - \tilde{F}(x) \right\}. \end{aligned}$$

where $\{X_n^*\}_{n\geq 0}$ is a stationary version of $\{X_n\}_{n\geq 0}$. For the technical details of this relation, see, for instance, Glynn (1990).

Now, we will present an alternative proof of the weak convergence of the empirical process $\beta_n(\cdot)$ substituting mixing assumptions by the condition that the inter-regeneration times (or length of the cycles) of the regenerative sequence be bounded. This resul is interesting because we show tightness of $\beta_n(\cdot)$ without to use estimates valid for mixing process, we obtain moment bounds for partial sums of regenerative sequences using definitions and properties we studied in Chapter 2.

The proof of the following estimative makes use of some ideas from Proposition 8 in Clémençon (2001).

Lemma 3.3.7. *Let $\{X_n\}_{n\geq 1}$ be an aperiodic, positive recurrent regenerative sequence with regeneration times $\{T_n\}_{n\geq 1}$ such that $|T_n - T_{n-1}| \leq M$ for every $n \geq 1$, where M is a positive* *constant. Then for $p \geq 2$*

$$E\left|\sum_{i=1}^{n}\varphi(X_i)\right|^p \leq C(p,M)\left\{\left(E_{\tilde{\pi}}\left\{\varphi^2\right\}\right)^{\frac{p}{2}} n^{\frac{p}{2}} + E_{\tilde{\pi}}\left\{|\varphi|^p\right\} n\right\} \forall \varphi \in \mathcal{L}^1(S,\mathcal{G},\tilde{\pi}), \tag{3.33}$$

where $C(p,M)$ is a constant depending only on p and M.

Proof. To make the notation easier, we will assume that all cycles of $\{X_n\}_{n\geq 0}$ have the same distribution. Let $N_n = \sup\{k : T_k \leq n < T_{k+1}\}$. Then we have

$$\sum_{i=1}^{n}\varphi(X_i) = \sum_{k=0}^{N_n-1} Y_k + \sum_{i=T_{N_n}}^{n}\varphi(X_i) \ \ \textit{where} \ \ Y_k = \sum_{i=T_k}^{T_{k+1}-1}\varphi(X_i).$$

Hence, we deduce

$$E\left|\sum_{i=1}^{n}\varphi(X_i)\right|^p \leq 2^{p-1}\left\{E\left|\sum_{k=0}^{N_n-1} Y_k\right|^p + E\left|\sum_{i=T_{N_n}}^{n}\varphi(X_i)\right|^p\right\}. \tag{3.34}$$

Recall that $\{Y_k\}_{k\leq 0}$ is a i.i.d. sequence with mean zero and thus by L^p- Doob inequality applied to the positive submartingale $\left(\left|\sum_{k=0}^{l} Y_k\right|\right)_{l\geq 1}$, write

$$E\left|\sum_{k=0}^{N_n-1} Y_k\right|^p \leq E\left(\max_{1\leq l\leq n}\left|\sum_{j=0}^{l} Y_k\right|^p\right) \leq \left(\frac{p}{p-1}\right)^p E\left|\sum_{j=1}^{n} Y_k\right|. \tag{3.35}$$

Then, apply Rosenthal inequality

$$\begin{aligned}
E\left|\sum_{j=1}^{n} Y_k\right| &\leq C_1(p)\left\{(E\left\{Y_i^2\right\})^{\frac{p}{2}} n^{\frac{p}{2}} + E\left\{|Y_i|^p\right\} n\right\} \\
&= C_1(p)\left\{\left(E\left\{\sum_{i=T_1}^{T_2-1}\varphi(X_j)\right\}^2\right)^{\frac{p}{2}} n^{\frac{p}{2}} + E\left|\sum_{i=T_1}^{T_2-1}\varphi(X_j)\right|^p n\right\} \\
&\leq C_1(p)\left\{\left(E\left\{(T_2-T_1)\sum_{i=T_1}^{T_2-1}\varphi(X_j)^2\right\}\right)^{\frac{p}{2}} n^{\frac{p}{2}} + E\left\{(T_2-T_1)^{p-1}\sum_{i=T_1}^{T_2-1}|\varphi(X_j)|^p\right\} n\right\} \\
&\leq C_1(p)\left\{M^{p/2} E\left\{\sum_{i=T_1}^{T_2-1}\varphi(X_j)^2\right\}^{\frac{p}{2}} n^{\frac{p}{2}} + M^{p-1} E\left\{\sum_{i=T_1}^{T_2-1}|\varphi(X_j)|^p\right\} n\right\} \\
&= C_1(p) M^p\left\{\left(E_{\tilde{\pi}}\left\{\varphi^2\right\}\right)^{\frac{p}{2}} n^{\frac{p}{2}} + E_{\tilde{\pi}}\left\{|\varphi|^p\right\} n\right\}.
\end{aligned} \tag{3.36}$$

In the last equality we have used (2.5). For the second term in (3.34) , we obtain the bound

$$\begin{aligned}
E\left|\sum_{i=T_{N_n}}^{n} \varphi(X_i)\right|^p &\leq E\left|\sum_{i=T_{N_n}}^{T_{N_n+1}-1} |\varphi(X_i)|\right|^p \\
&\leq E\left\{(T_{N_n+1} - T_{N_n})^{p-1} \sum_{i=T_{N_n}}^{T_{N_n+1}-1} |\varphi(X_i)|^p\right\} \\
&\leq M^{p-1} E\left\{\sum_{i=T_{N_n}}^{T_{N_n+1}-1} |\varphi(X_i)|^p\right\} \\
&= M^p E_{\tilde{\pi}}\left\{|\varphi|^p\right\}. \qquad (3.37)
\end{aligned}$$

Again, in the last equality we have used (2.5). Thus, by (3.34), (3.36) and (3.37) we have

$$\begin{aligned}
E\left|\sum_{i=1}^{n} \varphi(X_i)\right|^p &\leq \left(\frac{p}{p-1}\right)^p C_1(p) M^p \left\{\left(E_{\tilde{\pi}}\left\{\varphi^2\right\}\right)^{\frac{p}{2}} n^{\frac{p}{2}} + E_{\tilde{\pi}}\left\{|\varphi|^p\right\} n\right\} \\
&+ M^p E_{\tilde{\pi}}\left\{|\varphi|^p\right\} n \\
&\leq C_2(p) M^p \left\{\left(E_{\tilde{\pi}}\left\{\varphi^2\right\}\right)^{\frac{p}{2}} n^{\frac{p}{2}} + E_{\tilde{\pi}}\left\{|\varphi|^p\right\} n\right\} \qquad (3.38)
\end{aligned}$$

and (3.33) follows. □

From the previous estimate, we can obtain an inequality of type (3.14) and thus we can establish the tightness of $\beta_n(\cdot)$ as follows.

Theorem 3.3.8. *Let $\{X_n\}_{n\geq 1}$ be an aperiodic, positive recurrent regenerative sequence with regeneration times $\{T_n\}_{n\geq 1}$ such that the sequence $\{T_n - T_{n-1}\}_{n\geq 1}$ is bounded. Then we have*

$$\beta_n(x) \Rightarrow \tilde{B}_{\tilde{F}}(x) \text{ in } D(-\infty, \infty), \qquad (3.39)$$

where $\tilde{B}_{\tilde{F}}(x)$ is a zero-mean Gaussian process defined by (3.4).

