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GENERALIZED QUASILINEAR EQUATIONS WITH CRITICAL GROWTH AND NONLINEAR BOUNDARY CONDITIONS

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Abstract. We study the quasilinear problem

$$-\operatorname{div}(h^2(u)\nabla u) + h(u)h'(u)|\nabla u|^2 + u = -\lambda|u|^{q-2}u + |u|^{2\cdot 2^*-2}u \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \eta} = \mu g(x,u) \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with regular boundary $\partial\Omega$, $\lambda, \mu > 0$, $1 < q < 4$, $2\cdot 2^* = 12$, $\frac{\partial}{\partial\eta}$ is the outer normal derivative and g has a subcritical growth in the sense of the trace Sobolev embedding. We prove a regularity result for all weak solutions for a modified, and introducing a new type of constraint, we obtain a multiplicity of solutions, including the existence of a ground state.

1. Introduction

We study the quasilinear Schrödinger equation

$$i\partial_t\psi = -\Delta\psi + V(x)\psi - \eta(|\psi|^2)\psi - \kappa[\Delta\rho(|\psi|^2)]\rho'(|\psi|^2)\psi, \tag{1.1}$$

where $\psi:\mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, $V:\mathbb{R}^N \to \mathbb{R}$ is a given potential, $N \geq 1$, κ is a positive constant and $\rho, \eta:\mathbb{R}^+ \to \mathbb{R}$ are suitable functions. This equation arises in various branches of mathematical physics, see for example [28]. When $\kappa \neq 0$, (1.1) models phenomena in plasma physics and fluid mechanics [15, 16, 18, 21], laser theory [2, 29], and in condensed matter theory [24]. The case $\rho(s) = s$ occurs in theory of superfluids (see [15, 16, 19] and the references in [17]), whereas $\rho(s) = (1+s)^{1/2}$ appears in the self-channeling of a high-power ultra short laser in matter (see [3, 4]).

Looking for standing wave solutions for (1.1), one takes $\psi(t,x) := \exp(-iEt)u(x)$ with $E \in \mathbb{R}$ and $u:\mathbb{R}^N \to \mathbb{R}$ a function, which leads to consider the elliptic equation

$$-\Delta u + V(x)u - \kappa\Delta(\rho(u^2))\rho'(u^2)u = \mathfrak{g}(u), \quad \text{in } \Omega \subseteq \mathbb{R}^N, \tag{1.2}$$

where we have replaced $V(x) - E$ by $V(x)$ and $\mathfrak{g}(u) = \eta(u^2)u$.

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In this article, we are interested in the quasilinear problem

$$\begin{gathered} -\operatorname{div}(h^2(u)\nabla u)+h(u)h'(u)|\nabla u|^2+u=-\lambda|u|^{q-2}u+|u|^{2\cdot 2^*-2}u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \eta}=\mu g(x,u) \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with regular boundary $\partial\Omega$, $\lambda, \mu > 0$, $1 < q < 4$, $2 \cdot 2^* = 12$, and $\frac{\partial}{\partial \eta}$ is the outer normal derivative. Note that if we take

$$h^2(s) = 1 + \frac{1}{2}\Big(\frac{d}{ds}\rho(s^2)\Big)^2,$$

then equation (1.3) becomes (1.2), see [31].

We consider nonlinearities h, satisfying the following:

(A1) $h \in C^2(\mathbb{R}, (0, +\infty))$ is even, non-decreasing in $[0, +\infty)$ and

$$h_\infty := \lim_{t\to\infty} \frac{h(t)}{t} \in (0, +\infty). \tag{1.4}$$

(A2) It holds that

$$\beta := \sup_{t\in\mathbb{R}} \frac{th'(t)}{h(t)} \leq 1. \tag{1.5}$$

(A3) The mapping $t \mapsto h'(t)h(t)/t$ is non-increasing for $t > 0$.

Remark 1.1. Hypotheses (A1) and (A3) together imply that, for all $t > 0$,

$$\frac{t^2h''(t)}{h(t)} + \frac{t^2[h'(t)]^2}{h^2(t)} \leq \frac{th'(t)}{h(t)}.$$

Since h is an even function, we have that h' is an odd function and h'' is an even function. Therefore, the above inequality still holds for $t \leq 0$.

