REFERÊNCIA

COSTA, Paulo Henrique Pereira da; HÖGELE, Michael A.; RUFFINO, Paulo R. Stochastic n-point D-bifurcations of stochastic Lévy flows and their complexity on finite spaces. **arXiv**, 2021. Disponível em: https://arxiv.org/abs/1502.07915v5.

Stochastic n-point D-bifurcations of stochastic Lévy flows and their complexity on finite spaces

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August 17, 2021

At the occasion of Björn Schmalfuß's 65th birthday.

Abstract

This article refines the classical notion of a stochastic D-bifurcation to the respective family of n-point motions for homogeneous Markovian stochastic semiflows, such as stochastic Brownian flows of homeomorphisms, and their generalizations. This notion essentially detects at which level $k \leqslant n$ the support of the invariant measure of the k-point bifurcation has more than one connected component. Stochastic Brownian flows and their invariant measures which were shown by Kunita (1990) to be rigid, in the sense of being uniquely determined by the 1-and 2-point motions, and hence only stochastic n-point bifurcation of level $n = 1$ or $n = 2$ can occur. For general homogeneous stochastic Markov semiflows this turns out to be false. This article constructs minimal examples of where this rigidity is false in general on finite space and studies the complexity of the resulting n-point bifurcations.

Keywords: stochastic D-bifurcation, stochastic n-point motion, Markovian random dynamical system, stochastic Brownian flow, Markov chains, algorithmic bifurcation detection, Marcus canonical equation.

2010 Mathematical Subject Classification: 37H20, 37A50, 60J10, 60J10, 60J05, 60J25, 60J27, 60C05, 05A10, 05A15.

1 Motivation

This article refines the classical notion of a *stochastic dynamical bifurcation* or *stochastic D-bifurcation* given by L. Arnold in [4] to the respective n-point motion for homogeneous Markovian stochastic semiflows of (continuous) mappings, such as stochastic Brownian flows of homeomorphisms, and their generalizations. The generalizations include coalescing stochastic flows, which go back to the PhD thesis by Arriata [5], but are an active field of research [12, 18, 19, 20, 21, 28, 44, 53]. For this purpose we introduce the notion of a *stochastic n-point D-bifurcation* in this setting. It essentially detects the lowest level n at which the respective n-point motion of such a stochastic flow undergoes a stochastic D-bifurcation in the sense of a qualitative change of the support of the invariant measure. This concept is finer than the classical stochastic D-bifurcation, since trivially any stochastic D-bifurcation is equivalent to a stochastic 1-point D-bifurcation. To the best of our knowledge, the concept of a stochastic n-point D-bifurcation seems to be new in the literature and in our view it has the potential to provide interesting new insights

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into the tipping behavior of stochastic Lévy (semi-)flows and more general systems given by Lévy driven stochastic (partial) differential equations.

It is known for a long time that the stochastic dynamics of a stochastic differential equation is only understood partially by the respective Markov semigroups and their generators. Indeed, these objects do not take into account the effect of the dependence structure between the trajectories of n ensemble members with different initial conditions, that is, the respective n-point motion. This conceptual problem was solved with the introduction of the notion of the associated stochastic flow in the works by Elworthy, Baxendale, Bismut, Ikeda, Kunita, Watanabe among others (see e.g. [8, 13, 22, 29, 37, 39] and the references therein).

The motivation for the generalization of stochastic D-bifurcation to n-point motions is twofold:

I) The 1- and 2-point rigidity of the laws of stochastic Brownian flows of homeomorphisms.

II) Discretization procedures for homogeneous Markovian semiflows of continuous functions.

Motivation I) is rooted in a sequence of results for homogeneous stochastic Brownian flows of homeomorphisms in Kunita [37]. The distribution of such stochastic Brownian flow is uniquely determined by the laws of the families of the corresponding 1-point and 2-point motions, see Theorem 4.2.5 and formula (19) in Kunita [37]. This sort of 1- and 2-point rigidity of the law of the flow is due to the Gaussian nature of the marginal laws and their n-point extensions and can be read off directly from the structure of the infinitesimal generators. This rigidity carries over to any invariant distribution Π of the Brownian flow (in the sense that the flow is Π-preserving) in the following sense. The flow is Π-preserving if and only if Π is the invariant measure of the respective 1-point motion and $\Pi \otimes \Pi$ is the invariant measure of the 2-point motion (see Theorem 4.3.2(v) in [37]). In other words, the Gaussianity of Brownian flows imposes that the complete dependence structure of the n-point motion of the flow is uniquely determined by the respective infinitesimal covariances of 2-point motion contained in the n-point motion, that is to say, its 1 and 2-point characteristics. However, the 2-point characteristics can change the law of the flow, as shown by Baxendale in [10], who studies the ergodicity of the 1- and 2-point Brownian motion on a torus. Homogeneous Markov semiflows of homeomorphisms, which we denote for convenience as stochastic Lévy semiflows, generalize the notion of the respective stochastic Brownian flows by dropping the continuity assumption in time, cf. [2, 24, 25, 38]. However, in general, neither their (non-Gaussian) laws nor the respective invariant measures can be expected to be rigid in the sense of Brownian flows. This is due to the lack of continuity resulting in the non-local nature of the infinitesimal generator. Therefore, the law and the invariant measures of the respective n-point motion for $n > 2$ of the flow provide new and finer information about the law and the invariant measure of the flow. It is therefore natural to ask for examples of stochastic Lévy flows of homeomorphisms whose laws are underdetermined by their 2-point motions and which are minimal in some sense. While non-trivial stochastic Brownian flows are confined to spaces where Gaussian laws can be defined properly such as Hilbert spaces, stochastic Lévy flows exhibit jumps and have a rich behavior already on spaces of finite sets. In general, Lévy driven stochastic differential equations yield stochastic Lévy flows under rather restrictive conditions on the coefficients [1, 2, 38]. However, their easiest representatives are given as continuous homogeneous Markov chains with values in a finite state space. In order to settle ideas we focus in this article on finite state spaces.

Motivation II) is a bit more far-flung. There is an ever growing necessity to detect (stochastic) bifurcations or tipping of very high dimensional stochastic semiflows or even more general systems, such as in stochastically perturbed general circulation models in climatology (see for instance [30, 31]). The notion of a stochastic n-point D-bifurcation over finite points seems to be promising for the development of discretization procedures for systems in continuous time and space. The aim is to detect stochastic n-point bifurcations (of low order) of the original system via the respective detection in the discretized system. An initial step in this direction such discretizations is done in [35, 36]. Rigorous discretization

results for stochastic Lévy (semi-) flows, however, are beyond the scope of the current article and left for future research. However, for the realization of this it is primordial to understand the complexity of such discretized systems, the ground work of which laid in this article.
In this article we answer the following natural questions:

Q.1: (a) How can a stochastic n-point D-bifurcation be defined rigorously for homogeneous Markovian stochastic (semi-) flows of bijections and mappings? (b) What is the minimal setting in order to define a stochastic n-point D-bifurcation for n-point motions, for instance in situations which do not necessarily come from a stochastic flow?

Q.2: (a) What are a minimal examples (in the sense of the smallest spaces $\mathcal{M}$) of a stochastic Lévy flow in this type, whose invariant law is not determined by its 1- and 2-point motion (opposed to the rigidity of a stochastic Brownian flow)? (b) Are there are stochastic n-point motions of any level n for any state space cardinality m? (c) How can we detect the level of a respective n-point bifurcation given the invariant measure of the flow algorithmically?

Q.3: (a) How "complex" can the stochastic n-point D-bifurcations become over a given finite space? That is, how many possible flows of mappings do exist with the same given n-point characteristics? In the context of the rigidity of the laws of stochastic Brownian flows of homeomorphisms: (b) How many linearly independent restrictions are given of every level of n-point motion, and how many are necessary to completely determine the law of a respective stochastic flow?

The article is organized as follows. After a review on stochastic flows, Section 2 provides the setup in terms of a homogeneous Markov n-point system, which generalizes the respective notion of an n-point motion of a homogeneous Markov semiflow to consistent families of transition probabilities in the spirit of [44] and [45]. In Definition 12 of Subsection 2.2 we introduce the notion of a stochastic n-point D-bifurcation, which completely answers Q.1.
Section 3 is focussed on the special case of stochastic Lévy flows over finite sets $\mathcal{M}$ with $|\mathcal{M}| = m < \infty$. First we construct two important classes of examples of n-point bifurcations in finite spaces with m elements and show stochastic n-point D-bifurcations well beyond the well known examples mentioned above (for $n = 1, 2$) answering Q.2(a) and (b). In Subsection 3.2 we present a simple algorithm how to detect the precise level of a stochastic n-point D-bifurcation given the invariant measure of a stochastic Lévy (semi-) flow. We apply this algorithm to two examples of stochastic 3-point bifurcations, including the minimal one. This provides an answer to Q.2(c). The second part of Section 3 is devoted to the study of the complexity of stochastic n-point D-bifurcations in Q.3. For stochastic semiflows of mappings we give a complete answer to Q.3(a) in terms of a recursive formula, which is verified for low dimensions by hand and for large values computationally. In case of stochastic flows of bijections, which essentially are the discrete analogue of stochastic Brownian flows of homeomorphisms we conjecture based on extensive simulations with the data base [51] a combinatorially interesting, highly nontrivial triangular array of natural numbers $T(m, k)$ to what degree a stochastic flow is determined by its n-point characteristics ($n \leqslant m$). Finally, in Subsection 3.4 we embed a stochastic Lévy flow (and its stochastic n-point D-bifurcation) in continuous time and space, in terms of Marcus canonical equation, following the lines of [41].

2 The notion of a stochastic n-point D-bifurcation

We start with some preliminary notation. Let $\mathbb{T} \in \{\mathbb{N}_0, [0, \infty)\}$ and $\mathcal{M}$ be a Polish space, that is, a separable, topological space, whose topology is metrizable with respect to a complete metric ρ. This includes, $\mathcal{M}$ being a finite set or the Euclidean space $\mathbb{R}^d$. Denote by $\mathcal{C}_\mathcal{M} = \mathcal{C}(\mathcal{M}, \mathcal{M})$ the continuous mappings from $\mathcal{M} \to \mathcal{M}$. In case of $\mathcal{M}$ being discrete and equipped with the discrete topology this coincides with the self-maps $\{\mathcal{M} \to \mathcal{M}\}$. It is equipped with the metric

$$\rho_\infty(f, g) := \sup_{x\in\mathcal{M}} \rho(f(x), g(x)), \qquad f, g \in \mathcal{C}_\mathcal{M}.$$

We denote by $\mathcal{H}_\mathcal{M} := \mathrm{Homeo}(\mathcal{M}, \mathcal{M})$ the space of homeomorphisms from $\mathcal{M} \to \mathcal{M}$. Note that in case of $\mathcal{M}$ being discrete with the discrete topology this coincides with the permutations $\mathrm{Sym}(\mathcal{M}) = \{\sigma : \mathcal{M} \to \mathcal{M} \mid \text{ bijective } \}$. It is equipped with the metric

$$\rho^\infty(f, g) = \rho_\infty(f, g) + \rho_\infty(f^{-1}, g^{-1}), \qquad f, g \in \mathcal{H}_\mathcal{M}.$$

2.1 The setup and the main notation

We start with the standard setup, see for instance [37, 39, 44].

Definition 1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.*

1. *A family $(\varphi_{s,t})_{\substack{s,t\in\mathbb{T}\\ s\leqslant t}}$ of random self-maps on $\mathcal{M}$ is called a **Markovian stochastic semiflow of continuous self-maps of** $\mathcal{M}$, if there is $\mathcal{N} \in \mathcal{F}$ with $\mathbb{P}(\mathcal{N}) = 0$ such that the following is satisfied:*
   - *(a) For all $s, t \in \mathbb{T}$ with $s \leqslant t$ and $\omega \in \mathcal{N}^c$ it follows $\varphi_{s,t}(\omega) \in \mathcal{C}_\mathcal{M}$.*
   - *(b) For all $s, t, u \in \mathbb{T}$ with $s \leqslant u \leqslant t$, $\omega \in \mathcal{N}^c$ and $x \in \mathcal{M}$ it is valid $\varphi_{s,t}(\omega) = \varphi_{u,t}(\omega) \circ \varphi_{s,u}(\omega)$.*
   - *(c) For all $s \in \mathbb{T}$ and $\omega \in \mathcal{N}^c$ $\varphi_{s,s}(\omega) = id_\mathcal{M}$.*
   - *(d) For all $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}$ with $t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n$ the family of increments $(\varphi_{t_1,t_2}, \ldots, \varphi_{t_{n-1},t_n})$ is independent.*
   - *(e) For fixed $s', t' \in \mathbb{T}$ and $\omega \in \mathcal{N}^c$ the mappings $t \mapsto \varphi_{s',t}(\omega)$ and $s \mapsto \varphi_{s,t'}(\omega)$ are càdlàg (right-continuous with left limits).*
2. *A Markovian stochastic semiflow $(\varphi_{s,t})_{\substack{s,t\in\mathbb{T}\\ s\leqslant t}}$ of continuous self-maps on $\mathcal{M}$ is **homogeneous** if*
   - *(f) For all $s, t \in \mathbb{R}$ with $s \leqslant t$ and $h \in \mathbb{T}$ such that $s + h, t + h \in \mathbb{T}$ we have $\varphi_{s,t} \overset{d}{=} \varphi_{s+h,t+h}$.*