Proof. As in proof of Theorem (3.3.5) we treat the uniform case. Let $u_n(\cdot)$ the uniform empirical process defined in (3.31). By Theorem 3.3.3, the finite dimensional distributions of $u_n(\cdot)$ converge to the corresponding finite dimensional distributions of $\tilde{B}(t)$ defined by (3.32) . To prove that $u_n(t)$ is tight, fix $\epsilon > 0$ and $\eta > 0$ and let $s, t \in [0, 1]$. Take

$\varphi(x) = I_{[0,t]}(x) - I_{[0,s]}(x) - (t-s)$ and $p = 4$ in Lemma 3.3.7. Then

$$\begin{aligned} E\left|\sum_{i=1}^{n} I_{[0,t]}(U_i) - I_{[0,s]}(U_i) - (t-s)\right|^4 &\leq C[\left(E_{\tilde{\pi}}\left\{(I_{[0,t]}(x) - I_{[0,s]}(x) - (t-s))^2\right\}\right)^2 n^2 \\ &+ E_{\tilde{\pi}}\left\{\left|I_{[0,t]}(x) - I_{[0,s]}(x) - (t-s)\right|^4\right\} n] \\ &\leq C[|t-s|^2 n^2 + |t-s|n], \end{aligned} \tag{3.40}$$

where C depends on p and M alone. If $\frac{\epsilon}{n} \leq |t-s|$ we have,

$$E|u_n(t) - u_n(s)|^4 \leq \frac{2C}{\epsilon}(t-s)^2. \tag{3.41}$$

Note that estimative (3.41) is of type (3.14). So, tightness follows form Shao and Yu's tightness criterion (Theorem 3.2.7).

□

3.4 Convergence of the Empirical Quantile Process

Let $\{X_n\}_{n\geq 0}$ be an aperiodic and a positive recurrent regenerative sequence with values in $\mathbb{R}$ and regeneration times $\{T_n\}_{n\geq 0}$ such that with $\mu_T = E(T_2 - T_1) > 0$. As before, the canonical measure $\tilde{F}(x)$ and the empirical distribution function $F_n(x)$ are given by

$$\begin{aligned} \tilde{F}(x) &= \frac{1}{\mu_T} E\left(\sum_{j=T_1}^{T_2 - 1} I_{(-\infty,x]}(X_j)\right) \\ F_n(x) &= \frac{1}{n}\sum_{j=1}^{n} I_{(-\infty,x]}(X_j) \quad, x \in \mathbb{R},\ n \geq 1, \end{aligned}$$

Define the quantile function $\tilde{F}^{-1}$ of $\tilde{F}$ and the empirical quantile function F_n^{-1} of F_n by

$$\begin{aligned} \tilde{F}^{-1}(t) &= \inf\left\{x \in \mathbb{R} : \tilde{F}(x) > t\right\},\ 0 < t \leq 1,\ \tilde{F}^{-1}(0) = \tilde{F}^{-1}(0^+) \\ \tilde{F}_n^{-1}(t) &= \inf\left\{x \in \mathbb{R} : \tilde{F}_n(x) > t\right\},\ 0 < t \leq 1,\ \tilde{F}_n^{-1}(0) = \tilde{F}_n^{-1}(0^+) \end{aligned}$$

Just imitating (3.3) consider

$$q_n(t) = \sqrt{n}(F_n^{-1}(t) - \tilde{F}^{-1}(t)),\ 0 < t < 1,\ n \geq 1 \tag{3.42}$$

be the empirical quantile process associated to regenerative sequence $\{X_n\}_{n\geq 0}$ with regeneration times $\{T_n\}_{n\geq 0}$ and consider

$$v_n(t) = \sqrt{n}(U_n^{-1}(t) - t), \;\; 0 < t < 1, \, n \geq 1. \tag{3.43}$$

be the uniform quantile process associated to uniform regenerative sequence.

Remark 3.4.1. *If $\tilde{F}$ is continuous, $\tilde{F}(X_n)$ is uniformly distributed and if $U_n(t)$ is the empirical distribution function for $\tilde{F}(X_1), \tilde{F}(X_2), ..., \tilde{F}(X_n)$, and if the uniform empirical process is given by*

$$u_n(t) = \sqrt{n}(U_n(t) - t)$$

then, with probability 1*,*

$$u_n(\tilde{F}(x)) = \beta_n(x) \;\; \text{for all} \;\; x \tag{3.44}$$

where $\beta_n(x) = \sqrt{n}(F_n(x) - \tilde{F}(x))$. In this case we also have

$$\tilde{B}_{\tilde{F}}(x) = \tilde{B}(\tilde{F}(x)) \;\; \text{for all} \;\; x. \tag{3.45}$$

3.4.1 Pointwise Convergence of the Empirical Quantile Process

In this subsection, we prove that for t fixed, the process $q_n(t)$ converges in distribution to the random variable $-\dfrac{\tilde{B}(\cdot)}{\tilde{f}(\tilde{F}^{-1}(\cdot))}$ where $\tilde{f}(x) = \dfrac{d\tilde{F}(x)}{dx}$ and $\tilde{B}(\cdot)$ is given by (3.47) . For this, we need to obtain the convergence in distribution of $u_n(t)$ and $v_n(t)$.

Theorem 3.4.1. *Assume that Condition 3.3.4 is satisfied. Then*

$$u_n(\cdot) \xrightarrow{d} \tilde{B}(\cdot) \tag{3.46}$$

where $\tilde{B}$ is a zero-mean Gaussian process and

$$\begin{aligned} Cov(\tilde{B}(s), \tilde{B}(t)) &= s \wedge t - st \\ &+ \sum_{j=2}^{\infty} Cov\left\{ I_{[0,s]}(\tilde{F}(X_1)), I_{[0,t]}(\tilde{F}(X_j)) \right\} \\ &+ \sum_{j=2}^{\infty} Cov\left\{ I_{[0,t]}(\tilde{F}(X_1)), I_{[0,s]}(\tilde{F}(X_j)) \right\}. \end{aligned} \tag{3.47}$$

Proof. Since $\tilde{F}$ is a continuous function and by the relationships (3.4.1) and (3.45) we have $u_n(\tilde{F}(x)) = \beta_n(x)$ and $\tilde{B}_{\tilde{F}}(x) = \tilde{B}(\tilde{F}(x))$ for all x . Thus, the result follows by letting $t = \tilde{F}(x)$ in (3.27). □

Now, we prove that $v_n(t)$ converges in distribution to random variable $-\tilde{B}(t)$, for t fixed. The proof of this result is based on the Lemma 1.8.6 and Skorokhod's representation Theorem (Theorem 1.8.5).