We refer the reader to [26] and references therein, for a review of the semilinear case, i.e., problem (1.2) when $\kappa = 0$, in bounded domains $\Omega \subset \mathbb{R}^N$. Whether $\Omega = \mathbb{R}^N$ and again $\kappa = 0$, there are [22] and its references. The literature on the subcritical case of problem (1.2) with $\kappa \neq 0$ is extensive for $\Omega = \mathbb{R}^N$ (see [8, 20, 23, 25]), as well as a bounded domain $\Omega \subset \mathbb{R}^N$ (see [7, 10]). Furthermore, recent results concerning the case of the critical power in $\mathbb{R}^N$, $\mathfrak{g}(u) = u^p$ for $p = 2 \cdot 2^* = 4N/(N-2)$ are found in publications such as Deng et al. [9]. In their introduction they present a complete review for this class of problems.

We highlight the seminal papers [8, 19] in which the particular case $\rho(s) = s$, that is, $h(s) = (1+2s^2)^{1/2}$, was cleverly studied. Since the energy functional associated to the problem is not well defined in the whole Sobolev space, the authors considered the change of variables $u = f(v)$, where f is defined by

$$f'(t) := \frac{1}{\sqrt{1+2f^2(t)}} \quad \text{in } [0, +\infty) \quad f(t) := -f(-t) \quad \text{in } (-\infty, 0], \tag{1.6}$$

and for some adequate growth for function $\mathfrak{g}$, they applied variational methods to establish the existence of a nontrivial solution for (1.2). We point out that this change of variables has become a powerful tool for solving problem (1.2) when $\rho(s) = s$. For more details, see [1, 23, 30] and references therein.

Note that problem (1.2) in a bounded domain Ω is also relevant, for example, in physical models that describe electrons on lattices and applications to nanotubes [14]. Semilinear and quasilinear problems of this type in bounded domains, on either Dirichlet or Neumann boundary conditions, appear in [6, 26, 27] and its references.

To tackle problem (1.3), we use a new type of constraint for the energy functional related to a modified problem. Alternative to the usual method of Nehari (for example, [20]), we define, in Section 3, the constraint based on the change of variable that we will apply. One of the advantages of this definition is that we can consider values of q in the interval $(1,4)$ that may not be considered when applying the usual Nehari manifold as a constraint. In exchange, we restrict the approach to three dimensions because of technical issues related to the Sobolev embeddings, as explained in Remark 3.6. The lack of compactness issues, which naturally appear due to the critical exponent, are circumvent by proving that, for μ sufficiently large, there exists a (PS) sequence in the range $\big(0, \frac{4^{-N/2}(Sh_\infty)^{N/2}}{N}\big)$ where compactness holds (see Proposition 2.6 below). Here, S is the best constant to the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and $h_\infty > 0$ is defined in (1.4).

Next we give some examples of functions that appear in physics models satisfy conditions (A1)–(A3).

Lemma 1.2. *The following functions $h : \mathbb{R} \to (0,+\infty)$ satisfy* (A1)–(A3).

(a) $h(t) = \sqrt{1+2t^2}$;
(b) $h(t) = \sqrt{1 + \frac{t^2}{2(1+t^2)} + t^2}$;
(c) $h(t) = \sqrt{1 + \frac{3t^2}{1+t^2} + \ln(1+e^{t^2})}$;
(d) $h(t) = \sqrt{1 - \frac{e^{-t^2}}{2} + \frac{1}{2}\ln(1+e^{t^2})}$.

In this article, we will use either the notations $2\cdot 2^* = 4N/(N-2)$ and $2\cdot 2_* = 4(N-1)/(N-2)$, or respectively, 12 and 8 in dimension $N=3$.