   *A stochastic homogeneous Markovian semiflow $(\varphi_{s,t})_{\substack{s,t\in\mathbb{T}\\ s\leqslant t}}$ of continuous self-maps of $\mathcal{M}$ is called a **stochastic Lévy semiflow** of continuous self-maps.*
3. *A stochastic Lévy semiflow $(\varphi_{s,t})_{\substack{s,t\in\mathbb{T}\\ s\leqslant t}}$ of continuous self-maps of $\mathcal{M}$ is called a **stochastic Lévy flow of homeomorphisms** if the property (a) in item 1. is replaced by*
   - *(a') For all $s, t \in \mathbb{T}$ with $s \leqslant t$ and $\omega \in \mathcal{N}^c$ it follows $\varphi_{s,t}(\omega) \in \mathcal{H}_\mathcal{M}$.*
4. *A stochastic Lévy semiflow $(\varphi_{s,t})_{\substack{s,t\in\mathbb{T}\\ s\leqslant t}}$ of continuous self-maps of $\mathcal{M}$ is called **stochastic Brownian semiflow of continuous self-maps of** $\mathcal{M}$ or in Kunita's notation [37] a **Brownian motion with values in** $\mathcal{C}_\mathcal{M}$ if the property (e) in item 1. is replaced by*
   - *(e') For fixed $s', t' \in \mathbb{T}$, $s' \leqslant t'$, the mappings $t \mapsto \varphi_{s',t}(\omega)$ and $s \mapsto \varphi_{s,t'}(\omega)$ are continuous.*
5. *A stochastic Brownian semiflow $(\varphi_{s,t})_{\substack{s,t\in\mathbb{T}\\ s\leqslant t}}$ of continuous self-maps of $\mathcal{M}$ is called a **stochastic Brownian flow of homeomorphisms** if the property (a) in item 1. is replaced by (a') of item 3.*

6. *A stochastic Brownian semiflow of continuous self-maps of $\mathcal{M}$ is called* ***homogeneous*** *if it satisfies property (f) in item 2.*

Example 2. A stochastic Lévy semiflow of continuous self-maps in discrete time $\mathbb{T}$ can be written as the abstract random walk of a random i.i.d. sequence $(\xi_t)_{t\in\mathbb{T}}$ of self-maps in $\mathcal{M}$. Then, for a positive integer t, the flow $\varphi_t = \xi_t(\omega) \circ \ldots \circ \xi_1(\omega)$. See for instance Arnold [4], LeJan and Raimond [44] and references therein.

It is one of the main achievements of [37] that stochastic Brownian semiflows of mappings can be characterized as the solution flow of the Fisk-Stratonovich SDE in $\mathcal{C}_{\mathcal{M}}$. More general stochastic Lévy (semi-)flows are found to satisfy the same in case of Marcus canonical equations, see [38, 39].

Definition 3. *Given a stochastic homogeneous Lévy semiflow φ of continuous self-maps in $\mathcal{M}$, $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_n) \in \mathcal{M}^n$. The respective* ***stochastic n-point motion of*** *φ is defined by*

$$\varphi_{s,t}(x) := (\varphi_{s,t}(x_1), \ldots, \varphi_{s,t}(x_n)), \qquad t \geqslant s \geqslant 0.$$

For a detailed overview we refer the reader to Fujiwara and Kunita [25]. Following the lines of the proof of [37], Theorem 4.2.1, the respective transition probabilities

$$P^{(n)}_{s,t}(x, E) := \mathbb{P}(\varphi_{s,t}(x) \in E)$$

satisfy the Markov property with respect to the filtration $\mathcal{F}_{s,t}$ generated by the stochastic Lévy (semi)-flow $\varphi_{s,t}$. We denote by $\pi^k_\ell : \mathcal{M}^k \to \mathcal{M}^{k-1}$, for $1 \leqslant \ell \leqslant k$, the **projection along the ℓ-th coordiante**

$$\pi^k_\ell(x_1, \ldots, x_k) := (x_1, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_k) \in \mathcal{M}^{k-1}.$$

The following definition turns out to be crucial for the generalization of the n-point motions of a stochastic flow to situations of merely compatible Markovian families.

Definition 4 (Homogeneous n-point Markov System)**.** *Let $\mathcal{M}$ be a Polish space equipped with its Borel σ-algebra $\mathcal{B}(\mathcal{M})$ and $n \in \mathbb{N}$ satisfying $n \leqslant |\mathcal{M}|$. Consider a family $(P^k)_{1\leqslant k\leqslant n}$ of homogeneous transition kernels*

$$P^k : \mathbb{T} \times \mathcal{M}^k \times \mathcal{B}(\mathcal{M}^k) \to [0, 1],$$

in the following sense:

1. *For any $t \in \mathbb{T}$, $A \in \mathcal{B}(\mathcal{M}^k)$ the map $x \mapsto P^k_t(x, A)$ is measurable.*
2. *For any $t \in \mathbb{T}$, $x \in \mathcal{M}^k$ the map $A \mapsto P^k_t(x, A)$ is a probability measure.*
3. *For all $0 \leqslant s \leqslant t$, $x \in \mathcal{M}^k$ and $A \in \mathcal{B}(\mathcal{M}^k)$ the kernel satisfies the Chapman-Kolmogorov equation*

$$P^k_t(x, A) = \int_{\mathcal{M}^k} P^k_{t-s}(z, A)\ P^k_s(x, \mathrm{d}z).$$

In addition, the compatibility of $(P^k)_{1\leqslant k\leqslant n}$ in the sense that all the marginals of P^k_t are given by P^{k-1}_t, that is,

$$P^k_t(x, (\pi^k_\ell)^{-1}(A)) = P^{k-1}_t(\pi^k_\ell(x), A) \qquad \text{for all } t \in \mathbb{T},\ x \in \mathcal{M}^k, A \in \mathcal{B}(\mathcal{M}^{k-1}) \text{ and } 1 \leqslant \ell \leqslant k.$$

The pair $(\mathcal{M}, (P^k)_{1\leqslant k\leqslant n})$ is called a ***homogeneous n-point Markov system****.*

Example 5. In case of $\mathcal{M} = \{1, \ldots, m\}$, a stochastic Lévy semiflow of mappings $\mathcal{M} \to \mathcal{M}$ and given the respective homogeneous m-point Markov system $(P^k)_{1 \leqslant k \leqslant m}$ the law of the flow is uniquely determined by the laws of the respective m-point motions.

Example 6. Given a stochastic Lévy semiflow of mappings over $\mathcal{M}$, $|\mathcal{M}| = \{1, \ldots, m\}$, $n \leqslant m$, and the respective family of laws of the k-point motions for $1 \leqslant k \leqslant n$ forms a homogeneous n-point Markov system.

Example 7. Consider $\mathcal{M} = \{1, \ldots, m\}$ and $n < m$ a homogeneous n-point Markov system $(P^k)_{1 \leqslant k \leqslant n}$ over $\mathcal{M}$. Such laws are not necessarily the distributions of a n-point motion of a stochastic Lévy semiflow $\mathcal{M} \to \mathcal{M}$.
For instance: A Markov chain in $\mathcal{M} = \{0, 1\}$ with all transition matrix entries equal to $1/2$ and a lifted process in $\mathcal{M}^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ with all transition matrix entries equal to $1/4$. This system defines a homogeneous 2-point Markov system, but a look at the diagonal shows that this dynamics in $\mathcal{M}^2$ is clearly not generated by any semiflow of mappings.

The basic object of study in this article is the invariant measure of a homogeneous n-point Markov system and its (compatible) projections.

Lemma 8. *Let μ be an invariant measure for a Markov process X generated by the transition probabilities P^n in $\mathcal{M}^n$. If $\pi_k^n(X)$ is also a Markov process, then the induced measure $(\pi_k^n)_*\mu$ is an invariant measure for the process $\pi_k^n(X)$ in $\mathcal{M}^{n-1}$. Moreover, if μ is ergodic then $(\pi_k^n)_*\mu$ is ergodic in $\mathcal{M}^{k-1}$.*

Proof. For convenience we drop the superscript n whenever possible. It is enough to treat the case $t = 1$ which we omit in the sequel. Let $P^n(x, A)$ be the family of transition probabilities of the process X in $\mathcal{M}^n$ for $x \in \mathcal{M}^n$ and subsets $A \subset \mathcal{M}^n$. The fact that the projection $\pi_k(X)$ generates a Markov process in $\mathcal{M}^{n-1}$ means that the transition probabilities in $\mathcal{M}^{n-1}$, denoted by $P^{n-1}(\pi_k(x), B)$, is well defined for any $B \subset \mathcal{M}^{n-1}$ and it is given by

$$P^{n-1}(\pi_k(x), B) = P^n(x, \pi_k^{-1}(B))$$

for all $x \in \mathcal{M}^n$. Now, by the induced measure theorem:

$$\begin{aligned}
(\pi_k)_*\mu(B) &= \mu(\pi_k^{-1}(B)) = \int_{\mathcal{M}^n} P^n(x, \pi_k^{-1}(B)) \, d\mu(x) \\
&= \int_{\mathcal{M}^{n-1}} P^n(\pi_k^{-1}(y), \pi_k^{-1}(B)) \, d(p_k)_*\mu(y) = \int_{\mathcal{M}^{n-1}} P^{n-1}(y, B) \, d(\pi_k)_*\mu(y).
\end{aligned}$$

The ergodicity follows directly. □

Note that any homogeneous n-point Markov system $(\mathcal{M}, (P^k)_{1 \leqslant k \leqslant n})$ can be considered as a compatible extension of P^1 from $\mathcal{M}$ to many possible distributions P^k on $\mathcal{M}^k$. In the sequel we define the stochastic n-point motion of a family of a given homogeneous Markov transitions kernel P on $\mathcal{M}$ (i.e. the characteristics of a 1-point motion) as any homogeneous n-point Markov system $(\mathcal{M}, (P^k)_{1 \leqslant k \leqslant n})$ with $P^1 = P$ and an addition symmetry condition, which guarantees the indistinguishability of the particles.

This notion is more general than a stochastic n-point motions of a stochastic Lévy semiflow of mappings, since the latter not necessarily needs to exist as seen in Example 7, where the 2-point motion does not define a semiflow. All such stochastic n-point motions define a homogeneous n-point Markov system with an additional indistinguishability condition, which is also satisfied in case of a stochastic n-point motion coming from a homogeneous Lévy semiflow of mappings.

Definition 9. *Consider a family P of homogeneous Markov transition kernels $P_t(x, A)$, with $x \in \mathcal{M}$, $t \in \mathbb{T}$ and $A \in \mathcal{B}(\mathcal{M})$ which satisfies the classical Chapman-Kolmogorov equation*

$$P_t(x, A) = \int_{\mathcal{M}} P_{t-s}(z, A)\ P_s(x, \mathrm{d}z), \qquad 0 \leqslant s \leqslant t.$$

For any $n \leqslant |\mathcal{M}|$ a homogeneous n-point Markov system $(\mathcal{M}, (P^k)_{1\leqslant k\leqslant n})$ which satisfy

1. *the* ***indistinguishability condition*** *of the components*

$$P_t^n((x_1, \dots, x_n), B_1 \times \dots \times B_n) = P_t^n((x_{\sigma(1)}, \dots, x_{\sigma(n)}), B_{\sigma(1)} \times \dots \times B_{\sigma(n)}), \qquad \sigma \in S_n, \quad (2.1)$$

   where $(x_1, \dots, x_n) \in \mathcal{M}^n$ and $B_1, \dots, B_n$ are Borel sets in $\mathcal{M}$, and

2. *the* ***extension property***

$$P^1 = P,$$

is called a ***n-point motion of*** P ***(in law).***

Remark 10. Definition 4 naturally extends to the respective semigroups as follows. Given a homogeneous n-point Markov system $(\mathcal{M}, (P^k)_{1\leqslant k\leqslant n})$ and a continuous bounded function $f : \mathcal{M}^k \to \mathbb{R}$ the associated Markov semigroup is defined by

$$P_t^k f(x) = \int_{\mathcal{M}^k} f(y)\, P_t^k(x, \mathrm{d}y), \qquad t \in \mathbb{T},\ x \in \mathcal{M}^k.$$

The compatibility condition of the family $(P^k)_{1\leqslant k\leqslant n}$ in terms of the semigroup is equivalent to

$$P_t^\ell f(x_1, \dots, x_\ell) = P_t^k g(y_1, \dots, y_k) \qquad \text{for all } \ell \leqslant k,$$

whenever f and g are symmetric bounded continuous functions in the sense that

$$g(y_1, \dots, y_k) = f(y_{i_1}, \dots, y_{i_\ell})$$

for a fixed subset $\{i_1, \dots, i_\ell\} \subset \{1, \dots, k\}$ and $(x_1, \dots, x_\ell) = (y_{i_1}, \dots, y_{i_\ell})$. In the case of a compatible family of Feller semigroup, see also e.g. [44, Def.1.1].

Remark 11. The concept of n-point D-bifurcation for a stochastic n-point motion of a family of kernels P covers the following three levels of generality:

(1) The most restrictive situation is to assume that the random dynamics comes from a stochastic flow of (measurable) bijections, which are the discrete analogue of stochastic Brownian flows of homeomorphisms.

(2) An intermediate scenario is a stochastic Lévy semiflow of (measurable) mappings φ from $\mathcal{M}$ to $\mathcal{M}$, which allows for the coalescence of particles. As for the bijective case, the diagonal is still positively invariant.

(3) However, as shown in Example 7, in general, the existence of a semiflow cannot be guaranteed. Hence the third perspective consists of stochastic n-point motions of a family of homogeneous transition kernel P in the sense of Definition 9, independently of any flow in $\mathcal{M}$. As a consequence, sub-diagonals may also no longer be positively invariant, that is, particles can coalesce and "split".