Theorem 3.4.2. *Assume that Condition 3.3.4 is satisfied. Then*

$$v_n(t) \xrightarrow{d} -\tilde{B}(t). \tag{3.48}$$

Proof. From Theorem 3.4.1, for t fixed we have $\sqrt{n}(U_n(t) - t) = u_n(t) \xrightarrow{d} \tilde{B}(t)$. By Skorokhod representation (Theorem 1.8.5) exists random variables $u_n^*(t) \stackrel{d}{=} u_n(t)$ and $B^*(t) \stackrel{d}{=} \tilde{B}(t)$ such that $\sqrt{n}(U_n^*(t) - t) = u_n^*(t) \xrightarrow{a.s.} B^*(t)$ in Skorokhod topology and by Remark 1.8.1 since $B^*(t)$ is continuous this convergence is locally uniform, i.e.,

$$\lim_{n\to\infty} \sup_{0\le t\le 1} |\sqrt{n}(U_n^*(t) - t) - B^*(t)| \stackrel{a.s.}{=} 0.$$

By Lemma 1.8.6

$$\lim_{n\to\infty} \sup_{0\le t\le 1} |\sqrt{n}((U_n^*)^{-1}(t) - t) + B^*(t)| \stackrel{a.s.}{=} 0.$$

For t fixed this implies $\sqrt{n}((U_n^*)^{-1} - t) \xrightarrow{a.s.} -B^*(t)$. Since $(U_n^*)^{-1}(t) \stackrel{d}{=} U_n^{-1}(t)$ and $-B^*(t) \stackrel{d}{=} -\tilde{B}(t)$ we have

$$v_n(t) = \sqrt{n}(U_n^{-1}(t) - t) \xrightarrow{d} -\tilde{B}(t).$$

□

Next for $t \in (0, 1)$ fixed, we prove the convergence in distribution of the empirical quantile process $q_n(t)$ defined by (3.42).

Theorem 3.4.3. *Assume that Condition 3.3.4 is satisfied. If $\tilde{F}$ is absolutely continuous distribution function with a strictly positive density function $\tilde{f} = \tilde{F}'$ then*

$$q_n(t) \xrightarrow{d} -\frac{\tilde{B}(t)}{\tilde{f}(\tilde{F}^{-1}(t))}. \tag{3.49}$$

Proof. By definition of $v_n(t)$ and $q_n(t)$ we have

$$
\begin{aligned}
\left| \frac{q_n(t)}{\left(\tilde{F}^{-1}(t)\right)'} - v_n(t) \right| &= \left| \frac{\sqrt{n}(F_n^{-1}(t) - \tilde{F}^{-1}(t))}{\left(\tilde{F}^{-1}(t)\right)'} - \sqrt{n}(U_n^{-1}(t) - t) \right| \\
&= \left| \sqrt{n}(U_n^{-1}(t) - t) \left\{ \frac{F_n^{-1}(t) - \tilde{F}^{-1}(t)}{(U_n^{-1}(t) - t)\left(\tilde{F}^{-1}(t)\right)'} - 1 \right\} \right| \\
&= \left| \sqrt{n}(U_n^{-1}(t) - t) \left\{ \frac{\tilde{F}^{-1}(U_n^{-1}(t)) - \tilde{F}^{-1}(t)}{(U_n^{-1}(t) - t)\left(\tilde{F}^{-1}(t)\right)'} - 1 \right\} \right| \longrightarrow 0.
\end{aligned}
\tag{3.50}
$$

To see this, note that $\dfrac{\tilde{F}^{-1}(U_n^{-1}(t)) - \tilde{F}^{-1}(t)}{(U_n^{-1}(t) - t)} \stackrel{n\to\infty}{\longrightarrow} \left(\tilde{F}^{-1}(t)\right)'$ because $U_n^{-1}(t) \to t$ uniformly and by Theorem 3.4.2, $\sqrt{n}(U_n^{-1}(t) - t) \stackrel{d}{\longrightarrow} -\tilde{B}(t)$.
On the other hand, $\left(\tilde{F}^{-1}(t)\right)' = \dfrac{1}{\tilde{f}(\tilde{F}^{-1}(t))}$ because $\tilde{F}$ is absolutely continuous. Therefore,

$$
q_n(t) \stackrel{d}{\longrightarrow} -\left(\tilde{F}^{-1}(t)\right)' \tilde{B}(t) = -\frac{\tilde{B}(t)}{\tilde{f}(\tilde{F}^{-1}(t))}.
$$

□

3.4.2 Weak Convergence of the Empirical Quantile Process

Finally, as a consequence of the Bahadur representation of sample quantiles under α-mixing coefficients obtained by Xing, Yang, Liu et al. (2012), we derive weak convergence of the empirical quantile process $q_n(t)$ in the Skorokhod space D. First, we obtain weak convergence of the uniform quantile process $v_n(t)$ given by (3.43) using properties of locally uniformly aproximation of monotone functions (Lemma 1.8.6) together with Skorokhod's representation Theorem (Theorem 1.8.5).

Theorem 3.4.4. *Assume that Condition 3.3.4 is satisfied. Then*

$$
v_n(t) \Rightarrow -\tilde{B}(t) \stackrel{d}{=} \tilde{B}(t) \text{ in } D[0,1]. \tag{3.51}
$$

Proof. Using relationships (3.4.1) and (3.45) and by letting $t = \tilde{F}(x)$ in (3.30) we have

$$
u_n(t) = \sqrt{n}(U_n(t) - t) \stackrel{d}{\Rightarrow} \tilde{B}(t) \text{ in } D[0,1].
$$

Thus, the Skorokhod's representation (Theorem 1.8.5) gives random elements u_n^* and B^* defined on a new sample space such that $u_n^* \stackrel{d}{=} u_n,\ B^* \stackrel{d}{=} \tilde{B}$ and $u_n^*(t) \xrightarrow{a.s.} B^*(t)$ in $D[0,1]$. Because $B^*(t)$ has $a.s$ continuous sample paths, this implies that

$$\sup_{0\le t\le 1} |u_n^*(t) - B^*(t)| \xrightarrow{a.s.} 0. \tag{3.52}$$

Defining $U_n^*(t) := \dfrac{u_n^*(t)}{\sqrt{n}} + t$, we have $U_n^*(t) \stackrel{d}{=} U_n(t)$ and (3.52) is equivalent to

$$\sup_{0\le t\le 1} |\sqrt{n}(U_n^*(t) - t) - B^*(t)| \xrightarrow{a.s.} 0$$

By Vervaat's lemma (Lemma 1.8.6) we have $\sqrt{n}((U_n^*)^{-1}(t) - t) \xrightarrow{a.s.} -B^*(t)$ locally uniformly, or equivalently,

$$\sqrt{n}((U_n^*)^{-1}(t) - t) \xrightarrow{a.s.} -B^*(t) \ in\ D[0,1].$$

Since $(U_n^*)^{-1}(t) \stackrel{d}{=} U_n^{-1}(t)$ we obtain the desired weak convergence

$$v_n(t) = \sqrt{n}((U_n^{-1}(t) - t) \Rightarrow -\tilde{B}(t) \stackrel{d}{=} \tilde{B}(t).$$