We assume that the function $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$, satisfies the following hypotheses:

(A4) $g \in C^{1,\theta}(\partial\Omega \times \mathbb{R}, \mathbb{R})$ for some $\theta \in (0,1)$;
(A5) Let $G(x,s) = \int_0^s g(x,t)dt$. There exists a constant σ satisfying $\frac{1}{2\cdot 2_*} < \sigma \le \frac{1}{4}$ such that
$$\sigma g(x,s)s \ge G(x,s) > 0$$
for all $s \neq 0$ and almost every $x \in \partial\Omega$;
(A6) $\lim_{s\to 0} \frac{g(x,s)}{s^3} = 0$ and $\lim_{|s|\to+\infty} \frac{|g(x,s)|}{|s|^{p-1}} = g_\infty(x)$ uniformly for $x \in \partial\Omega$, for some $g_\infty \in L^\infty(\partial\Omega)$, and $4 \le p < 2\cdot 2_*$;
(A7) The function defined by $s \mapsto g(x,s)/s^3$ for $s \in (-\infty,0)\cup(0,+\infty)$ is non-decreasing for almost every $x \in \partial\Omega$;
(A8) There exist $c_1, c_2 > 0$ such that
$$|g'(x,s)| \le c_1|s|^{p-2} + c_2.$$

Remark 1.3. We note that hypothesis (A5) includes the 3-asymptotically linear case, that is, it may occurs that
$$\lim_{|s|\to+\infty} \frac{|g(x,s)|}{|s|^3} = g_\infty(x)$$
uniformly on $x \in \partial\Omega$.

If we consider the functions $g(x,s) = g_\infty(x)|s|^{p-2}s$ or $g(x,s) = g_\infty(x)\frac{s^5}{1+s^2}$, where $g_\infty \in L^\infty(\partial\Omega)$ such that $0 < g_0 \le g_\infty(x) \le g_\infty$ almost everywhere $x \in \partial\Omega$ and $4 \le p < 2\cdot 2_*$, then g satisfies all conditions $(g_1) - (g_5)$.

The first difficulty in directly applying variational methods to solve problem (1.3) is that the energy functional associated with this problem may not be well defined in the whole space $H^1(\Omega)$. Precisely, the functional $T_{\lambda,\mu} : H^1(\Omega) \to \mathbb{R}$ associated with equation (1.3), given by

$$\begin{aligned} T_{\lambda,\mu}(u) = \frac{1}{2}\int_\Omega (h^2(u)|\nabla u|^2 + u^2)dx + \frac{\lambda}{q}\int_\Omega |u|^q dx \\ - \frac{1}{2\cdot 2^*}\int_\Omega |u|^{2\cdot 2^*}dx - \mu\int_{\partial\Omega} G(x,u)d\sigma_x, \end{aligned} \tag{1.7}$$

for $u \in H^1(\Omega)$, where $d\sigma_x$ is the measure on the boundary, is not well defined, because the term $\int_\Omega h^2(u)|\nabla u|^2 dx$ is not finite for all $u \in H^1(\Omega)$ and for all h that we are considering. Indeed, without loss of generality, assume $B_2(0) \subset \Omega$ and let $h(t) = \sqrt{1+2t^2}$ (item a) from Lemma 1.2), and $\phi \in C_0^\infty(\Omega,[0,1])$ be such that $\phi \equiv 1$ in $B_1(0) = \{x \in \Omega;\ |x| < 1\}$ and $\phi \equiv 0$ in $\Omega \setminus B_2(0) = \{x \in \Omega;\ |x| \geq 2\}$. Now taking $u(x) = |x|^{\frac{-1}{4}}\phi(x)$ for $x \neq 0$, it is easy to see that $u \in H^1(\Omega)$, however

$$\int_\Omega h^2(u)|\nabla u|^2 dx \geq 2\int_\Omega u^2|\nabla u|^2 dx = +\infty.$$

To overcome this difficulty, the main idea is to take the primitive $H(s) := \int_0^s h(t)dt$, and consider the change of variable $w = H(u)$, then look for critical point of the functional $I_{\lambda,\mu} : H^1(\Omega) \to \mathbb{R}$ defined by

$$I_{\lambda,\mu}(u) := T_\lambda, \mu(H^{-1}(w))$$

for $w \in H^1(\Omega)$. It can be proved that $w \in H^1(\Omega)$ is a critical point of $I_{\lambda,\mu}$ if, and only if, $u = H^{-1}(w)$ is a weak solution of problem (1.3).