Our concept of n-point bifurcation applies to all these three cases covered by the notion of a n-point motion in the sense of Definition 9.

2.2 The definition of a stochastic n-point D-bifurcation

In the theory of deterministic dynamical systems a bifurcation occurs when the change of a parameter ε of the flow affects the support of the invariant measure topologically, such as for instance splitting into two or more disconnected invariant domains. This is well described in classical dynamical systems, where the precise definition is based on breaking local topological equivalences of the flows (see e.g. Katok and Hasselblatt [33] and the references therein). For stochastic systems generated by Itô-Stratonovich equations, the bifurcation is mostly considered as the sign change of the top Lyapunov exponent, see e.g. L. Arnold [4], [7] or recently [43]. These two situations have in common the fact that they are observing a breaking in the topology of the support of invariant measures, but at different levels: In the deterministic case, the invariant measures are considered in $\mathcal{M}^1$, with trivial extension to $\mathcal{M}^n$ as the respective product measure; in the stochastic case, the sign of the Lyapunov exponents points to properties of the invariant measures in $\mathcal{M}^2$. See the explicit example by Baxendale [9], where a bifurcation happens for Brownian motions in the torus: i.e. the top Lyapunov exponent change the sign but the law of the 1-point motion is not affected, see also [10].

We extend these 1 and 2-point phenomena to n-point motions and introduce the following natural generalization of a stochastic D-bifurcation for more general stochastic flows in Polish spaces.

Definition 12 (Stochastic n-point D-bifurcation)**.** *Let $\mathcal{M}$ be a Polish space and $((P^{k,\varepsilon})_{1\leqslant k\leqslant N})_{\varepsilon\in I}$ be a family of homogeneous N-point Markov systems in $\mathcal{M}$ indexed by a parameter ε taking values in a real interval I. We say, that the family $((P^{k,\varepsilon})_{1\leqslant k\leqslant N})_{\varepsilon\in I}$ exhibits a* ***stochastic n-point D-bifurcation at the bifurcation point*** *$\varepsilon_D \in I$ for a certain level $n \leqslant N$ if it satisfies the following:*

1. *For any $\bar{x} \in \mathcal{M}^k$, $1 \leqslant k \leqslant n$, the mapping $\varepsilon \mapsto P^{k,\varepsilon}(\bar{x},\cdot)$ is continuous with respect to the weak topology on the space of probability measures $\mathcal{P}(\mathcal{M}^k)$.*

2. *For any $\varepsilon \in I$ there exists an invariant distribution μ^ε with respect to $P^{n,\varepsilon}$ on $\mathcal{M}^n$ satisfying the following.*

   (a) For any $\varepsilon > \varepsilon_D$ the measure μ^ε is ergodic and all sets of the family $(supp(\mu^\varepsilon))_{\varepsilon>\varepsilon_D}$ are topologically equivalent among each other, but

   $$supp(\mu^\varepsilon) \text{ is not topologically equivalent to } supp(\mu^{\varepsilon_D}).$$

   (b) For any sequence of projections $\pi^2_{k_2}, \ldots, \pi^n_{k_n}$ where $k_i \in \{1,\ldots,i\}$, $i \in \{2,\ldots,n\}$

   $$supp\big((\pi^2_{k_2}\circ\cdots\circ\pi^n_{k_n})_*\mu^\varepsilon\big) \text{ is topologically equivalent to } supp\big((\pi^2_{k_2}\circ\cdots\circ\pi^n_{k_n})_*\mu^{\varepsilon_D}\big).$$

Examples are given in Subsections 3.1.1, 3.1.2 and 3.1.3. Since each invariant measure on $\mathcal{M}^1$ can have many lifts to invariant measures in higher levels $\mathcal{M}^k$, these lifts can exhibit more than one stochastic n-point D-bifurcation, at different levels k. Moreover, the same bifurcation on k-point motion can have projections into different invariant measures on $\mathcal{M}^1$ (depending on the sequence of projections). The precise level n of the n-point D-bifurcation is detected algorithmically in Subsection 3.2. In Subsection 3.3 the respective numbers of linear restrictions (and its complementary degrees of freedom) are quantified.

Remark 13. 1. As for comparison (following Kunita [37]), we consider a homogeneous stochastic Brownian flow φ in the group of diffeomorphisms in Euclidean space $\mathcal{M} = \mathbb{R}^d$ with the infinitesimal mean

$$b(x) = \lim_{h\to 0+} \frac{1}{h}\left[\mathbb{E}\left[\varphi_h(x)\right] - x\right], \qquad \forall\, x \in \mathbb{R}^d,$$

and the infinitesimal covariance

$$a(x,y) = \lim_{h\to 0+} \frac{1}{h} \left[\mathbb{E}\left[\big(\varphi_h(x)-x\big)\big(\varphi_h(y)-y\big)^*\right]\right], \qquad \forall\, x,y \in \mathbb{R}^d.$$

Given certain regularity conditions on this parameters (satisfied for instance by flows of SDE generated by smooth vector fields with bounded derivatives) the law of φ in the group of diffeomorphisms is determined by $a(x,y)$ and $b(x)$ [37, Thm. 4.2.5, p. 126]. In other words, the law of a (homogeneous) stochastic Brownian semiflow (hence the law of its n-point motion, with $n \geqslant 2$) is fully determined by the laws of its 1-point motion and its 2-point motion. This result tells us that classical stochastic flows for SDEs driven by Brownian motion generically do not furnish the richness of flows differing only on higher n-point motion with $n > 2$.

2. There are several notions of bifurcation in the literature, which are of rather independent nature (for a discussion we refer to [4], Section 9.1). An alternative definition of a bifurcation would be the sign change of the (leading) Lyapunov exponent. In [10] the author shows a sign change for the 2-point motion of the Lyapunov exponent, while leaving the 1-point characteristics invariant, for a stochastic Brownian motion on the torus. While item 1) tells us, that there are at most stochastic 2-point D-bifurcations for homogeneous stochastic Brownian flows of homeomorphisms, this result indicates that, in fact, there are 2-point bifurcations for homogeneous stochastic Brownian flows of homeomorphisms.

Remark 14. The problem we are addressing here is also related to the recent results [32] by Jost, Kell and Rodrigues where they study conditions under which the transition probabilities (1-point motion) in a manifolds can be represented by families of random maps. In the same article, they consider further conditions for regularity and representations by diffeomorphisms. For this kind of problem in the context of flows which are merely measurable we refer to Kifer [34] and Quas [42].

Remark 15. The flows we are interested in here are also related to the flow of measurable mappings of Le Jan and Raimond [44] (see also [45]) in the following sense: their flows are constructed from a family of Feller compatible semigroups in $C(\mathcal{M}^n)$, $n \geqslant 1$, which preserves the diagonal. They are also constructed based on the observation of the statistics of the n -point motion, for $n \geqslant 1$. Problems related to synchronization can also be considered in the context for 2- and n-point motions [27, 49].

3 Stochastic n-point D-bifurcations in finite space

3.1 Examples in finite space

In this section we construct different examples that exhibit a stochastic n-point D-bifurcation of some level $n \geqslant 2$. The purpose is twofold: We construct classes of examples of arbitrarily large cardinality, which are interesting in its own right, but also yield the example of a minimal space $\mathcal{M}$ with $m = |\mathcal{M}| = 4$ announced in the Introduction.

3.1.1 Minimal example of a stochastic 2-point D-bifurcation (without any semiflow)

Initially we show an example for $m = 2$. The novelty here is, that it does not come from a semiflow.

Example 16. Let (X_n) be a Markov chain with state space $\mathcal{M} = \{0, 1\}$ and 1-point transition probability matrix given by

$$P^1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Note that $\mu^1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ is the unique invariant measure associated with this system. Consider the following 2-point motion associated to this system on $\mathcal{M}^2 = \{(0,0),(0,1),(1,0),(1,1)\}$ (in lexicographical order) with the transition probability matrix given below

$$P^2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

This system has an ergodic measure given by $\mu^2 = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$ whose support is equal to $\{(0,0),(1,1)\}$. On the other hand, for $\varepsilon \in (0, \frac{1}{4})$ the system

$$P^{2,\varepsilon} = \begin{pmatrix} \frac{1}{4}+\varepsilon & \frac{1}{4}-\varepsilon & \frac{1}{4}-\varepsilon & \frac{1}{4}+\varepsilon \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4}+\varepsilon & \frac{1}{4}-\varepsilon & \frac{1}{4}-\varepsilon & \frac{1}{4}+\varepsilon \end{pmatrix} \qquad \text{defines a 2-point motion of} \qquad P^{1,\varepsilon} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

In addition, it exhibits a stochastic a 2-point D-bifurcation in the sense of Definition 12, Indeed, it is straightforward that while $\varepsilon > \varepsilon_D = \frac{1}{4}$, $\mathcal{M}^2 = supp(\mu^\varepsilon) \neq supp(\mu^{\varepsilon_D})$, since

$$\mu^\varepsilon = \frac{1}{1-2\varepsilon}\left(\frac{1}{4}, \frac{1}{4}-\varepsilon, \frac{1}{4}-\varepsilon, \frac{1}{4}\right).$$

3.1.2 Minimal example of a stochastic 3-point D-bifurcation for a stochastic Lévy flow of bijections

Recall that one original motivation of this article is the result by Kunita [37] Theorem 4.2.5 that all stochastic Brownian flows of homeomorphisms in $\mathbb{R}^d$ under mild conditions on the coefficients are determined by its 1- and 2-point characteristics, and hence excludes the existence of 3-point D-bifurcations. The first result of this subsection shows that the smallest cardinality of the state space $\mathcal{M}$, which allows for a 3-point D-bifurcation is $m = |\mathcal{M}| = 4$.

Lemma 17. *Over $\mathcal{M}$ with $m = |\mathcal{M}| = 4$ there exists a stochastic flow of bijections which exhibits a 3-point D-bifurcation, while for $m = 3$ there are no 3-point D-bifurcations, only 2 and 1-point D-bifurcations.*

The proof is given by the following example.

Example 18 (Proof of Lemma 17)**.** Consider $m = 4$ and G the symmetric group S_4 over $\mathcal{M} = \{1,2,3,4\}$. Let $H < G$ be the alternating subgroup of G which is given by the even permutations

No. of transpositions	H	$G\backslash H$	No. of transpositions
0	id	(12)	1
2	(123) = (12)(23)	(13)	1
2	(132) = (13)(32)	(14)	1
2	(124) = (12)(24)	(23)	1
2	(142) = (14)(42)	(24)	1
2	(134) = (13)(34)	(34)	1
2	(143) = (14)(43)	(1234) = (12)(23)(34)	3
2	(234) = (23)(34)	(1432) = (14)(43)(32)	3
2	(243) = (24)(43)	(1324) = (13)(32)(24)	3
2	(12)(34)	(1423) = (14)(42)(23)	3
2	(13)(24)	(1243) = (12)(24)(43)	3
2	(14)(23)	(1342) = (13)(34)(42)	3

We define the uniform distributions $\Delta^H = U(H)$ on H and $\Delta^{G\backslash H} = U(G\backslash H)$ and consider the discrete flow φ^ε associated to the increment distribution given by

$$\Delta_\varepsilon := \Delta^H + \varepsilon \left[\Delta^{G\backslash H} - \Delta^H\right]. \tag{3.1}$$

For $\varepsilon > 0$ the invariant distribution is given by μ_ε, which is the uniform distribution along the orbits of G, while for $\varepsilon = \varepsilon_D := 0$ we have μ_{ε_D} is the uniform distribution along the orbits of H. Since for each couple of pairs $(i,j),(k,\ell) \in \mathcal{M}^2$, $i \neq j$, $k \neq \ell$ we have that the transition probability $P^\varepsilon_{ij,k\ell}$ of the 2-point motion inherited by the flow which is generated by Δ_ε does not depend on ε. In fact, for $\varepsilon \geqslant 0$ we have

$$P^\varepsilon_{ij,k\ell} = \frac{1-\varepsilon}{12} + \frac{\varepsilon}{12} = \frac{1}{12} \tag{3.2}$$

since there are only two bijections in G such that $\mathcal{M} \setminus \{i,j\} \to \mathcal{M} \setminus \{k,\ell\}$, one with an even and one with an odd number of transpositions, that is, one in H and exactly one in $G\backslash H$. Hence the effect of the ε-perturbation on the transition probabilities in (3.1) cancels out.

Therefore $\varepsilon = 0$ and any initial condition outside the subdiagonals the trajectories of φ^0 live almost sure in exactly $12 = |H|$ points such that the invariant measure satisfies $|supp(\mu^0)| = |H| = 12$. However, for $\varepsilon > 0$ the same argument yields $|supp(\mu^\varepsilon)| = |G| = 24$. Hence there is an n-point bifurcation of level $n = 3$ or $n = 4$. Note that for $m = 3$ there is no stochastic n-point bifurcation for $n > 2$ since no proper subgroup H of S_3 preserves the transition probabilities of the 2-point motion. In conclusion, $m = 4$ represents the minimal number of points over which the dynamics of bijections exhibits a stochastic n-point bifurcation with $n > 2$, which vastly contrasts the rigidity of Brownian flows of diffeomorphisms and their invariant measures.
This fact is confirmed in Subsection 3.3.2 quantitatively. The upper left corner of Table 35 reads as follows:

$T(m,k)$	$k=1$	$k=2$	$k=3$	$k=4$
$m=2$	1	2	-	-
$m=3$	1	5	6	-
$m=4$	1	14	23	24

Here the number of linear restrictions which imposes the properties of a homogeneous n-point Markov system is given by the numbers $T(m,k)$, where $n = k$ in this setting. In the line for $m = 3$, we see that for $n = k = 3 > 2$, that is, for fixed 2-point characteristics, the 3 point motion over a space of 3 elements is (trivially) equal to $6 = m!$. In other words, the matching of the number of variables and linear equations uniquely determines the entire flow. However, in the line of $m = 4$ we see that the law of the 3-point motion does not determine the entire flow, since $T(4,3) = 23 < 24 = 4! = m!$.