□

Using the previous result we prove weakly convergence of the empirical quantile process $q_n(t)$. Observe that $q_n(t) = \sqrt{n}(\tilde{F}^{-1}(U_n^{-1}(t)) - \tilde{F}^{-1}(t))$ and if $\tilde{F}'$ exists, using the mean value theorem, we can write

$$q_n(t) = \sqrt{n}(U_n^{-1}(t) - t)\left(\tilde{F}^{-1}(\xi_n)\right)' \ for\ \ t \wedge U_n^{-1}(t) \le \xi_n \le t \vee U_n^{-1}(t).$$

On the other hand, let $\phi(s) = \left(\tilde{F}^{-1}(\xi_n)\right)' s$ then $q_n(t) = \left(\tilde{F}^{-1}(\xi_n)\right)' v_n(t)$. By (3.51) and since $\phi(s)$ is a continuous function by the continuous mapping Theorem 1.8.1, we have

$$q_n(t) = \left(\tilde{F}^{-1}(\xi_n)\right)' v_n(t) = \phi(v_n(t)) \Rightarrow \phi(-B(t)) = -\left(\tilde{F}^{-1}(\xi_n)\right)' \tilde{B}(t).$$

Therefore, if F is absolutely continuous with positive density

$$q_n(t) \Rightarrow -\frac{\tilde{B}(t)}{\tilde{f}(\tilde{F}^{-1}(\xi_n))} \ in\ D[0,1],\ for\ t \wedge U_n^{-1}(t) \le \xi_n \le t \vee U_n^{-1}(t).$$

In this sense, using a Bahadur representation for quantiles of $\alpha-$mixing samples we can obtain the weak convergence of $q_n(t)$ in a more general way.

Theorem 3.4.5. *Assume that Condition 3.3.4, i) is satisfied. Let $\tilde{F}(x)$ be an absolutely continuous distribution function with a strictly positive density function $\tilde{f} = \tilde{F}'$ such that $\tilde{f}'$ is bounded in some neighborhood of $\tilde{F}^{-1}(t)$. Then we have the weak convergence*

$$q_n(t) \Rightarrow -\frac{\tilde{B}(t)}{\tilde{f}(\tilde{F}^{-1}(t))} \text{ in } D[0,1]. \tag{3.53}$$

Proof. By Bahadur representation (1.22) we have

$$\begin{aligned}\sqrt{n}(F_n^{-1}(t) - \tilde{F}^{-1}(t)) &= \frac{1}{\tilde{f}(\tilde{F}^{-1}(t))}\sqrt{n}\left(t - F_n(\tilde{F}^{-1}(t))\right) + \sqrt{n}R_n \\ &= \frac{1}{\tilde{f}(\tilde{F}^{-1}(t))}\sqrt{n}(t - U_n(t)) + \sqrt{n}R_n,\end{aligned}$$

where $R_n = O(n^{-3/4}\log n)$. From our Theorem 3.3.5 and from Slutsky's theorem follow that the first term on the right side converges weakly in $D[0,1]$ to $-\frac{\tilde{B}(t)}{\tilde{f}(\tilde{F}^{-1}(t))}$. The second term vanishes as $n \to \infty$. Again by Slutsky's theorem we have

$$q_n(t) = \sqrt{n}(F_n^{-1}(t) - \tilde{F}^{-1}(t)) \Rightarrow -\frac{\tilde{B}(t)}{\tilde{f}(\tilde{F}^{-1}(t))} \text{ in } D[0,1].$$

This complete the proof. □

Remark 3.4.2. *From Corollary 3.2.5 we have that the process empirical associated to an $\mathcal{L}$-geometrically ergodic Markov chain $\{X_n\}_{n\geq 0}$ with values in $\mathbb{R}$ satisfies the invariance principle of Theorem 3.2.2. Moreover, $\{X_n\}_{n\geq 0}$ is $\alpha-$mixing (see, Bradley(2005)). So, from Bahadur representation (1.22) we can obtain the weak convergence (3.53) of the quantile process $q_n(t)$ associated to the chain.*

Chapter 4

Similarity Tests

4.1 Introduction

Let a random sample of random variables $X_1, X_2, ..., X_n$ with common distribution function F. We consider two types of goodness-of-fit problems: i) test the null hypothesis $F = F_0$ for a fixed distribution function F_0 and ii) $F \in \mathcal{G}_G$ where $\mathcal{G}_G$ is a suitable location-scale family. One way to study the problem i) consists of employing a functional distance to measure the discrepancy between the hypothesized distribution function F_0 and the empirical distribution function F_n. In this sense two statistics have received special attention in the literature:

$$D_n = \sqrt{n}\,\|F_n - F\|_\infty \qquad \text{(Kolmogorov-Smirnov)}$$
$$W_n^2 = n\int_{-\infty}^{\infty}(F_n(x) - F(x))^2 dF(x) \qquad \text{(Cramér-von Mises)}$$

where $\|F_n - F\|_\infty = \sup_x |F_n(x) - F(x)|$.

Asymptotic null distributions of D_n and W_n^2 are commonly handled by using empirical process techniques and weak convergence theory on the metric spaces. For the i.i.d. case, knowing the weak convergence of the empirical process $\sqrt{n}(F_n - F)$ to the Brownian bridge B and under the null hypothesis we have as $n \to \infty$,

$$D_n \xrightarrow{d} \|B\|_\infty \ \ \textit{and} \ \ W_n^2 \xrightarrow{d} \int_0^1 B(t)^2 dt. \tag{4.1}$$

On the other hand, del Barrio et al. (1990; 2000) proposed a new approach for goodness-of-fit tests based on the 2nd-order Mallows distance between the empirical distribution and the distribution F. The statistic used was

$$\sqrt{n}d_2(F_n, F) = \left(n\int_0^1 (F_n^{-1}(t) - F^{-1}(t))^2 dt\right)^{1/2}. \tag{4.2}$$

For i.i.d. observations, Samworth and Johnson (2008) showed that under "regularity conditions" the 2nd-order Mallows distance $d_2(F_n, F)$ satisfies

$$\sqrt{n}d_2(F_n, F) \xrightarrow{d} \left(\int_0^1 \frac{B^2(t)}{f^2(F^{-1}(t))}dt\right)^{1/2}. \tag{4.3}$$

Recent literature on statistics based on the 2nd-order Mallows distance has focused on goodness-of-fit tests for location-scale families

$$\mathcal{G}_G = \left\{H : H(x) = G\left(\frac{x-\mu}{\sigma}\right),\ \mu \in \mathbb{R},\ \sigma > 0\right\}.$$

To test $F \in \mathcal{G}_G$ del Barrio et al. (1999) proposed the used of the statistics

$$R_n = 1 - \frac{\left(\int_0^1 F_n^{-1}(t)G^{-1}(t)dt\right)^2}{\hat{\sigma}_n^2} \tag{4.4}$$

where F_n is the usual empirical distribution and $\hat{\sigma}_n^2$ the sample variance.