We list below the main properties of the change of variable which will be used throughout this work.

Lemma 1.4. *The function $H^{-1} : \mathbb{R} \to \mathbb{R}$ satisfies the following properties:*

(1) $H^{-1} \in C^1(\mathbb{R},\mathbb{R})$;
(2) $0 < \frac{d}{dt}\Big(H^{-1}(t)\Big) = \frac{1}{h(H^{-1}(t))} \leq \frac{1}{h(0)}$ *for all* $t \in \mathbb{R}$;
(3) $|H^{-1}(t)| \leq \frac{|t|}{h(0)}$ *for all* $t \in \mathbb{R}$;
(4) $\frac{H^{-1}(t)}{t} \to \frac{1}{h(0)}$ *as* $t \to 0$;
(5) $1 \leq \frac{H^{-1}(t)h(H^{-1}(t))}{t} \leq 2$ *for all* $t \neq 0$.
(6) $\left|\frac{t}{h(t)}\right| \leq \frac{1}{h_\infty}$ *for all* $t \in \mathbb{R}$;
(7) $\frac{H^{-1}(t)}{\sqrt{t}}$ *is non-decreasing in* $(0,+\infty)$ *and* $|H^{-1}(t)| \leq (2/h_\infty)^{1/2}\sqrt{|t|}$ *for all* $t \in \mathbb{R}$;
(8) $\frac{H^{-1}(t)}{\sqrt{t}} \to \sqrt{\frac{2}{h_\infty}}$ *as* $t \to +\infty$;
(9) $|H^{-1}(t)| \geq H^{-1}(1)\sqrt{|t|}$ *for all* $|t| \geq 1$;
(10) $\frac{1}{2}(H^{-1}(t))^2 \leq H^{-1}(t)(H^{-1})'(t)t \leq (H^{-1}(t))^2$ *for all* $t \in \mathbb{R}$.

Proof. Properties (1)–(9) can be found in [12, Lemma 2.1]. Property (10) follows from property (5) and the fact that h is even, H is odd and so H^{-1} as well. □

After the change of variable $u = H^{-1}(w)$ in (1.7), we obtain

$$I_{\lambda,\mu}(w) = \frac{1}{2}\int_\Omega (|\nabla w|^2 + |H^{-1}(w)|^2)dx + \frac{\lambda}{q}\int_\Omega |H^{-1}(w)|^q dx$$

$$- \frac{1}{2 \cdot 2^*} \int_\Omega |H^{-1}(w)|^{2 \cdot 2^*} dx - \mu \int_{\partial\Omega} G(x, H^{-1}(w)) d\sigma_x,$$

and the functional $I_{\lambda,\mu}$ is associated with the problem

$$\begin{gathered} -\Delta w + H^{-1}(w)(H^{-1})'(w) = \mathfrak{p}(w), \quad \text{in } \Omega, \\ \frac{\partial w}{\partial \eta} = \mu g(x, H^{-1}(w))(H^{-1})'(w), \quad \text{on } \partial\Omega \end{gathered} \tag{1.8}$$

for $w \in H^1(\Omega)$, where

$$\mathfrak{p}(w) = -\lambda |H^{-1}(w)|^{q-2} H^{-1}(w)(H^{-1})'(w) + |H^{-1}(w)|^{2\cdot 2^* - 2} H^{-1}(w)(H^{-1})'(w).$$

To show that $I_{\lambda,\mu}$ is well defined and belongs to $C^1(H^1(\Omega), \mathbb{R})$, we use that, for every $\varepsilon > 0$, by conditions $(g_1) - (g_3)$, there exists $C_\varepsilon := C(\varepsilon, q, \sigma) > 0$ such that

$$|g(x,s)| \le \varepsilon |s|^3 + C_\varepsilon |s|^{p-1} \quad \text{and} \quad |G(x,s)| \le \varepsilon |s|^4 + C_\varepsilon |s|^p \tag{1.9}$$

for all $s \in \mathbb{R}$ and $x \in \partial\Omega$. Here, we may choose $4 \le p = 1/\sigma < 2 \cdot 2_*$ (see Lemma 2.1 below). Then, it is enough to use (1.9), properties (1) and (7) and the Sobolev embeddings to conclude that $I_{\lambda,\mu}$ is continuous and is well defined in $H^1(\Omega)$. The C^1 regularity of $I_{\lambda,\mu}$ follows from Lemma 1.4, the properties of the functions H^{-1} and $(H^{-1})'$.