The preceding example extends naturally to the following proposition which shows that the bifurcation in higher levels is rather typical.

Lemma 19. *Given a finite state space $\mathcal{M} = \{1, 2, \ldots, m\}$, there exist stochastic dynamics with n-point D-bifurcation at level $n = m-1$ or $n = m$ which preserves the transition probabilities of the $(m-2)$-motion.*

Proof. We repeat the construction of the preceding example with $H = A_m$, being the so-called alternating group, that is the subgroup $H < S_m$ with an even number of transpositions. There are exactly two elements in $G = S_m$, which sends any fixed tuple without repetition $(i_1, \ldots, i_{m-2})$ to another fixed tuple without repetition $(j_1, \ldots, j_{m-2})$ in $\mathcal{M}^{m-2}$. One of them in H (with an even number of transpositions) and one in $G \setminus H$ (with an odd number of transpositions). Hence analogously to Example 18 the effect of the ε-perturbation on the $(m-2)$-point transition probabilities cancels out and the bifurcation of the invariant measures follows analogously. □

3.1.3 A conceptual class of stochastic n-point bifurcations

Following the spirit of the previous subsection we construct another family of examples starting with the following basic construction extended in the sequel to arbitrary pairs of subgroups $H < G \lneqq S_m$.

Example 20. We adopt the notation $f_{i_1 i_2 i_3 i_4 i_5 i_6}$ for the function

$$(1, 2, 3, 4, 5, 6) \mapsto (i_1, i_2, i_3, i_4, i_5, i_6)$$

with $i_1, \ldots, i_m \in \mathcal{M}$. The following pairwise notation is convenient for our example:

$$(1, 2, 3, 4, 5, 6) = ((1, 2), (3, 4), (5, 6)) = (a, b, c)$$

and denote by $\bar{a}, \bar{b}$ and $\bar{c}$ the flips of each double entry, for example $(\bar{a}, \bar{b}, c) = (2, 1, 4, 3, 5, 6)$. Consider the group

$$G = \{f_{abc}, f_{\bar{a}bc}, f_{a\bar{b}c}, f_{ab\bar{c}}, f_{\bar{a}\bar{b}c}, f_{\bar{a}b\bar{c}}, f_{a\bar{b}\bar{c}}, f_{\bar{a}\bar{b}\bar{c}}\}$$

with multiplication given by the composition, and its proper subgroup

$$H = \{f_{abc}, f_{\bar{a}\bar{b}c}, f_{\bar{a}b\bar{c}}, f_{a\bar{b}\bar{c}}\}.$$

The 6-point motion of a flow φ^0 generated by composition of i.i.d. bijections with law concentrated on this subgroup, say

$$\frac{1}{4}\left[\delta_{f_{abc}} + \delta_{f_{\bar{a}\bar{b}c}} + \delta_{f_{\bar{a}b\bar{c}}} + \delta_{f_{a\bar{b}\bar{c}}}\right], \tag{3.3}$$

has random trajectories with the following property: for each initial condition in $\mathcal{M}^6$ the corresponding random orbit of the process is concentrated on at most 4 points out of 6^6 possible elements of $\mathcal{M}^6$. For an initial condition which does not belong to any subdiagonal (i.e. such that their entries are all different from each other), the support of the invariant measure is concentrated on exactly 4 elements. On the other hand, the orbits of elements outside any sub-diagonal of the 6-point motion φ^ε generated by the ε-perturbation in the law, with $\varepsilon > 0$,

$$\frac{1}{4}\left[\delta_{f_{abc}} + \delta_{f_{\bar{a}\bar{b}c}} + \delta_{f_{\bar{a}b\bar{c}}} + \delta_{f_{a\bar{b}\bar{c}}}\right] + \frac{\varepsilon}{4}\left[\delta_{f_{\bar{a}bc}} + \delta_{f_{a\bar{b}c}} + \delta_{f_{ab\bar{c}}} + \delta_{f_{\bar{a}\bar{b}\bar{c}}} - \delta_{f_{abc}} - \delta_{f_{\bar{a}\bar{b}c}} - \delta_{f_{\bar{a}b\bar{c}}} - \delta_{f_{a\bar{b}\bar{c}}}\right], \tag{3.4}$$

has invariant measures supported on exactly 8 elements. Moreover, one easily checks by inspection that, due to appropriate cancellations, the transition probability of jumps from a pair of points to any other pair of points does not depend on ε, i.e the law in $\mathcal{M}^2$ is constant. The same happens for the law in $\mathcal{M}^1$. The splitting on the number of connected components of the support of the invariant measure implies that there must exist an n-point bifurcation for $3 \le n \le 6$. In the next section we shall construct an algorithm to find out exactly at which level n the bifurcation occurs.

Extending the construction: For a positive even integer k, take the uniform partition of the set $\mathcal{M} = \{1, 2, \dots, m\}$ with $m = k(k+1)$ into $(k+1)$ subsets of the form

$$\mathcal{M} = \{1, 2, \dots, k\} \ \dot\cup \ \{(k+1), (k+2), \dots 2k\} \dot\cup \ \dots \ \dot\cup \{k^2+1, \dots, k^2+k\} = \bigcup_{\ell=1}^{k+1} \{\ell k - k + 1, \dots, \ell k\}.$$

For each $\ell \in \{1, 2, \dots, (k+1)\}$, let b_ℓ denote the cyclic permutations of the corresponding interval of integers $((\ell k - k + 1), \dots, \ell k)$. We consider the Abelian group G of compositions of these cyclic permutations of $\mathcal{M}$ which preserves the subsets of the partition, i.e.:

$$G = \left\{ b_1^{i_1} \circ b_2^{i_2} \ \circ \cdots \ \circ b_{k+1}^{i_{k+1}} \ \mid \quad \text{with exponents} \ \ i_1, i_2, \dots i_{k+1} \in \{0, 1, \dots k-1\} \ \right\}.$$

This group has order $|G| = k^{(k+1)}$. Consider the proper subgroup $H < G$ given by

$$H = \left\{ b_1^{i_1} \ \circ b_2^{i_2} \ \circ \cdots \ \circ b_{k+1}^{i_{k+1}} \ \mid \quad \text{such that} \ \ (i_1 + i_2 + \dots \ + i_{k+1}) \text{ is even} \ \right\},$$

whose order is $|G|/2$. We take the uniform distribution Δ^H in H, i.e. the sum of normalized Dirac measures at each element of H. Analogously we denote by $\Delta^{G \setminus H}$ the sum of the normalized Dirac measures on the $k^{k+1}/2$ elements of its complementary set $G \setminus H$. With this notation, consider the distribution Δ_ε in G given by

$$\Delta_\varepsilon = \Delta^H + \varepsilon \left[\Delta^{G \setminus H} - \Delta^H \right]. \tag{3.5}$$

Consider the discrete stochastic flow φ^ε generated by the composition of i.i.d. random elements in G according to the above law. The invariance of the transition probabilities at order less than or equal to k is guaranteed by the following lemma.

Lemma 21 (There exist stochastic n-point bifurcations of any order)**.** *The transition probabilities of the k-point motion in $\mathcal{M}^k$ induced by the discrete flow φ^ε defined above do not depend on $\varepsilon > 0$. Moreover φ^ε exhibits a stochastic n-point D-bifurcation for some $n > k$.*

Proof. Fix an element $u = (i_1, i_2, \dots, i_k)$ outside the subdiagonals in $\mathcal{M}^k$. Since the partition of $\mathcal{M}$ has $(k+1)$ subsets and u has k components, there exists at least one block b_ℓ, $\ell \in \{1, 2, \dots, (k+1)\}$ whose domain has no intersection with $\{i_1, i_2, \dots, i_k\}$. Every element $g = b_1^{i_1} \circ b_2^{i_2} \circ \dots \circ b_\ell^{i_\ell} \circ \dots \circ b_{k+1}^{i_{k+1}} \in G$ acts on u with the following property:

1) There are $k/2$ elements $b_1^{i_1} \circ \dots \circ b_\ell^{\alpha} \circ \dots \circ b_{k+1}^{i_{k+1}} \in H$, where $\alpha \in \{0, \dots, k-1\}$ satisfies the positive parity condition of the exponents, and
2) $k/2$ elements $b_1^{i_1} \circ \dots \circ b_\ell^{\beta} \circ \dots \circ b_{k+1}^{i_{k+1}} \in G \setminus H$ for $\beta \in \{0, \dots, k-1\}$, where β satisfies the respective negative parity conditions,

such that for any α and β given in item 1) and 2) we have

$$g \cdot u = b_1^{i_1} \circ \dots \circ b_\ell^{\alpha} \circ \dots \circ b_{k+1}^{i_{k+1}} \cdot u = b_1^{i_1} \circ \dots \circ b_\ell^{\beta} \circ \dots \circ b_{k+1}^{i_{k+1}} \cdot u.$$

Therefore, when one subtracts the probability of the action of elements in H in (3.4) the same probability is added to elements in $G \setminus H$ whose action at u is exactly the same. Hence, summing up the independent probabilities that u is sent to any other element in $\mathcal{M}^k$ does not depend on $\varepsilon > 0$. The bifurcation phenomenon of the invariant measures follows analogously as above. □

3.2 How to detect the level of a stochastic n-point D-bifurcation

We consider the finite space $\mathcal{M} = \{1, \ldots, m\}$ with $m \geqslant 2$. The purpose of this section is to answer the following question. Given two invariant measures of the m-point motion, whose projections coincide from a level $k \leqslant m$ downwards. What is the lowest level $n \in \{k, \ldots, n\}$ of projections at which they differ? The answer of this question yields the level of the stochastic n-point D-bifurcation. Of course, this can be done along each sequence of projections

$$\pi^k_{j_k} \circ \cdots \circ \pi^{m-1}_{j_{m-1}} \circ \pi^m_{j_m} \text{ for } j_\ell \in \{1, \ldots, \ell\}, \ell \in \{k, \ldots, n\}$$

and the detected level will depend on it. The minimal level $n > k$ then determines the stochastic n-point bifurcation of the entire flow.

In the sequel we present an algorithm to find at which level n the bifurcation happens along a given sequence of projections for a given initial condition. We apply this algorithm to Example 20 when $k = 2$ exhibits a stochastic n-point D-bifurcation at levels $n = 3$ and $n = 5$ for different projections. This gives another example of a stochastic 3-point bifurcation as motivated in the Introduction, however, over a larger than minimal set of points $\mathcal{M}$ for $m = |\mathcal{M}| = 6$.

3.2.1 The projection algorithm

For each $1 \leqslant n \leqslant m$ denote by p^n be the $m^n \times m^n$ stochastic matrix, whose entries are the transition probabilities among the elements of $\mathcal{M}^n$, in the lexicographical order. By definition, homogeneous n-point Markov systems have transition probabilities which are compatible with projections. Hence p^{n-1} can be obtained from the projections π^n_r for any $r \in \{1, \ldots, n\}$ defined in Lemma 8. More precisely, for all $1 \leqslant r \leqslant n$ and all $(i_1, \ldots, i_n), (j_1, \ldots, j_n) \in \mathcal{M}^n$ we have

$$p^{n-1}_{\pi^n_r(i_1,\ldots i_n),\pi^n_r(j_1,\ldots j_n)} = \sum_{\ell \in \mathcal{M}} p^n_{(i_1,\ldots,i_n),(j_1,\ldots,j_{r-1},\ell,j_{r+1},\ldots j_n)}. \tag{3.6}$$

Recall that $\pi^n_r(i_1, \ldots, i_r, \ldots i_n) = (i_1, \ldots, i_{r-1}, i_{r+1}, \ldots i_n) \in \mathcal{M}^{n-1}$. This procedure defines a projection π^n_r of P^n onto P^{n-1} and can be expressed algebraically as follows. For each fixed r and $i_r \in \mathcal{M}$, there exists a pair of matrices (R_{n-1}, Q_{n-1}) which, according to formula (3.6), satisfies the equation

$$p^{n-1} = R_{n-1} \cdot p^n \cdot Q_{n-1},$$

where R_{n-1} is a $(m^{n-1} \times m^n)$-dimensional matrix with zero entries except exactly a unique entry 1 in each row and Q_{n-1} is an $(m^n \times m^{n-1})$-dimensional matrix with zero entries except, again, a unique 1 in each row.

Example 22. For $m = 2$, assuming compatibility of the matrix of probability transitions p^2 in $\mathcal{M}^2$, we calculate $R_1 \in \mathbb{R}^{2\times4}$ and $Q_1 \in \mathbb{R}^{4\times2}$ for different choices of (r, i_r). For $r = 1$ and $i_r = 1$ we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11,11} & p_{11,12} & p_{11,21} & p_{11,22} \\ p_{12,11} & p_{12,12} & p_{12,21} & p_{12,22} \\ p_{21,11} & p_{21,12} & p_{21,21} & p_{21,22} \\ p_{22,11} & p_{22,12} & p_{22,21} & p_{22,22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{bmatrix}.$$

For $r = 1$ and $i_r = 2$:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11,11} & p_{11,12} & p_{11,21} & p_{11,22} \\ p_{12,11} & p_{12,12} & p_{12,21} & p_{12,22} \\ p_{21,11} & p_{21,12} & p_{21,21} & p_{21,22} \\ p_{22,11} & p_{22,12} & p_{22,21} & p_{22,22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{bmatrix}.$$

Instead, for $r = 2$ and $i_r = 1$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11,11} & p_{11,12} & p_{11,21} & p_{11,22} \\ p_{12,11} & p_{12,12} & p_{12,21} & p_{12,22} \\ p_{21,11} & p_{21,12} & p_{21,21} & p_{21,22} \\ p_{22,11} & p_{22,12} & p_{22,21} & p_{22,22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p_{1,1} & p_{1,2} \\ p_{2,1} & p_{2,2} \end{bmatrix}.$$

Note that in all these examples permuting simultaneously lines of R_1 and columns of Q_1 leaves the product invariant.