In this chapter, we study the asymptotic null distribution of the statistics D_n, W_n^2 and R_n for a regenerative sample. In our results, we replace the common distribution F and the Brownian Bridge B of the i.i.d. case by the canonical measure $\tilde{F}$ given by (3.1) and by the zero-mean Gaussian process $\tilde{B}_{\tilde{F}}$ given by (3.4). In this sense, in section 4.2, our Lemma 4.3.2 provides sufficient conditions to obtain the asymptotic null distribution of the Kolmogorov-Smirnov and Cramér-von Mises statistics for a regenerative sequence $\{X_n\}_{n\geq 1}$.

Finally, in section 4.3 we use the 2nd-order Mallows distance between the empirical distribution and the canonical measure $\tilde{F}$ to study the statistics $\sqrt{n}d_2(F_n, \tilde{F})$ and R_n defined by (4.2) and (4.4), respectively. So, our Lemma 4.4.2 provides sufficient conditions to obtain the convergence (4.3) for a regenerative sample. In our Lemma 4.4.3 we establish the limiting distribution of the statistics nR_n under the null hypothesis, thai is, when the canonical measure $\tilde{F}$ is a member of the location-scale family being tested.

The results obtained in this chapter follow from the weak convergence of the empirical and quantile process associated to X_n. In this sense, we know that any Harris chains $\{X_n\}_{n\geq 1}$ on a general state space that possess an atom A is a regenerative process with limiting distribution F_{lim}. By Kac's Theorem we have $F_{lim} = \tilde{F}$ where $\tilde{F}$ is the canonical distribution given by

$$\tilde{F}(x) = \frac{1}{E_A(T_A)} E_A \left\{ \sum_{j=0}^{T_A-1} I_{(-\infty,x]}(X_j) \right\},\ x \in \mathbb{R},$$

where $T_A = \inf\{n \geq 1, X_n \in A\}$ the hitting time on A. Thus, our invariance principle holds valid for Harris Markov chains and we may use the statistics described above to test $H_0 : \tilde{F} = F_0$ or $\tilde{F} \in \mathcal{G}_G$. In Subsection 3.2.1, we discuss the empirical process associated with a $\mathcal{L}$-geometrically ergodic Markov chain $\{X_n\}_{n\geq 0}$. Under some assumptions on the Markov transition function it was shown that the invariance principle of Theorem 3.2.2 holds. Thus, all the similarity tests proposed in this chapter can be applied for this type of Markov chains.

In order to prove our results we need to introduce some notation. Let $C[0,1]$ denote the space of continuous functions on the interval [0, 1], endowed with the supremum norm and the space $D[0,1]$ (respectively $(-\infty,\infty)$) denote the space of all real functions on [0, 1] (resp. on $(-\infty,\infty)$) which are right-continuous and have left limits, endowed with the Skorohod distance (see Billingsley 1986).

4.2 Harris Markov chains

In Chapter 2 we showed that any Harris recurrent chain is regenerative and perhaps this example is the most important examples of regenerative processes. If $\{X_n\}_{n\geq 0}$ is a Harris irreducible Markov chain on a general state space that possess an atom A. we may define hitting time on A by

$$T_A = \inf\{n \geq 1, X_n \in A\}$$

and the successive return times to A by

$$T_k(A) = \inf\{n : n \geq T_{k-1}(A), X_n \in A\}, \ k \geq 2, \ (T_1(A) := T_A).$$

And let $E_A(\cdot)$ be the expectation conditioned on $X_0 \in A$. Also assume that $\{X_n\}_{n\geq 0}$ is Harris recurrent, so, for any initial distribution, the probability of returning infinitely often to the atom A is equal to one. By the strong Markov property it follows that, for any initial distribution μ, the sample paths of the chain can be divided into i.i.d. blocks of random length corresponding to consecutive visits to A, i.e., this type of chain is regenerative according to the Definition 2.2.1 (see, Meyn and Tweedie (1996) for a detailed review and references). The cycles can be defined by

$$\eta_1 = (X_{T_1(A)}, X_{T_1(A)+1}, ..., X_{T_2(A)-1}), ..., \eta_k = (X_{T_k(A)}, X_{T_k(A)+1}, ..., X_{T_{k+1}(A)-1}).$$

For Harris recurrent chains the stochastic stability properties of the chain amount to properties concerning the speed of return time to the atom only. For instance, the following result show that exist an unique stationary measure and this measure is given by the occupation probability measure (2.4).

Theorem 4.2.1. *[Meyn and Tweedie (1996), Kac's Theorem] The Harris Markov chain $\{X_n\}_{n\geq 0}$ is positive recurrent if and only if $E_A(T_A) < \infty$. The unique stationary measure $\tilde{\pi}$ is the occupation probability measure given by*

$$\tilde{\pi}(B) = \frac{1}{E_A(T_A)} E_A \left\{ \sum_{j=0}^{T_A - 1} I_A(X_j) \right\}, \ B \in \mathcal{S}. \tag{4.5}$$

Remark 4.2.1. *If the chain is positive Harris recurrent, it follows from Theorem 2.2.3, i) that*

$$\frac{1}{n} \sum_{j=0}^{n} \varphi(X_j) \xrightarrow{a.s.} E_{\tilde{\pi}} \{\varphi\}, \tag{4.6}$$

for any integrable function φ. Moreover, if the chain is aperiodic and positive Harris recurrent, it follows from Theorem 2.2.3, ii) that $X_n \xrightarrow{d} X$ as $n \to \infty$, where X is distributed according to $\tilde{\pi}$. Thus $\{X_n\}_{n\geq 0}$ converges in distribution to a unique invariant probability measure. In the real valued case, we will denote the limiting distribution by

$$\tilde{F}(x) = \frac{1}{E_A(T_A)} E_A \left\{ \sum_{j=0}^{T_A - 1} I_{(-\infty, x]}(X_j) \right\}, \ x \in \mathbb{R}. \tag{4.7}$$

On the other hand, as a consequence of our Corollary 3.3.6 and our Theorem 3.4.5 we obtain weak convergence in the Skorokhod space D of the empirical process $\beta_n(x) = \sqrt{n}(F_n(x) - \tilde{F}(x))$, $x \in \mathbb{R}$ and the quantile process $q_n(t) = n^{1/2}(F_n^{-1}(t) - \tilde{F}^{-1}(t))$, $t \in (0,1)$ associated to the Harris Markov chain $\{X_n\}_{n\geq 0}$, where $\tilde{F}$ is the limiting distribution given by (4.7). We call $\tilde{F}$ as the canonical distribution.