In this article, let $\| \cdot \|$ denote the norm $u \mapsto \sqrt{\int_\Omega (|\nabla u|^2 + u^2) dx}$ in $H^1(\Omega)$ and $| \cdot |_r$ denote the usual norm in the Lebesgue space $L^r(\Omega)$ for $r \ge 1$. The main contributions of this article are the following.

Theorem 1.5. *Under assumptions* (A1)–(A8)*, for $\lambda > 0$, there exists $\mu_\lambda > 0$ such that, for every $\mu \ge \mu_\lambda$, one of the following cases occurs:*

1. *Problem* (1.8) *has two solutions, one of which is nonnegative and ground state solution and the other is non-positive;*
2. *Problem* (1.8) *has two solutions, one of which is non-positive and ground state solution and the other is nonnegative.*

Corollary 1.6. *Let $u_{\lambda,\mu} \ge 0$ and $v_{\lambda,\mu} \le 0$ be the solutions given in Theorem 1.5. It holds that $I_{\lambda,\mu}(u_{\lambda,\mu}) \to 0$ and $I_{\lambda,\mu}(v_{\lambda,\mu}) \to 0$ as $\mu \to +\infty$ uniformly on λ in a bounded set.*

Theorem 1.7. *Under assumptions* (A1)–(A8)*, every weak solution $w \in H^1(\Omega)$ for problem* (1.8) *is a classical solution in the sense that $w \in C^{2,\gamma}(\overline{\Omega})$, for some $\gamma \in (0,1)$, and w satisfies pointwisely equation* (1.8)*.*

2. A compactness result

The next lemma is a direct consequence of hypothesis (A6) and Remark 1.3.

Lemma 2.1. *Let $p \le \tau < 2 \cdot 2_*$. For all $\varepsilon > 0$, there exists a positive constant $C_\varepsilon > 0$ such that*

$$\begin{gathered} |g(x,s)| \le \varepsilon |s|^3 + C_\varepsilon |s|^{\tau - 1}, \\ |G(x,s)| \le \varepsilon |s|^4 + C_\varepsilon |s|^\tau \end{gathered}$$

for all $s \in \mathbb{R}$ and $x \in \partial\Omega$.

In what follows, we show that H^{-1} has an appropriate behavior at the origin and at infinity in order to use a general theorem due to Brezis-Lieb, involving the change of variable H^{-1}, result that will be essential to demonstrate the Palais-Smale condition ((PS) condition) for functional $I_{\lambda,\mu}$.

Lemma 2.2. *For each $\varepsilon > 0$, there exists $\hat{C}_\varepsilon = \hat{C}(\varepsilon) > 0$ such that $|H^{-1}(t)| \le \varepsilon|t|^{1/2} + \hat{C}_\varepsilon|t|^{1/2}$ for all $t \in \mathbb{R}$. Moreover, $(\hat{C}_\varepsilon)$ is uniformly bounded for ε in a bounded set.*

Proof. Let $\varepsilon > 0$ be any positive real number. By property (3), we have $\frac{|H^{-1}(t)|}{|t|^{1/2}} \to 0$ as $t \to 0$ and then there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|H^{-1}(t)| \le \varepsilon|t|^{1/2} \quad \text{for all } |t| < \delta. \tag{2.1}$$

Property (8) ensures that there exists $\gamma = \gamma(\varepsilon) > 0$ such that

$$|H^{-1}(t)| \le \Big(\varepsilon + \sqrt{\frac{2}{h_\infty}}\Big)|t|^{1/2} \quad \text{for all } |t| > \gamma. \tag{2.2}$$