For higher levels, with $m > 2$ and $r = i_r = 1$ fixed we get the following. Thanks to the lexicographical order on the entries of the matrices $A^{(n)}$ we have a standard way of performing the projections of transition probabilities, using that:

$$R_{n-1} = \left[\ \mathcal{I}_{m^{n-1}} \ \middle|\ 0 \ \middle|\ \ldots\ \middle|\ 0\ \right]_{m^{n-1} \times m^n},$$

where '0' above represents the null (m^{n-1})-square matrices, whereas

$$Q_{n-1} = \left[\ \mathcal{I}_{m^{n-1}} \ \middle|\ \mathcal{I}_{m^{n-1}} \ \middle|\ \vdots \ \middle|\ \mathcal{I}_{m^{n-1}}\ \right]^*_{m^{n-1} \times m^n}. \tag{3.7}$$

Lemma 8 implies that given a (left) eigenvector $v_n \in \mathbb{R}^{m^n}$ of $A^{(n)}$, its projection v_{n-1} in $\mathbb{R}^{m^{n-1}}$ is again an eigenvector of $A^{(n-1)}$. More precisely we have the following representation.

Lemma 23. *Given v_n an invariant measure for a compatible Markov chain in the product space $\mathcal{M}^n$ represented as a (row) vector in $\mathbb{R}^{m^n}$, then*

$$v_{n-1} = v_n\ Q_{n-1} \tag{3.8}$$

is an invariant measure in $\mathcal{M}^{n-1}$ represented as a vector in $\mathbb{R}^{m^{n-1}}$.

Proof. Since formula (3.8) represents the projection $(\pi_1^n)_*$ in Lemma 8. This is straightforwardly, in fact, each column of Q_{n-1} acts on a fixed configuration $(i_2, i_3, \ldots, i_n) \in \mathcal{M}^{n-1}$, whose sum with the first parameter i_1 ranging from 1 to m gives the desired projection. □

We stress that the matrix Q_{n-1} in formula (3.8) is not unique and a different choice of Q_{n-1} may lead to a different distribution v_{n-1}. Nevertheless, the choice of $r = 1$ and $i_r = 1$ leads to the simplest version given by (3.7).

3.2.2 Example 20 continued: algorithmic detection of a stochastic 3-point bifurcation

We go back to Example 20 and apply Proposition 23 to find at which level $3 \leqslant n \leqslant 6$ the bifurcation occurs given an invariant measure for the 6-point motion.

Lemma 24. *Example 20 exhibits a stochastic 3-point D-bifurcations.*

The proof is given by the following example.

Example 25 (Example 20 continued)**.** For the sake of notation, we denote by v_ℓ^0 an invariant measure at ℓ-points for the unperturbed system φ^0 $(\varepsilon = 0)$ and by v_ℓ^ε an invariant measure of the perturbed system φ^ε, $\varepsilon > 0$, respectively. We start with $\ell = 6$ and compare v_ℓ^0 and v_ℓ^ε, for different initial invariant measures.

First we consider the invariant measures of both systems which contain the point $(1,2,3,4,5,6)$. In column representation we obtain

$$v_6^0 = \begin{bmatrix} 1_{123456} \\ 1_{124365} \\ 1_{213465} \\ 1_{214356} \end{bmatrix}_{6^6\times 1} \qquad \text{and} \qquad v_6^\varepsilon = \begin{bmatrix} 1_{123456} \\ 1_{123465} \\ 1_{124356} \\ 1_{124365} \\ 1_{213456} \\ 1_{213465} \\ 1_{214356} \\ 1_{214365} \end{bmatrix}_{6^6\times 1},$$

where the symbol $1_{i_1i_2i_3i_4i_5i_6}$ in the column vector means that the entry $(i_1,i_2,i_3,i_4,i_5,i_6)$ is strictly positive while all omitted entries are zero. In addition, we assume as in Example 20 that the distributions v_6^0 and v_6^ε are uniform (on their respective support).

The projections of the invariant measures can easily be performed along the first coordinate $(\pi_1^\ell)_*$, $\ell \in \{2,\dots 6\}$ as in Proposition 23. This means, that according to formula (3.8), one just has to exclude the first entry of a nonzero entry $(i_1,\dots,i_r)$, $2 \leqslant r \leqslant m$, in v_j^i, and rearrange, if necessary, in such a way that the order in which they appear in the column matrix of the reduced level corresponds to the lexicographic order again. With this method we generate a sequence of vectors, $v_6^0, v_5^0, \dots, v_1^0$, which represent each the invariant distributions of a certain level, and $v_6^\varepsilon, v_5^\varepsilon, \dots, v_1^\varepsilon$ for the unperturbed and the ε-perturbed system accordingly. This yields in column vector notation

$$v_5^0 = Q_5^T v_6^0 = Q_5^T \begin{bmatrix} 1_{123456} \\ 1_{124365} \\ 1_{213465} \\ 1_{214356} \end{bmatrix}_{6^6\times 1} = \begin{bmatrix} 1_{13465} \\ 1_{14356} \\ 1_{23456} \\ 1_{24365} \end{bmatrix}_{6^5\times 1}.$$

The complete sequence of projections reads as follows.

Projections	v_6^0	v_5^0	v_4^0	v_3^0	v_2^0	v_1^0
Shape of projections	$\begin{pmatrix} 1_{123456} \\ 1_{124365} \\ 1_{213465} \\ 1_{214356} \end{pmatrix}$	$\begin{pmatrix} 1_{13465} \\ 1_{14356} \\ 1_{23456} \\ 1_{24365} \end{pmatrix}$	$\begin{pmatrix} 1_{3456} \\ 1_{3465} \\ 1_{4356} \\ 1_{4365} \end{pmatrix}$	$\begin{pmatrix} 1_{356} \\ 1_{365} \\ 1_{456} \\ 1_{465} \end{pmatrix}$	$\begin{pmatrix} 1_{56} \\ 1_{65} \end{pmatrix}$	$\begin{pmatrix} 1_{5} \\ 1_{6} \end{pmatrix}$
Dimension	$6^6\times 1$	$6^5\times 1$	$6^4\times 1$	$6^3\times 1$	$6^2\times 1$	6×1

For the ε-perturbed system we carry out the same algorithm and find the invariant measures

$$v_5^\varepsilon = Q_5^T v_6^\varepsilon = Q_5^T \begin{bmatrix} 1_{123456} \\ 1_{123465} \\ 1_{124356} \\ 1_{124365} \\ 1_{213456} \\ 1_{213465} \\ 1_{214356} \\ 1_{214365} \end{bmatrix}_{6^6\times 1} = \begin{bmatrix} 1_{13456} \\ 1_{13465} \\ 1_{14356} \\ 1_{14365} \\ 1_{23456} \\ 1_{23465} \\ 1_{24356} \\ 1_{24365} \end{bmatrix}_{6^5\times 1} \neq \begin{bmatrix} 1_{13465} \\ 1_{14356} \\ 1_{23456} \\ 1_{24365} \end{bmatrix}_{6^5\times 1} = v_5^0,$$

$$\text{while } v_j^\varepsilon = v_j^0 \qquad \text{for all} \qquad 1 \leqslant j \leqslant 4.$$

This shows that the flow exhibits a stochastic 5-point bifurcation for the invariant measure which contains the initial value $(1, 2, 3, 4, 5, 6)$.
Taking a different initial invariant measure whose support contains the point $(1, 2, 1, 4, 1, 6)$ we have the following projections. The sequence of the unperturbed system is given as

Projections	v_6^0	v_5^0	v_4^0	v_3^0	v_2^0	v_1^0
Shape	$\begin{pmatrix} 1_{121416} \\ 1_{121315} \\ 1_{212425} \\ 1_{212326} \end{pmatrix}$	$\begin{pmatrix} 1_{21416} \\ 1_{21315} \\ 1_{12425} \\ 1_{12326} \end{pmatrix}$	$\begin{pmatrix} 1_{1416} \\ 1_{1315} \\ 1_{2425} \\ 1_{2326} \end{pmatrix}$	$\begin{pmatrix} 1_{416} \\ 1_{315} \\ 1_{425} \\ 1_{326} \end{pmatrix}$	$\begin{pmatrix} 1_{16} \\ 1_{15} \\ 1_{25} \\ 1_{26} \end{pmatrix}$	$\begin{pmatrix} 1_{5} \\ 1_{6} \end{pmatrix}$

For the ε-perturbed system we obtain

Projections	v_6^ε	v_5^ε	v_4^ε	v_3^ε	v_2^ε	v_1^ε
Shape	$\begin{pmatrix} 1_{121416} \\ 1_{121415} \\ 1_{121316} \\ 1_{121315} \\ 1_{212426} \\ 1_{212425} \\ 1_{212326} \\ 1_{212325} \end{pmatrix}$	$\begin{pmatrix} 1_{12426} \\ 1_{12425} \\ 1_{12326} \\ 1_{12325} \\ 1_{21416} \\ 1_{21415} \\ 1_{21316} \\ 1_{21315} \end{pmatrix}$	$\begin{pmatrix} 1_{1416} \\ 1_{1415} \\ 1_{1316} \\ 1_{1315} \\ 1_{2426} \\ 1_{2425} \\ 1_{2326} \\ 1_{2325} \end{pmatrix}$	$\begin{pmatrix} 1_{416} \\ 1_{415} \\ 1_{316} \\ 1_{315} \\ 1_{426} \\ 1_{425} \\ 1_{326} \\ 1_{325} \end{pmatrix}$	v_2^0	v_1^0
Dimension	$6^6 \times 1$	$6^5 \times 1$	$6^4 \times 1$	$6^3 \times 1$	$6^2 \times 1$	6×1

This shows that the flow exhibits in fact also a stochastic 3-point D-bifurcation over $m = 6$ points, in comparison to the minimal Example 18 with $m = 4$.

3.3 The complexity of n-point D-bifurcations for stochastic Lévy semiflows

In this subsection we study how many linearly independent equations do fixed k-point characteristics P^k for all $1 \leqslant k \leqslant n \leqslant m$ for some fixed n impose on the laws of the respective flow of self-maps and bijections over $\mathcal{M}$ (and hence its invariant measure). This combinatorial question is first carried out for the easier case of flows of self-maps $\{\mathcal{M} \to \mathcal{M}\}$ and then in the second case of $\{\mathcal{M} \to \mathcal{M} \mid \text{bijective}\} = S_m$, $m = |\mathcal{M}| < \infty$. In the first case we prove a recursion formula, which is illustrated numerically. In the second case we conjecture - based on numerical experiments - that the respective triangular numbers are given by a well-known (complicated) combinatorial quantity introduced in [26], for which to date no closed formula has been found. In the appendix we state some explicit formulas for special cases and some asymptotics found there.

3.3.1 The complexity of n-point D-bifurcations for stochastic Lévy semiflows of random self-maps

The purpose of this section is to find formulas for the dimensions of the vector space of distributions of i.i.d. random self-maps of $\mathcal{M} = \{1, 2, \dots, m\}$ such that the law of the respective flow of random mappings respects the prescribed k-point characteristics P^k for $0 \leqslant k \leqslant n \leqslant m$ for some $n \leqslant m$.
Recall the notation $f_{i_1 \dots i_m}$ for the function $(1, \dots, m) \mapsto (i_1, \dots, i_m)$ for $i_1, \dots, i_m \in \mathcal{M}$. The stochastic flow of maps $(\varphi_n)_{n \geqslant 0}$ in $\mathcal{M}$ is generated by i.i.d. random variables in the space of maps with the following discrete probability distribution

$$\nu = \sum_{i_1, \dots, i_m = 1}^{m} \alpha_{i_1 \dots i_m} \delta_{f_{i_1 \dots i_m}}, \tag{3.9}$$

where $\delta_{f_{i_1 \dots i_m}}$ is a Dirac measure centered on the mapping $f_{i_1 \dots i_m}$. The non-negative coefficients $\alpha_{i_1 \dots i_m} \in \mathbb{R}$ are ordered lexicographically by the sub-indices. We denote

$$p_{u_1 \dots u_k,\ v_1 \dots v_k} := P_1^k((u_1, \dots, u_k), \{(v_1, \dots, v_k)\}), \qquad (u_1, \dots, u_k), (v_1, \dots, v_k) \in \mathcal{M}^k.$$

1. The first linear restriction on the m^m coefficients $(\alpha_{i_1, \dots, i_m})_{(i_1, \dots, i_m) \in \mathcal{M}^m}$ comes from the fact that they determine the distribution of a random variable, hence

$$\sum_{i_1, \dots, i_m = 1}^{m} \alpha_{i_1 \dots i_m} = 1. \tag{3.10}$$

   We call this the 0-level restriction for the coefficients.