Corollary 4.2.2. *Let $\{X_n\}_{n\geq 0}$ be an aperiodic and positive Harris recurrent Markov chain on $\mathbb{R}$ with an accessible atom A. Assume that the canonical distribution $\tilde{F}$ is continuous. If $E(T_A^{\theta+\epsilon}) < \infty$ for some $\theta \geq 2 + \sqrt{2}$ and $\epsilon > 0$. then we have*

$$\beta_n(\cdot) \Rightarrow \tilde{B}(\tilde{F}(\cdot)) \text{ in } D(-\infty, \infty) \tag{4.8}$$

where $\tilde{B}(\cdot)$ is a zero-mean Gaussian processes and

$$\begin{aligned} Cov(\tilde{B}(s), \tilde{B}(t)) &= s \wedge t - st \\ &+ \sum_{j=1}^{\infty} Cov\left\{I_{[0,s]}(\tilde{F}(X_0)), I_{[0,t]}(\tilde{F}(X_j))\right\} \\ &+ \sum_{j=1}^{\infty} Cov\left\{I_{[0,t]}(\tilde{F}(X_0)), I_{[0,s]}(\tilde{F}(X_j))\right\} \end{aligned} \tag{4.9}$$

Corollary 4.2.3. *Let $\{X_n\}_{n\geq 0}$ be an aperiodic and positive Harris recurrent Markov chain on $\mathbb{R}$ with an accessible atom A. Assume that $\tilde{F}$ satisfies the conditions of Theorem 3.4.5 with $\tilde{F}' = \tilde{f}$. If $E(T_A^{\theta+\epsilon}) < \infty$ for some $\theta \geq 2 + \sqrt{2}$ and $\epsilon > 0$ then we have*

$$q_n(t) \Rightarrow \frac{\tilde{B}(t)}{\tilde{f}(\tilde{F}^{-1}(t))} \text{ in } D[0,1] \tag{4.10}$$

where $\tilde{B}(\cdot)$ is the Gaussian process given by (4.9).

The similarity tests proposed below are based on the weak convergence of the empirical and quantile process. So, as a consequence of Corollary 4.2.2 and Corollary 4.2.3 we will obtain the asymptotic null distributions for the classical statistics of Kolmogorov-Smirnov and Crámer-von Mises under the null hypothesis $\tilde{F} = F_0$. We will also obtain the asymptotic null distributions of tests based on the 2nd-order Mallows distance include similarity tests of location-scale families for Harris Markov chain with atom.

4.3 Kolmogorov-Smirnov and Cramér-von Mises Tests

The well-known global measures of discrepancy are given by

$$\sqrt{n}\,\|F_n - F\|_\infty \qquad \text{(Kolmogorov-Smirnov)}$$
$$n\int_{-\infty}^{\infty}(F_n(x)-F(x))^2 dF(x) \qquad \text{(Cramér-von Mises)}$$

where $\|F_n - F\|_\infty = \sup_x |F_n(x) - F(x)|$.

For a regenerative sequence $\{X_n\}_{n\geq 1}$ the common distribution F of the i.i.d. sequence is replaced by the canonical measure $\tilde{F}$.

Our Theorem 3.3.5 showed that under regularity conditions the empirical process

$$\beta_n(x) = \sqrt{n}(F_n(x) - \tilde{F}(x))$$

converges weakly to a zero-mean Gaussian process $\tilde{B}_{\tilde{F}} = \left\{\tilde{B}(\tilde{F}(x)) : x \in \mathbb{R}\right\}$ with covariance function

$$\begin{aligned} Cov(\tilde{B}(s),\tilde{B}(t)) &= s\wedge t - st \\ &+ \sum_{j=1}^{\infty} Cov\left\{I_{[0,s]}(\tilde{F}(X_0)), I_{[0,t]}(\tilde{F}(X_j))\right\} \\ &+ \sum_{j=1}^{\infty} Cov\left\{I_{[0,t]}(\tilde{F}(X_0)), I_{[0,s]}(\tilde{F}(X_j))\right\}. \end{aligned} \tag{4.11}$$

Now, we obtain the asymptotic null distribution of the Kolmogorov-Smirnov and Cramér-von Mises statistics for a Harris Markov chain with atom and for a $\mathcal{L}$-geometrically ergodic Markov chain $\{X_n\}_{n\geq 0}$. As in the i.i.d. case this results are based on the convergence of the empirical processes associated to $\{X_n\}_{n\geq 1}$. In this sense, our Corollary 3.3.6 and Corollary 3.2.5 suggest the following conditions.

Condition 4.3.1. *i) Let $\{X_n\}_{n\geq 0}$ be an aperiodic and positive Harris recurrent Markov chain on $\mathbb{R}$ with an accessible atom A satisfying $E(T_A^{\theta+\epsilon}) < \infty$ for some $\theta \geq 2 + \sqrt{2}$ and $\epsilon > 0$. And assume that the canonical distribution $\tilde{F}$ is continuous, or* *ii) Let $\{X_n\}_{n\geq 0}$ be an $\mathcal{L}$-geometrically ergodic Markov chain with values in $\mathbb{R}$. Assume that the distribution function $\tilde{F}$ of X_0 is continuous and satisfies*

$$\omega_{\tilde{F}}(\delta) \leq D|\log(\delta)|^{-\gamma} \ \ for \ \ some \ \ D > 0 \ \ and \ \ \gamma > 2. \tag{4.12}$$

with $\omega_{\tilde{F}}$ given by (3.8).

Lemma 4.3.2. *Assume that conditions 4.3.1 is satisfied. Then*

$$\sqrt{n}\left\|F_n - \tilde{F}\right\|_\infty \xrightarrow{d} \left\|\tilde{B}_{\tilde{F}}\right\|_\infty \tag{4.13}$$

and

$$n\int_{-\infty}^{\infty}(F_n(x) - \tilde{F}(x))^2 d\tilde{F}(x) \xrightarrow{d} \int_{-\infty}^{\infty}\tilde{B}^2_{\tilde{F}}(x)d\tilde{F}(x) \tag{4.14}$$

Proof. Under Condition 4.3.1 we have the hypotheses of our Corollary 3.3.6 satisfied. Thus $\beta_n \Rightarrow \tilde{B}_{\tilde{F}}$ on $(D, \mathcal{D}, \|\cdot\|_\infty)$. On the other hand, we have that the mappings $z \longrightarrow \int z^2(x)d\tilde{F}(x)$ and $z \longrightarrow \|z\|_\infty$ from D in $\mathbb{R}$ are continuous and $P(\tilde{B}_{\tilde{F}} \in C) = 1$. And the results follows from the continuous mapping Theorem 1.8.1. □

To finish this section, it is worth pointing out again that if $\{X_n\}_{n\geq 1}$ is a Markov chain with general state space, positive Harris recurrent and aperiodic that posses an atom and limiting distribution F_{lim} then $F_{lim} = \tilde{F}$ where

$$\tilde{F}(x) = \frac{1}{E_A(T_A)}E_A\left\{\sum_{j=0}^{T_A-1} I_{(-\infty,x]}(X_j)\right\},\ x \in \mathbb{R},$$

is the canonical distribution. Also, it is worth mentioning that Merlevede and Rio (2015) obtained the KMT (Komlós, Major and Tusnády) strong aproximation of empirical processes associated to an Harris recurrent geometrically ergodic Markov chain $\{X_n\}_{n\geq 1}$,

$$P\left(\sup_{1\leq k\leq n}|S_k - \tilde{\sigma}W_k| \geq c\log n + x\right) \leq a\exp(-bx). \tag{4.15}$$

where $S_k = \sum_{j=0}^{k} X_j$, W_k is a sequence of Brownian motions, $\tilde{\sigma}^2 = \frac{E\left\{\left(\sum_{j=0}^{T_A-1} X_j\right)^2\right\}}{E_A(T_A)}$ and a, b and c are positives constants conveniently chosen. And this could well be used to eventually derive rates of convergence similar to i.i.d. case.

4.4 Similarity tests based on Mallows distance

For the i.i.d. sequence with empirical distribution F_n and a common distribution function F, Samworth and Johnson (2008) showed that under "regularity conditions" the 2nd-order Mallows distance $d_2(F_n, F)$ satisfies

$$\sqrt{n}d_2(F_n, F) \xrightarrow{d} \left(\int_0^1 \frac{B^2(t)}{f^2(F^{-1}(t))}dt\right)^{1/2} \tag{4.16}$$

where B is the Brownian bridge and f the density function of F.

Using the same type of arguments as in Samworth and Johnson (2008) we will extend the use of statistics (4.16) to regenerative sequences.

Condition 4.4.1. *Assume that condition 4.3.1 is satisfied. And suppose that the canonical distribution $\tilde{F}$ possesses a density $\tilde{f}$ such that $\tilde{f}(\tilde{F}^{-1}(t))$ is positive and continuous for $0 \leq t \leq 1$ and that $\lim_{t\downarrow 0} \tilde{F}^{-1}(t)$ and $\lim_{t\uparrow 1} \tilde{F}^{-1}(t)$ are finite.*

Essentially Condition 4.4.1 requires that de density $\tilde{f}$ is positive and has a bounded support. In this case we do not need to worry about existence of 2nd moment ou higher moments.

Lemma 4.4.2. *Assume that Condition 4.4.1 holds. Then*

$$\sqrt{n}d_2(F_n, \tilde{F}) \xrightarrow{d} \left(\int_0^1 \frac{\tilde{B}^2(t)}{\tilde{f}^2(\tilde{F}^{-1}(t))}dt\right)^{1/2} \tag{4.17}$$

where $\tilde{B}$ is given by (4.11).

Proof. The representation result, Theorem 1.5.2, allows us to write

$$nd_2^2(F_n, \tilde{F}) = \int_0^1 n|F_n^{-1}(t) - \tilde{F}^{-1}(t)|^2 dt = \int_0^1 q_n(t)^2 dt \tag{4.18}$$

where $q_n(t)$ is the empirical quantile process

$$q_n(t) = \sqrt{n}(F_n^{-1}(t) - \tilde{F}^{-1}(t)),\ 0 < t < 1. \tag{4.19}$$

Our Theorem 3.4.5 show that $q_n(t) \Rightarrow \tilde{B}_q$ where $\tilde{B}_q = \dfrac{\tilde{B}(t)}{\tilde{f}(\tilde{F}^{-1}(t))}$.

Note that Condition 4.4.1 guarantees the maximum of $\tilde{f}$ is positive and the right-hand side of (4.17) is well defined. Since $P(\tilde{B} \in C) = 1$ we can apply the continuous mapping Theorem to the function $\int z^2(t)dt$ and (4.17) follows. $\square$

As pointed out in del Barrio et al.(1999) the right side of (4.17) is not easy to handle. For i.i.d. case where instead of $\tilde{B}$ we have the classical Brownian bridge B with $E(B^2(t)) = t(1-t)$, the integral

$$\int_0^1 \frac{E(B(t)^2)}{f^2(F^{-1}(t))}dt = \int_0^1 \frac{t(1-t)}{f^2(F^{-1}(t))}dt \tag{4.20}$$

is not finite even if F is Gaussian. . If (4.20) is finite and F has second moment finite was shown in Cuesta et al, (2000) that

$$nd_2^2(F_n, F) \xrightarrow{d} \int_0^1 \frac{B^2(t)}{f^2(F^{-1}(t))}dt.$$

Clearly, the condition that F has a second finite moment restricts the use of this result to distributions of light tail. To weaken this hypothesis and extend these techniques for heavy tail distributions one alternative is to use weighted Mallows distance as in Csörgő (2003), del Barrio et al. (2005) or Dorea and Lopes (2016).

Let $w : [0,1] \longrightarrow [0,1]$, $w(t) \geq 0$ and $\int_0^1 w(t)dt = 1$. Considerer the weighted Mallows distance

$$d_{2,w}^2(F_n, \tilde{F}) = \int_0^1 (F_n^{-1}(t) - \tilde{F}^{-1}(t))^2 w(t)dt$$

and

$$nd_{2,w}^2(F_n, \tilde{F}) = \int_0^1 q_n^2(t)w(t)dt. \tag{4.21}$$

By properly choose the weight function w, one should expect

$$\sqrt{n}d_{2,w}(F_n, \tilde{F}) \xrightarrow{d} \left(\int_0^1 \frac{\tilde{B}^2(t)}{\tilde{f}^2(\tilde{F}^{-1}(t))}w(t)dt\right)^{1/2}. \tag{4.22}$$

Remark 4.4.1. *Let B be a Brownian bridge, in the i.i.d. case under certain standard conditions we have:*

i) If $\int_0^1 \frac{t(1-t)}{f^2(F^{-1}(t))}w(t)dt < \infty$ *then* $nd_{2,w}^2(F_n, F) \xrightarrow{d} \int_0^1 \frac{B^2(t)}{f^2(F^{-1}(t))}w(t)dt.$

ii) If $\int_0^1\int_0^1 \frac{(s\wedge t - st)^2}{f^2(F^{-1}(s))f^2(F^{-1}(t))}w(t)w(s)dt < \infty$ *but* $\int_0^1 \frac{t(1-t)}{f^2(F^{-1}(t))}w(t)dt = \infty$ *then there exists* $\{a_n\}_{n\geq 1}$ *such that*

$$nd^2_{2,w}(F_n, F) - a_n \xrightarrow{d} \int_0^1 \frac{B^2(t) - E(B^2(t))}{f^2(F^{-1}(t))}w(t)dt.$$

In Gaussian case the used weight function was $w(t) \equiv 1$ *in ii), for distributions as Weibull, Gamma, Lognormal and Gumbel among others,*

$$w(t) = \frac{1}{I}\frac{L'(F^{-1}(t))}{F^{-1}(t)}, 0 < t < 1$$

where

$$I = \int_{-\infty}^{\infty} L^2(x)f(x)dx \;\; with \;\; L(x) = -1 - \frac{xf'(x)}{f(x)}, \;\; f(x) = F'(x).$$

And in the $\alpha-$*stable case*

$$w(t) = \begin{cases} k_* t^{-\beta}, & 0 < t < t_* \\ k_*(1-t)^{-\beta}, & t_* \leq t < 1. \end{cases}$$

where $0 < \alpha < 2$, $\beta < -2/\alpha$ *and* $k_* = \dfrac{1-\beta}{t_*^{1-\beta} + (1-t_*)^{1-\beta}}$.