2. In general, at the k-level, for a given family of transition probability in k-point motion $p_{u_1, \dots, u_k, v_1, \dots, v_k}$, these characteristics determine linear restrictions for the coefficients $\alpha_{i_1, \dots, i_m}$ given by:

$$\sum_{(i_1, \dots, i_{m-k}) \in \mathcal{M}^{m-k}} \alpha_{(i_1, \dots, i_{m-k}) \triangleleft \binom{v_1, \dots v_k}{u_1, \dots, u_k}} = p_{u_1 \dots u_k,\ v_1 \dots v_k}, \tag{3.11}$$

   where the expression $(i_1, \dots, i_{m-k}) \triangleleft \binom{v_1, \dots v_k}{u_1, \dots, u_k}$ is the shorthand notation for the following vector

$$(i_1, \dots, i_{u_1 - 1}, v_1, i_{u_1 + 1}, \dots i_{u_2 - 1}, v_2, i_{u_2 + 1}, \dots, i_{u_k - 1}, v_k, i_{u_k + 1}, \dots, i_m).$$

   In other words, at position u_i of $(i_1, \dots, i_{m-k})$ the vector v_i is introduced.

Obviously, the degree of freedom (dimension of subspaces which preserve the k-point characteristics) is given by m^m minus the number of linearly independent restrictions for the coefficients $\alpha_{i_1, \dots, i_m}$. The following lemma yields a complete recursive description of the number of linearly independent restrictions.

Given a finite space $\mathcal{M} = \{1, 2, \dots, m\}$, $1 \leqslant k \leqslant n \leqslant m$ and a homogeneous n-point Markov system $(P^k)_{1 \leqslant k \leqslant n}$ we denote by $R_k^{n,m}$ the number of linearly independent restrictions imposed simultaneously on the coefficients $(\alpha_{i_1 \dots i_m})$ (in the sense of (3.11)) by all P^ℓ, $\ell \leqslant k$, over the alphabet of size $m = |\mathcal{M}|$.

Theorem 26 (Recursion formula for the number of restrictions)**.** *In the preceding setting, the triangular numbers*

$$R_0^{m,m} \leqslant R_1^{m,m} \leqslant \dots \leqslant R_n^{m,m}$$

satisfy the following recursion formula: For all given $1 \leqslant k \leqslant n \leqslant m$ we have

$$R_k^{n,m} = R_{k-1}^{n,m} + \binom{n}{k}(m^k - R_{k-1}^{k,m}), \qquad R_0^{n,m} = 1. \tag{3.12}$$

Remark 27. Note that $R_m^{m,m} = m^m$, since the law of the flow of self-maps is uniquely determined, that is, the number of variables equals the number of linearly independent equations.

Proof of Theorem 26: We prove by induction over $0 \leqslant k \leqslant n$. First note that when $k = 0$, it means that in formula (3.11) the right-hand side is equal to 1 and on the left-hand side the sum is over all $\alpha_{(i_1,\dots,i_m)}$. In other words, there are no further restrictions other than equation (3.10), that is, $R_0^m = 1$.

Assume that the formula holds for $R_{k-1}^{n,m}$, for all $n \in \{k, \dots, m\}$. The number of restrictions $R_k^{n,m}$ at the k-th level depends of the characteristics of the level $(k-1)$, i.e. it is a sum of $R_{k-1}^{n,m}$ plus some new linearly independent restrictions depending exactly on characteristics at level k. This justifies the first summand on the right hand side of equation (3.12). It remains to describe these new restrictions depending exactly on characteristics at level k. By formula (3.11), considering the projection at level k from level n means that there is a subset of positions $\{\tilde u_1, \dots, \tilde u_k\} \subseteq \{u_1, \dots, u_n\}$ such that for all $v_1, \dots, v_k \in \mathcal{M}$ we have

$$\sum_{(i_1,\dots,i_{n-k})\in\mathcal{M}^{n-k}} \tilde\alpha_{(i_1,\dots,i_{n-k})\lhd\binom{v_1,\dots,v_k}{\tilde u_1,\dots,\tilde u_k}} = p_{\tilde u_1\dots\tilde u_k,\ v_1\dots v_k}. \tag{3.13}$$

The number of possible subsets with k elements $\{\tilde u_1, \dots, \tilde u_k\}$ out of the "positions" $\{u_1, \dots, u_n\}$ of the n-point motion yields $\binom{n}{k}$ many "blocks" of equations in (3.12) which only vary in $v_1, \dots, v_k \in \mathcal{M}$. However not all of them linearly independent, since P^k inherits dependencies from P^{k-1}. The number of linearly dependent equations over an alphabet $\mathcal{M}$ with $|\mathcal{M}| = m$ at level k inhered form level $k-1$ and below can be expressed in terms of the $R_{k-1}^{k,m}$ for which the recursion assumption holds. Subtracting $R_{k-1}^{k,m}$ from m^k yields the desired recursion. □

The following pseudocode algorithm shows how to determine the number of linearly independent restrictions directly.

Algorithm 1: Direct algorithmic calculation of number of linearly independent restrictions to complete a homogeneous semiflow of self-maps over m-points

Data: homogeneous n-point Markov systems of random self-maps over $\mathcal{M} = \{1, \dots, m\}$.
Result: Number of linearly independent restrictions imposed simultaneously on the coefficients $(\alpha_{i_1\dots i_m})$ in the sense of (3.11) imposed by all P^k $k \leqslant n$.

For $k = 1$ **to** n **do**;
  Set $A_k =$ **Matrix**{coefficients $\alpha_{(i_1,\dots,i_{m-k})\lhd\binom{v_1,\dots,v_k}{u_1,\dots,u_k}}$ of equations (3.11)}
  Set $M_k =: \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{pmatrix}$
  Set $R_k^{n,m} = \mathrm{rank}(M_k)$
Return $(R_1^{n,m}, \dots, R_1^{n,m})$.

The following table yields the results of the computational illustration of the recursion formula obtained in Theorem 26. The unboxed values of the table have been obtained by the recursion formula of Theorem 26. The $\boxed{\text{boxed}}$ values have been calculated by Algorithm 1 and coincide with the values of the recursion formula of Theorem 26.

Table 28 (Computational illustration of the recursion formula (3.12)).

$R_n^{m,m}$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
$m=1$	1	1	-	-	-	-	-	-	-	-	-
$m=2$	1	3	4	-	-	-	-	-	-	-	-
$m=3$	1	7	19	27	-	-	-	-	-	-	-
$m=4$	1	13	67	175	256	-	-	-	-	-	-
$m=5$	1	21	181	821	2101	3125	-	-	-	-	-
$m=6$	1	31	406	2906	12281	31031	46656	-	-	-	-
$m=7$	1	43	799	8359	53719	217015	543607	823543		-	-
$m=8$	1	57	1429	20637	188707	1129899	4424071	11012415	16777216	-	-
$m=9$	1	73	2377	45385	561481	4690249	26710345	102207817	253202761	387420489	-
$m=10$	1	91	3736	91216	1469026	16349374	127951984	701908264	2639010709	6513215599	10^{10}

The 31 values in boxes have been checked numerically.

Example 29 (Semiflow of mappings over $m = 4$ elements)**.** We illustrate the arguments used in the proof of Theorem 26. We consider here flows in $\mathcal{M} = \{1, 2, 3, 4\}$ with $4^4 = 256$ coefficients $\alpha_{i_1 i_2 i_3 i_4}$.
0-point motion: The number of restrictions is obviously $R_0^4 = 1$ since it only consists of

$$\sum_{ijk\ell} \alpha_{ijk\ell} = 1.$$

1-point motion: We have $\binom{4}{1}$ blocks, each block with $m^1 = 4$ new equations:

$$\sum_{ijk} \alpha_{uijk} = p_{1,u}, \quad u \in M, \qquad \sum_{ijk} \alpha_{iujk} = p_{2,u}, \quad u \in M,$$
$$\sum_{ijk} \alpha_{ijuk} = p_{3,u}, \quad u \in M, \qquad \sum_{ijk} \alpha_{ijku} = p_{4,u}, \quad u \in M.$$

In each block we have $R_0^{1,4} = 1$ linearly dependent equations which has to be subtracted from the total number of equations in the block. Hence, the number of linear independent restrictions is given by

$$\binom{4}{0} + \binom{4}{1}[4^1 - \binom{1}{0}4^0] = 1 + 4 * (4 - 1) = 13.$$

2-point motion: We have $\binom{4}{2} = 6$ blocks, each block with $m^2 = 16$ new equations:

$$\sum_{ij} \alpha_{uvij} = p_{12,uv}, \quad u, v \in M, \qquad \sum_{ij} \alpha_{iuvj} = p_{23,uv}, \quad u, v \in M,$$
$$\sum_{ij} \alpha_{ijuv} = p_{34,uv}, \quad u, v \in M, \qquad \sum_{ij} \alpha_{uivj} = p_{13,uv}, \quad u, v \in M,$$
$$\sum_{ij} \alpha_{iujv} = p_{24,uv}, \quad u, v \in M, \qquad \sum_{ij} \alpha_{uijv} = p_{14,uv}, \quad u, v \in M.$$

In each block we have $R_1^{2,4} = 7$ linearly dependent equations (obtained by putting together the reduction from 2-point motion to 1-point and 0-point motion) which has to be subtracted from the total number of equations of the block. Hence, linearly independent restrictions are given by

$$\binom{4}{0}[4^0] + \binom{4}{1}[4^1 - \binom{1}{0}4^0] + \binom{4}{2}[4^2 - [\binom{2}{0}4^0 + \binom{2}{1}[4^1 - \binom{1}{0}4^0]]] \quad = 67.$$

Remaining degrees of freedom $4^4 - 67 = 256 = 189$.
3-point motion: We have $\binom{4}{3} = 4$ blocks, each block with $m^3 = 64$ new equations:

$$\sum_i \alpha_{uvwi} = p_{123,uvw}, \qquad u,v,w \in M, \qquad \sum_i \alpha_{uviw} = p_{124,uvw}, \qquad u,v,w \in M,$$
$$\sum_j \alpha_{uivw} = p_{134,uvw}, \qquad u,v,w \in M, \qquad \sum_i \alpha_{iuvw} = p_{234,uvw}, \qquad u,v,w \in M.$$

In each block we have $R_2^3 = 37$ linearly dependent equations (obtained by putting together the reduction from 3-point motion to 2-point, 1-point and 0-point motion) which has to be subtracted from the total number of equations of the block. Hence, linearly independent restrictions are given by:

$$\binom{4}{0}[4^0]+$$
$$\binom{4}{1}[4^1 - \binom{1}{0}[4^0]]+$$
$$\binom{4}{2}[4^2 - (\binom{2}{2}4^0 + \binom{2}{1}[4^1 - \binom{1}{0}[4^0])]+$$
$$\binom{4}{1}[4^3 - (\binom{3}{3}4^0 + \binom{3}{2}[4^1 - \binom{1}{0}[4^0]] + \binom{3}{1}[4^2 - [\binom{2}{0}4^0 + \binom{2}{1}[4^1 - \binom{1}{0}[4^0]]]]] = 175.$$

4-point motion: We have a single $\binom{4}{4} = 1$ block, with m^4 new equations:

$$\alpha_{uvwx} = p_{1234,uvwx},, \qquad u,v,w,x \in M,$$

In this single block, we have $R_3^4 = 175$ linearly dependent equations (obtained by putting together the reduction from 4-point motion to 3-point, 2-point, 1-point and 0-point motion) which has to be subtracted from the total number of equations of the block. Hence, linearly independent restrictions are given by the following equation, which one easily sees that has a telescopic cancellation:

$$\binom{4}{0}[4^0]+$$
$$\binom{4}{1}[4^1 - \binom{1}{0}[4^0]]+$$
$$\binom{4}{2}[4^2 - [\binom{2}{0}4^0 + \binom{2}{1}[4^1 - \binom{1}{0}[4^0]]]+$$
$$\binom{4}{3}[4^3 - [\binom{3}{0}4^0 + \binom{3}{1}[4^1 - \binom{1}{0}[4^0]] + \binom{3}{2}[4^2 - [\binom{2}{0}4^0 + \binom{2}{1}[4^1 - \binom{1}{0}[4^0]]]]]+$$
$$\binom{4}{4}[4^4 - [\binom{4}{0}[4^0] + \binom{4}{1}[4^1 - \binom{1}{0}[4^0]] + \binom{4}{2}[4^2 - [\binom{2}{0}4^0 - \binom{2}{1}[4^1 - \binom{1}{0}]]$$
$$+ \binom{4}{3}[4^3 - [\binom{3}{0}4^0 + \binom{3}{1}[4^1 - \binom{1}{0}[4^0]] + \binom{3}{2}[4^2 - [\binom{2}{0}4^0 + \binom{2}{1}[4^1 - \binom{1}{0}[4^0]]]]]]] = 4^4.$$

The results of this example illustrate the line of $m = 4$ in Table 28 for $R_n^{m,m}$, which carries the sequence $(1, 13, 67, 175, 256)$.

3.3.2 Combinatorial conjecture on the complexity of n-point D-bifurcations for stochastic Lévy semiflows of random bijections over m points

In this subsection we conjecture the solution of the number of restrictions on the coefficients for flows of bijections based on a algorithmic computer experiment and *The On-Line Encyclopedia of Integer Sequences* [51]. If our conjecture turn out to be true, there is no hope to date to derive an analogous recursion formula to Theorem 26.
Recall the notation for bijective mappings $f_{i_1\dots i_m} : \mathcal{M} \to \mathcal{M}$. The stochastic Lévy flow of bijections $(\varphi_n)_{n\geqslant 0}$ in $\mathcal{M}$ is generated by i.i.d. random variables in the space of permutations with the following distribution:

$$\nu = \sum_{(i_1,\dots,i_m)\in \mathrm{Sym}(\{1,\dots,m\})} \alpha_{i_1\dots i_m} \delta_{f_{i_1\dots i_m}} \tag{3.14}$$

1. The first linear restriction on these $m!$ coefficients comes from the fact that they determine the distribution of a random variable, hence

$$\sum_{(i_1,\dots,i_m)\in \mathrm{Sym}(\{1,\dots,m\})} \alpha_{i_1\dots i_m} = 1. \tag{3.15}$$

   We call this the 0-level restriction for the coefficients.