iii) An essential result to obtain convergences in i) and ii) is the following approximation (Theorem 6.2.1 in Csörgő and Horváth (1993))

$$n^{1/2-v} \sup_{\frac{1}{n+1}\leq t\leq 1-\frac{1}{n+1}} \frac{|f(F^{-1}(t))q_n(t) - B_n(t)|}{(t(1-t))^v} = \begin{cases} O_P(\log n) & \text{if } v = 0 \\ O_P(1) & \text{if } 0 < v \leq 1/2 \end{cases} \quad (4.23)$$

where $\{B_n(t)\}_{n\geq 1}$ *is an sequence of Brownian bridges*

For regenerative samples, $w(t)$ must be such

$$q_n(\cdot)\sqrt{w(\cdot)} \Rightarrow \frac{\tilde{B}(\cdot)}{\tilde{f}(\tilde{F}^{-1})(\cdot)}\sqrt{w(\cdot)}$$

and then (4.22) follows by continuous mapping Theorem. For work on this direction, we refer to Csörgő and Yu (1996). On the other hand, in future works we hope to use the KMT strong approximation (4.15) to obtain an similar approximation as (4.23) for regenerative sequences and then we could use the techniques of the i.i.d. case to obtain the convergence (4.22). In this situation $w(t)$ must be such

$$\int_0^1 \frac{E(\tilde{B}^2(t))}{\tilde{f}^2(\tilde{F}^{-1}(t))}w(t)dt < \infty.$$

4.4.1 Similarity Tests for Location-Scale Families

Consider the location-scale family generated by a distribution G with zero-mean and unit-variance

$$\mathcal{G}_G = \left\{ H : H(x) = G\left(\frac{x-\mu}{\sigma}\right), \, \mu \in \mathbb{R}, \, \sigma > 0 \right\}. \tag{4.24}$$

For a given distribution F we want to test $F \in \mathcal{G}_G$.

Based on the 2nd-order Mallows distance and for a sequence of i.i.d. random variables with a common distribution F, del Barrio et al. (1999) proposed the used of the statistics

$$R_n = 1 - \frac{\left(\int_0^1 F_n^{-1}(t)G^{-1}(t)dt\right)^2}{\hat{\sigma}_n^2} \tag{4.25}$$

where F_n is the usual empirical distribution and $\hat{\sigma}_n^2$ the sample variance. As shown in del Barrio et al. (1999) the use of statistics (4.25) is fully justified by nothing that if

$$d_2(F, \mathcal{G}_G) := \inf \{ d_2(F, H) : H \in \mathcal{G}_G \} \tag{4.26}$$

then the infimum is attained by taking H with mean $\mu_H = \mu_F$ and $\sigma_H^2 = \left(\int_0^1 G^{-1}(t)F^{-1}(t)dt\right)^2$. Indeed, let $H(x) = G\left(\frac{x-\mu_H}{\sigma_H}\right)$ and μ_F and σ_F^2 the mean and variance of F. Then we have

$$H^{-1}(t) = \sigma_H G^{-1}(t) + \mu_H, \quad \int_0^1 (F^{-1}(t))^2 dt = \sigma_F^2 + \mu_F^2, \quad \int_0^1 (H^{-1}(t))^2 dt = \sigma_H^2 + \mu_H^2$$

and

$$\begin{aligned}
d_2^2(F, H) &= \int_0^1 \left(F^{-1}(t) - H^{-1}(t)\right)^2 dt \\
&= \sigma_F^2 + \mu_F^2 + \sigma_H^2 + \mu_H^2 - \int_0^1 F^{-1}(t)\left(\sigma_H G^{-1}(t) + \mu_H\right) dt \\
&= \sigma_F^2 + \mu_F^2 + \sigma_H^2 + \mu_H^2 - 2\mu_F\mu_H - 2\sigma_H \int_0^1 F^{-1}(t)G^{-1}(t)dt \\
&= (\mu_F - \mu_H)^2 + \sigma_F^2 + \left(\sigma_H - \int_0^1 F^{-1}(t)G^{-1}(t)dt\right)^2 - \left(\int_0^1 F^{-1}(t)G^{-1}(t)dt\right)^2.
\end{aligned}$$

By taking $\mu_H = \mu_F$ and $\sigma_H^2 = \left(\int_0^1 F^{-1}(t)G^{-1}(t)dt\right)^2$ we attain the infimum and

$$\begin{aligned}\frac{d_2^2(F,\mathcal{G})}{\sigma_F^2} &= 1 - \frac{\left(\int_0^1 F^{-1}(t)G^{-1}(t)dt\right)^2}{\sigma_F^2}\\ &= 1 - \frac{\left(\int_0^1 \left(F^{-1}(t) - \mu_F\right) G^{-1}(t)dt\right)^2}{\sigma_F^2}.\end{aligned}$$

The last equality shows that $\frac{d_2^2(F,\mathcal{G}_G)}{\sigma_F^2}$ is invariant with respect to location or scale changes. Hence, under null hypotheses $F \in \mathcal{G}_G$ we may take F with zero-mean and unit-variance. Now replacing F by the empirical distribution F_n and σ_F^2 by the sample variance $\hat{\sigma}_n^2$ we get R_n.

Moreover, we can write under null hypothesis

$$n\hat{\sigma}_n^2 R_n = A_n - B_n - C_n \tag{4.27}$$

where

$$A_n = \int_0^1 q_n^2(t)dt, \;\; B_n = \left(\int_0^1 q_n(t)dt\right)^2 \;\; \text{and} \;\; C_n = \left(\int_0^1 q_n(t)\tilde{F}^{-1}(t)dt\right)^2$$

being $q_n(\cdot)$ the empirical quantile process associated to $\tilde{F}$ and defined in (4.19).

To establish the limiting distribution of the statistics nR_n enough to derive the limiting distribution of A_n, B_n and C_n (see, del Barrio et al. (2005) for details). Next we extend the use of statistics nR_n to regenerative sequences.

Lemma 4.4.3. *Assume that Condition 4.4.1 holds. Then the statistics*

$$nR_n = 1 - \frac{\left(\int_0^1 F_n^{-1}(t)\tilde{F}^{-1}(t)dt\right)^2}{\hat{\sigma}_n^2} \tag{4.28}$$

converges to a non degenerated distribution given by

$$\int_0^1 \frac{\tilde{B}^2(t)}{\tilde{f}^2(\tilde{F}^1(t))}dt - \left(\int_0^1 \frac{\tilde{B}(t)}{\tilde{f}(\tilde{F}^1(t))}dt\right)^2 - \left(\int_0^1 \frac{\tilde{B}(t)\tilde{F}^{-1}(t)}{\tilde{f}(\tilde{F}^1(t))}dt\right)^2 \tag{4.29}$$

Proof. We make use of Theorem 3.4.5:

$$q_n(t) \Rightarrow \frac{B(t)}{\tilde{f}(\tilde{F}^{-1}(t))}$$

and the continuous mapping Theorem 1.8.1. Also, under null hypothesis it is assumed that $\tilde{F}$ possesses unit-variance. Thus $\hat{\sigma}_n^2 \xrightarrow{a.s.} 1$ by the SLLN for regenerative sequences (Theorem 2.2.3, (i)). Since we are assuming that $\tilde{f}$ has bounded support, questions concerning existence of moments do not arise. Also being $\tilde{f}$ positive we have the result directly by applying continuous mapping Theorem. □

Remark 4.4.2. *As in the previous section one should explore the use of convenient weight function in order to weaker the assumptions.*

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