2. As before, at the k-level, for a (compatible) family of transition probability in k-point motion $p_{u_1\dots u_k,\ v_1\dots v_k}$, they determine linear restrictions for the coefficients $\alpha_{i_1,\dots,i_m}$ given by

$$\sum_{(i_1,\dots,i_{m-k})\in\ \mathrm{Sym}(\{1,\dots,m\}\setminus\{v_1,\dots,v_k\})} \alpha_{(i_1,\dots,i_{m-k})\lhd\binom{v_1,\dots,v_k}{u_1,\dots,u_k}} = p_{u_1\dots u_k,v_1\dots v_k}, \tag{3.16}$$

   where the sum is taken over $(m-k)!$ indices. As before, varying $v_1,\dots,v_k$ in the expression above generates a block of $m!/(m-k)!$ equations.

In any level k, the diagonal and its complementary are invariant sets for the dynamics of random permutations. Moreover, for flows of bijections in a finite space, given the sub-maximal $(m-1)$-point transition probabilities, they already determine uniquely the maximal m-point transition probabilities, hence they also determine the $m!$ coefficients $\alpha_{i_1\dots i_m}$.

Let $u = (u_1,\dots,u_k)$ and $v = (v_1,\dots,v_k)$ be elements in $\mathcal{M}^k$. Since the order of the entries of the elements in $\mathcal{M}^k$ does not matter in a flow, then, if σ is a permutation of k elements, then by the indistinguishability condition (2.1) the transition probabilities satisfy:

$$p_{u_1\dots u_k,v_1\dots v_k} = p_{u_{\sigma(1)}\dots u_{\sigma(k)},v_{\sigma(1)}\dots v_{\sigma(k)}}.$$

That is, the entries $(u_1,\dots,u_k)$ can be assumed to be strictly ordered.

Remark 30 (A hidden symmetry)**.** Consider now $u' = (u'_1,\dots,u'_{(m-k)})$ and $v' = (v'_1,\dots,v'_{(m-k)})$ elements in $\mathcal{M}^{(m-k)}$ such that, as subsets, they complement u and v respectively, i.e. $\{u\}\cup\{u'\} = \{v\}\cup\{v'\} = \mathcal{M}$. Then

$$\sum_{\sigma\in\Delta} p_{u_1\dots u_k,v_{\sigma(1)}\dots v_{\sigma(k)}} = \sum_{\xi\in\Delta'} p_{u'_1\dots u'_k,v'_{\xi(1)}\dots v'_{\xi(k)}}. \tag{3.17}$$

where Δ are permutations on k elements and Δ' are permutations in $(m-k)$ elements. This is obvious from the observation that in a flow of bijections, the whole set $\{u\}$ is sent to $\{u'\}$ (independently of the order), if and only if its complementary $\{v\}$ is sent to $\{v'\}$, the complementary of $\{u'\}$. For example:

$$p_{1,1} = \sum_{\xi\in\mathrm{Sym}(\{2,3,\dots,m\})} p_{2\dots m\ ,\ \xi(2)\xi(3)\dots\xi(m)}.$$

Remark 31 (*The Birkhoff polytopes problem)***.** The celebrated Birkhoff-von Neumann theorem states that $m \times m$ bi-stochastic matrices lay in the convex hull generated by the $m!$ matrices of permutations in $\mathcal{M} = \{1, 2, \dots, m\}$. For any $m \in \mathbb{N}$ this convex set is called the Birkhoff polytope P_m. There are several proofs of this theorem in the literature, for a simple and elementary proof we refer to Mirsky [48]. This theory has many interesting application, and although already intensely studied, it still offers some interesting open problems, see Pak [50]. For instance despite its relevance there is no formula for the volume of P_m for higher dimensional Birkhoff polytopes P_m. Only recently an asymptotic formula was obtained by Canfield and McKay [15].

In the context of our article concerning the random dynamics generated by i.i.d. random mappings it means that a stochastic flow in $\mathcal{M}$ is a flow of permutations if and only if the matrices of transition probabilities of 1-point motion is not only a stochastic matrix (a matrix whose nonnegative lines entries sum up to 1), but a bi-stochastic matrix (a matrix whose nonnegative lines and column entries sum up to 1). Moreover, in the Birkhoff polytope language, what we are exploring in this article is the fact that, in general, except for elements in the wedges of the polytope P_m, the bi-stochastic matrices has a non-unique representation as a linear combination of the vertices of P_m (in fact, P_m is contained in a $(m-1)^2$-dimensional subspace and has $m!$ vertices).

The flow of bijections induced in the k-point level sends each whole fibre (component) into a whole fiber. By the Birkhoff-von Neumann theorem the matrix of transition probabilities in k-point level is again bi-stochastic for all $1 \leq k \leq m$. As we have pointed out in the Introduction, in our context, when one deals with permutations, it means that one enters in the theory of Birkhoff polytopes, with many open problems. For the particular case of $n = 2$ we have the following formula.

Proposition 32. *For a finite space $\mathcal{M} = \{1, 2, \dots, m\}$, given probability transitions of 1-point motion, the number of linearly independent restrictions for the coefficients $(\alpha_{i_1 \dots i_m})$ is given by*

$$R_1^{2,m} = (m-1)^2 + 1.$$

Proof. The bi-stochastic $m \times m$-matrix of transition probabilities of the 1-point motion has $2m - 1$ redundancies by definition. These redundancies correspond to linearly dependent equations of type (3.16)) with $k = 1$. Hence the restrictions are given by $[m^2 - (2m-1)]$ l.i. equations added to the 0-level restriction. □

The arguments in the proofs of Theorem 26 and of Proposition 32 are not easily extensible to higher levels k in the case of bijections. This is due to the fact that, in this case, for $k > 1$ there are further restrictions which involve crossed equations coming from different blocks of equations generated by each fixed $u_1 u_2 \dots u_k$ in equation (3.16)) (in contrast to the previous case of arbitrary self-maps). Moreover, for level $k \geq m/2$, new restrictions, coming from equation (3.17)) which represents further dependence on lower levels $(m-k) \leq k$, arises (again different from the case of self-maps). Therefore, combinatorially, it looks non trivial to control the restrictions coming from different properties with non-empty intersections. As far as our knowledge, in the case of flow of random bijections, the problem of number of restrictions on the coefficients, given the transition probabilities of k-point motion, for $k > 1$, is still open. However the respective numbers of linearly independent restrictions can be calculated numerically.

Algorithm 2: Calculation of the linearly independent restrictions of the n-point motion of a Lévy flow of bijections

Data: stochastic Lévy flow of bijections over a finite space $\mathcal{M} = \{1, \ldots, m\}$.
Result: Number of linearly independent restrictions imposed simultaneously on the coefficients $(\alpha_{i_1 \ldots i_m})$ in the sense of (3.16) imposed by all P^k $k \leqslant n$.

For $k = 1$ **to** n **do**;
  Set $A_k =$ **matrix**{coefficients $\alpha_{(i_1,\ldots,i_{m-k}) \lhd \binom{v_1,\ldots,v_k}{u_1,\ldots,u_k}}$ of equations (3.16)}
  Set $M_k = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{pmatrix}$
  Set $R_k^{n,m} = \mathrm{rank}(M_k)$;
return $(R_k^{1,m}, \ldots, R_n^{n,m})$.

Remark 33. A word about the computational complexity of Algorithms 1 and 2. We note that the total amounts of entries of A^k of the Algorithm 1 and 2 grow extremely fast.

1. For the self-maps in Subsection 3.3.2: the total number of variables $(\alpha_{i_1 \ldots i_n})$ (total number of columns) is n^n. Moreover for each P^k we have $k! \cdot \binom{n}{k}$ blocks of equations (3.11) and the amount of (not necessarily linearly independent) equation of each block is n^k. In total the number of equation are $k! \cdot \binom{n}{k} \cdot n^k$ consequently the matrix A^k has $k! \cdot \binom{n}{k} \cdot n^{n+k}$ entrances. This number shows the difficulty to run computations for large values.

2. For the bijections: the total number of variables $(\alpha_{i_1 \ldots i_n})$ (i.e. columns) is $n!$. Moreover for each P^k we have $k! \cdot \binom{n}{k}$ blocks of equation and the amount of (not necessarily linearly independent) equation of each block is
$$n \cdot \ldots \cdot (n-k+1) = k! \cdot \binom{n}{k} = \frac{n!}{(n-k)!}.$$
Then the total number of equation are $\left(\frac{n!}{(n-k)!}\right)^2$ consequently the matrix A^k has $\left(\frac{n!}{(n-k)!}\right)^2 \cdot n!$ entrances. In comparison, the numbers in the second case are slightly smaller and hence allow for more values to be verified numerically (31 for self-maps vs. 37 for the bijections).

However despite these computational issues we conjecture with the help of *The On-Line Encyclopedia of Integer Sequences*[51] the numbers $R_k^{n,m}$ for the stochastic Lévy flows of bijections.

Conjecture 34. *For all* $1 \leqslant k \leqslant m$ *we have* $R_k^{m,m} = T(m,k)$*, where the elements of the triangular array* $(T(m,k))_{1 \leq k \leq m}$ *are defined in [51] as*

"Triangle of numbers $T(m,k) =$ *number of permutations of* n *things with longest increasing subsequence of length* $\leqslant k$ $(1 \leqslant k \leqslant m)$*."*

The numbers $T(m,k)$ were introduced in [26] and turn out to be highly nontrivial. To date, no recursion or other simple formula is know in the literature for $T(m,k)$. In the Appendix of [26] some asymptotics are derived. On [52] the triangular array $T(m,k)$ are calculated for $m = 1, \ldots, 45$. By Remark 33, it is plausible, why the algorithmic verification seems to be unfeasible for large values.
The following table represents the numbers of $T(m,k)$ as given in [52]. The 37 $\boxed{\text{boxed}}$ values in the table below have been also calculated by Algorithm 2 and coincide with Conjecture 34.

Table 35 (Algorithmic verification of the triangle numbers $R_k^{m,m} = T(m,k)$)**.**

$T(m,k)$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
$m=2$	1	2	-	-	-	-	-	-	-	-
$m=3$	1	5	6	-	-	-	-	-	-	-
$m=4$	1	14	23	24	-	-	-	-	-	-
$m=5$	1	42	103	119	120	-	-	-	-	-
$m=6$	1	132	513	694	719	720	-	-	-	-
$m=7$	1	429	2761	4582	5003	5039	5040	-	-	-
$m=8$	1	1430	15767	33324	39429	40270	40319	40320	-	-
$m=9$	1	4862	94359	261808	344837	361302	362815	362879	362880	-
$m=10$	1	16796	586590	2190688	3291590	3587916	3626197	3628718	3628799	3628800

The 37 boxed values have been verified numerically to coincide with the respective values $T(m,k)$.

3.4 Embedding of stochastic n-point D-bifurcations on finite space in continuous time and space

Stochastic Lévy semiflows of continuous mappings coming from Lévy driven SDE are obtained (under rather restrictive conditions, such as finite variation paths) with the help of Marcus canonical equations, the analogue of the Stratonovich differential equations for a class. We refer to Kunita [38], Section 3, and [1], Chapter 6, more details. In discrete time, however, they are given by the random walk representation of Example 2. We show in the following example how to embed mutatis mutantis stochastic n-point D-bifurcations in (possibly high-dimensional) Euclidean space.

Example 36. In the context of the previous examples with $H < G \leqq S_m$ we use the classical representation of the permutations of S_m as elements in the orthogonal Lie group $SO(m+1,\mathbb{R}) \hookrightarrow \mathrm{Diff}(\mathbb{R}^{m+1}, \mathbb{R}^{m+1})$ acting on the first m elements $(e_1, \ldots, e_m)$ of the canonical basis of the Euclidean space $\mathbb{R}^{m+1}$. To each permutation $f_i \in G$, $i \in \{1, \ldots, |G|\}$ we associate the unique rotation U_i which sends $(e_1, e_2, \ldots, e_m)$ to $(e_{f_i(1)}, \ldots, e_{f_i(m)})$.

By definition, any representation in $SO(m+1,\mathbb{R})$ necessarily preserves positive orientation. However, a generic element $f_i \in G$ may have a negative sign, that is, the corresponding U_i has the shape

$$\left[\begin{array}{c|c|c|c|c} & & & & 0 \\ e_{f_i(1)} & e_{f_i(2)} & \cdots & e_{f_i(m)} & \vdots \\ & & & & 0 \\ & & & & \mathrm{sign}(f_i) \end{array}\right] \in \mathbb{R}^{(m+1)\times(m+1)}.$$

The last element of the canonical basis e_{m+1} is sent to $\mathrm{sign}(f_i) \cdot e_{m+1}$. The group $SO(m+1,\mathbb{R})$ is a connected compact Lie group whose Lie algebra is the vector space $\mathfrak{so}(m+1)$ of skew-symmetric matrices. Its compactness guarantees that the exponential of elements in the Lie algebra is surjective on $SO(m+1,\mathbb{R})$. Hence for each $U_i \in SO(m+1,\mathbb{R})$, there exists a skew-symmetric matrix $\mathfrak{X}_i$ in the Lie algebra of $\mathfrak{so}(m+1,\mathbb{R})$ such that, at time one, the exponential satisfies $U_i = \exp \mathfrak{X}_i$ for all $i \in \{1, 2, \ldots, |G|\}$.

Consider now the following linear Marcus canonical stochastic differential equation [46, 47] in the generalized Stratonovich sense as in Kurtz, Pardoux, Protter [41]

$$dx_t \;=\; \mathfrak{X}_1\, x_t \diamond dZ_t^1 + \ldots + \mathfrak{X}_{|G|}\, x_t \diamond dZ_t^{|G|}$$

where the $(Z^i)_{i=1,\ldots,|G|}$ is an i.i.d. family of Poisson process (with unitary increment $+1$) of intensity $1/|G|$ each. As a consequence the process $(x_t)_{t\geqslant 0}$ is a compound Poisson process with jump intensity

$|G| \cdot 1/|G| = 1$ with a uniform increment distribution on the set $\{U_1, \ldots, U_{|G|}\}$. Note that Marcus canonical equations are the jump noise equivalent of the Stratonovich stochastic integral since it satisfies the Leibniz formula for change of variables and hence preserves the flow without additional drift terms, see [41], Proposition 4.2.

In the sequel we follow the lines of the construction of a stochastic n-point D-bifurcation of Subsection 3.1.3 and keep the respective notation. For $H < G$ and $\varepsilon = 0$ we consider the flow embedding as above

$$dx_t^0 \quad = \quad \mathfrak{X}_1\, x_t^0 \diamond dZ_t^{0,1} + \ldots + \mathfrak{X}_{|H|}\, x_t^0 \diamond dZ_t^{0,|H|} \tag{3.18}$$

where $(Z^{0,i})_{i \in \{1,\ldots,|H|\}}$ an i.i.d. family of Poisson process with intensity $1/|H|$ each. That is, the process x^0 is a compound Poisson process with jump intensity 1 and uniform increment distribution on $\{U_1, \ldots, U_{|H|}\}$. For $\varepsilon \in (0, 1]$ we denote by $(Z^{\varepsilon,i})_{i\in\{1,\ldots,|H|\}}$ an i.i.d. family of Poisson processes with intensity $(1 - \varepsilon)/|H|$ and by $(\tilde{Z}^{\varepsilon,i})_{i\in\{1,\ldots,|G|-|H|\}}$ an i.i.d. family of Poisson processes with intensity $\varepsilon/(|G| - |H|)$ and consider

$$\begin{aligned} dx_t^\varepsilon = \quad & \mathfrak{X}_1 x_t^\varepsilon \diamond dZ_t^{\varepsilon,1} + \ldots \quad + \mathfrak{X}_{|H|} x_t^\varepsilon \diamond dZ_t^{\varepsilon,|H|} \\ & + \mathfrak{X}_{|H|+1} x_t^\varepsilon \diamond d\tilde{Z}_t^{\varepsilon,1} + \ldots + \mathfrak{X}_{|G|-|H|} x_t^\varepsilon \diamond d\tilde{Z}_t^{\varepsilon,|G|-|H|}. \end{aligned}$$

Then x^ε is a compound Poisson process with intensity $|H| \cdot (1 - \varepsilon)/|H| + (|G| - |H|) \cdot \varepsilon/(|G| - |H|) = 1$ and increment distribution

$$\Delta_\varepsilon := \Delta^H + \varepsilon \left[\Delta^{G \setminus H} - \Delta^H\right]. \tag{3.19}$$

We conclude that the stochastic flow generated by (3.18) with continuous time and space has a 3-point bifurcation at $\epsilon = 0$.

Acknowledgement: The authors would like to thank Prof. Pedro J. Catuogno for valuable discussions. The authors acknowledge financial support by DFG within the IRTG 1740: *Dynamical Phenomena in Complex Networks: Fundamentals and Applications*. M.A.H. was supported by the FAPA project "Stochastic dynamics of Lévy driven systems" at the School of Sciences of Universidad de los Andes. M.A.H. also greatfully acknowledges the financial support of Colciencias, the Colombian Administrative Department of Science, Technology and Innovation for travel support to UNICAMP in July 2019 in the framework of the Stic Math AmSud2019 project: "Stochastic analysis of non-Markovian phenomenta", where this work was completed. M.A.H. also thanks his colleagues Carolina Benedetti and Tristram Bogart from the Departamento de Matemáticas at Universidad de los Andes for their helpful comments on the project. P.R.R. is partially supported by FAPESP nr. 2015/50122-0, nr. 2020/04426-6 and CNPq nr. 305212/2019-2.

References

[1] D. Applebaum – *Lévy Processes and Stochastic Calculus.* 2nd edition. Cambridge University Press, 2009.

[2] D. Applebaum, H. Kunita – Invariant measures for Lévy flows of diffeomorphisms. Proceedings of the Royal Society of Edinburgh, 130A , 925–946, 2000.

[3] D. Applebaum, H. Kunita – Lévy flows on manifolds and Lévy processes on Lie groups. J. Math. Kyoto Univ. (JMKYAZ) 33-4 (1993) 1103-1123.

[4] Ludwig Arnold – *Random Dynamical Systems.* Springer-Verlag, Berlin 1998.

[5] R.A. Arratia – *Coalescing Brownian Motions on the Line.* (Ph.D. thesis), University of Wisconsin, 1979.

[6] D. K. Arrowsmith, C. M. Place – *Dynamical systems - Differential equations, maps, chaotic behavior* Chapman and Hall, London, 1992.

[7] Ludwig Arnold – Recent progress in stochastic bifurcation theory. IUTAM Symposium on Nonlinearity and Stochastic Structural Dynamics (Madras, 1999), 15–27, *Solid Mech. Appl.*, 85, Kluwer Acad. Publ., 2001.

[8] Peter Baxendale – Brownian motion in the diffeomorphism group I., Compositio Mathematica, 53:19–50, 1984.

[9] Peter Baxendale – Asymptotic behavior of stochastic flows of diffeomorphisms. Stochastic processes and their applications (Nagoya, 1985), 1–19, *Lecture Notes in Math.*, 1203, Springer-Verlag, 1986.

[10] Peter Baxendale – Statistical equilibrium and two-point motion for a stochastic flow of diffeomorphisms. *Spatial stochastic processes*, 189–218, Progr. Probab., 19, Birkhäuser, 1991.

[11] Baxendale P.H. – Stability Along Trajectories at a Stochastic Bifurcation Point. In: Stochastic Dynamics. Springer, New York, NY (1999).

[12] N. Berestycki, Ch. Garban, A. Sen – *Coalescing Brownian flows: a new approach,* Ann. Probab. 43 (6) (2015) 3177–3215.

[13] J.-M. Bismut. – Flots stochastiques et formule de Ito-Stratonovitch généralisée. C. R. Acad. Sci. Paris, Ser. I, 290: 483–486, 1980.

[14] Böttcher, Björn and Schilling, René and Wang, Jian. Lévy matters. III, Lecture Notes in Mathematics, Vol. 2099, Springer, Cham, 2013.

[15] E. Rodney Canfield and Brendan D. McKay – The asymptotic volume of the Birkhoff polytope. *Online J. Anal. Comb.* No. 4 (2009), 4 pp.

[16] D.N. Cheban, P.E. Kloeden, B. Schmalfuß – The relationship between pullback, forwards and global attractors of nonautonomous dynamical systems - Nonlinear Dyn. Syst. Theory, 2002.

[17] Henk A. Dijkstra. – Nonlinear climate dynamcis. Cambridge University Press, (2013).

[18] A. A. Dorogovtsev, G. V. Riabov, Björn Schmalfuß. – Stationary points in coalescing stochastic flows on $\mathbb{R}$ Stochastic Processes and their Applications 130 (2020) 4910–4926

[19] A.A. Dorogovtsev. – *Some remarks on a Wiener flow with coalescence.* Ukrainian Math. J. 57 (2005) 1550–1558.

[20] A.A. Dorogovtsev, Ia. Korenovska. – *Essential sets for random operators constructed from Arratia flow.* Commun. Stoch. Anal. 11 (3) (2017) 301–312.

[21] A.A. Dorogovtsev, M.B. Vovchanskii. – Arratia flow with drift and the Trotter formula for Brownian web. Commun. Stoch. Anal. 12 (1) (2018) 89–108.

[22] Elworthy, K. D. – Stochastic dynamical systems and their flows. In: M. Pinsky and V. Wihstutz, editors, Stochastic analysis pp. 79–95, Academic Press, New York-London, 1978.

[23] J. Guckenheimer, Ph. Holmes. – Nonlinear Oscillations, Dynamical systems and Bifurcations of vector fields Springer.

[24] T. Fujiwara – Stochastic differential equations of jump type on manifolds and Lévy flows. J. Math. Kyoto Univ. (JMKYAZ) 31-1 (1991) 99–119.

[25] T. Fujiwara, H. Kunita – Stochastic differential equations of jump type and Lévy processes in diffeomorphisms group. J. Math. Kyoto Univ. (JMKYAZ) 25-1 (1985) 71–106.

[26] I.M. Gessel – Symmetric functions and P-recursiveness. Journal of Combinatorial Theory, Series A, Volume 53, Issue 2, (1990) 257–285.

[27] F. Flandoli, B. Gess, M. Scheutzow – Synchronization by noise. Probability Theory and Related Fields 168, 511–556 (2017).

[28] Th.E. Harris. – *Coalescing and noncoalescing stochastic flows in* $\mathbb{R}^1$, Stochastic Process. Appl. 17 (1984) 187–210.

[29] N. Ikeda; S. Watanabe – Stochastic differential equations and diffusion processes. North-Holland Mathematical Library, 24. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1989.

[30] P. Imkeller, A. Monahan. Conceptual stochastic climate models. Stochastics and Dynamics, 2:311–326, 2002.

[31] P. Imkeller. Energy balance models: Viewed from stochastic dynamics. Prog. Prob., 49:213–240, 2001.

[32] J. Jost, Ma. Kell and Ch. Rodrigues – Representation of Markov chains by random maps: existence and regularity conditions. *Calculus of Variations and Partial Differential Equations*, 2015.

[33] Anatole Katok and Boris Hasselblatt – *Introduction to the Modern Theory of Dynamical Systems.* Cambridge University Press, 1995.

[34] Yu. Kifer – Ergodic theory of random transformations, Birkhäuser, Boston, 1986.

[35] T. Kosenkova – Strong Markov approximation of Lévy processes and their generalizations in a scheme of series. Theory of Probability and Mathematical Statistics 86 (2013).

[36] T. Kosenkova – Weak convergence of the series scheme of Markov chains to the solution of Lévy driven SDE. Theory of Stochastic Processes 18(34) 1 (2012).

[37] H. Kunita – *Stochastic Flows and Stochastic Differential Equations.* Cambridge University Press, 1990.

[38] H. Kunita – Stochastic differential equations based on Lévy processes and stochastic flows of diffeomorphisms. Real and Stochastic Analysis, 305–373, Trends Math., Birkhäuser Boston, MA, 2004.

[39] H. Kunita – Stochastic Flows and Jump-Diffusions, Springer Singapore, 2019.

[40] Yu. Kifer – Ergodic theory of random transformations., Birkhäuser, Boston, 1986.

[41] T. Kurtz, E. Pardoux e P. Protter – Stratonovich stochastic differential equations driven by general semimartingales. Annales de l'I.H.P., section B, tome 31, n. 2, 1995. p. 351-377.

[42] A. N. Quas – On representation of Markov chains by random smooth maps, Bulletin of the London Mathematical Society 23 (1991), no. 5, 487–492.

[43] M. Engel, J.S.W. Lamb, M. Rasmussen – Bifurcation analysis of a stochastically driven limit cycle. Communications in Mathematical Physics, 365, 935–942 (2019).

[44] Y. Le Jan and Olivier Raimond – Flows, coalescence and noise. *Ann. Probab.* 32 (2004), no. 2, 1247–1315.

[45] Y. Le Jan and Olivier Raimond – Stochastic flows on the circle. *Probability and partial differential equations in modern applied mathematics*, 151–162, IMA Vol. Math. Appl., 140, Springer, New York, 2005.

[46] Marcus, S.I. – Modeling and analysis of stochastic differential equations driven by Poisson point processes. IEEE Transactions on Information theory **24**(3), 164–172 (1978).

[47] Marcus, S.I. – Modeling and analysis of stochastic differential equations driven by semimartingales. Stochastics **4**(3), 223–245 (1981).

[48] Mirsky, L. – Proofs of two theorems on doubly-stochastic matrices. *Proc. Amer. Math. Soc.* no. 9, 1958, 371–374.

[49] Newman, J. – Necessary and sufficient conditions for stable synchronization in random dynamical systems. Ergodic Theory and Dynamical Systems (2018), 38, 1857–1875.

[50] Pak, Igor – Four questions on Birkhoff polytope. *Annals of Combinatorics* 4 (2000) no. 1, 83–90.

[51] OEIS Foundation Inc. (2021), The On-Line Encyclopedia of Integer Sequences, https://oeis.org/A047887. retrieved: Jul 23, 2021, 10:10h am.

[52] Heinz, Alois P. – Rows n = 1..45, flattened. On the webpage *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org/A047887 Particular file with the listed values of $T(m,k)$ https://oeis.org/A047887/b047887.txt retrieved: Jul 23, 2021, 10:11h am.

[53] G.V. Riabov. – *Random dynamical systems generated by coalescing stochastic flows on R.* Stoch. Dyn. 18 (4) (2018) 1850031.