Since from property (7) we have

$$|H^{-1}(t)| \le (2/h_\infty)^{1/2}|t|^{1/2} \quad \text{for all } \delta \le |t| \le \gamma,$$

it follows from (2.1) and (2.2) that, for all $t \in \mathbb{R}$,

$$|H^{-1}(t)| \le \varepsilon|t|^{1/2} + (\varepsilon + (2/h_\infty)^{1/2})|t|^{1/2} + (2/h_\infty)^{1/2}|t|^{1/2},$$

that is,

$$|H^{-1}(t)| \le \varepsilon|t|^{1/2} + \hat{C}_\varepsilon|t|^{1/2}$$

for all $t \in \mathbb{R}$, where $\hat{C}_\varepsilon = \varepsilon + 2(2/h_\infty)^{1/2} > 0$. Clearly $(\hat{C}_\varepsilon)$ is uniformly bounded for ε in a bounded set, and the lemma follows. □

Lemma 2.3. *Let $j : \mathbb{R} \to \mathbb{R}$ be defined as $j(t) = |H^{-1}(t)|^{2\cdot 2^*}$. Given $\varepsilon > 0$, there exist two nonnegative continuous functions $\varphi_\varepsilon, \psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that, for all $a, b \in \mathbb{R}$, it holds*

$$|j(a+b) - j(b)| \le \varepsilon\varphi_\varepsilon(a) + \psi_\varepsilon(b).$$

Proof. We apply the same arguments as in [11, Lemma 3.2], replacing f for H^{-1}, using property (6),the Mean Value Theorem, Young inequality, and Lemma 2.2. Then, for all $\varepsilon > 0$ and $a, b \in \mathbb{R}$, we obtain the existence of some constants $C, B_\varepsilon > 0$ such that

$$|j(a+b) - j(b)| \le \varepsilon\varphi_\varepsilon(a) + \psi_\varepsilon(b),$$

where $\hat{C}_\varepsilon = \varepsilon + 2(2/h_\infty)^{1/2} > 0$ is given in Lemma 2.2 and $\varphi_\varepsilon(a) = C\varepsilon^{2\cdot 2^*-2}(1 + \hat{C}_\varepsilon)|a|^{2^*}$ and $\psi_\varepsilon(b) = (\varepsilon^{2\cdot 2^*-2} + \hat{C}_\varepsilon + B_\varepsilon)|b|^{2^*}$ are the two nonnegative continuous functions required. The lemma is proved. □

Lemma 2.4. *Given $\varepsilon > 0$, let $(w_n) \subset H^1(\Omega)$ be a sequence that converges weakly to w in $H^1(\Omega)$ and let $j, \varphi_\varepsilon, \psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ be as in Lemma 2.3. Then*

- (i) *$j(w) \in L^1(\Omega)$;*
- (ii) *$\int_\Omega \varphi_\varepsilon(w_n - w)dx \le C < +\infty$ for some constant $C > 0$, which does not depend on $0 < \varepsilon < 1$ and $n \in \mathbb{N}$;*
- (iii) *$\int_\Omega \psi_\varepsilon(w)dx < +\infty$ for all $\varepsilon > 0$.*

Proof. Items (i) and (iii) follow directly from Sobolev embedding. To prove (ii), we have from Lemma 2.2 that $\hat{C}_\varepsilon = \varepsilon + 2(2/h_\infty)^{1/2} < 1 + 2(2/h_\infty)^{1/2}$ for all $0 < \varepsilon < 1$, whence

$$\varphi_\varepsilon(w_n - w) = C\varepsilon^{2\cdot 2^*-2}(1+\hat{C}_\varepsilon)|w_n - w|^{2^*} \le C(2 + 2(2/h_\infty)^{1/2})|w_n - w|^{2^*}.$$

Thus, item (ii) is proved since (w_n) is also bounded in $L^{2^*}(\Omega)$ by hypothesis. □

Proposition 2.5. *Let $(w_n) \subset H^1(\Omega)$ be a sequence that converges weakly to w in $H^1(\Omega)$. Then*

$$\int_\Omega \Big| |H^{-1}(w_n - w)|^{2\cdot 2^*} - |H^{-1}(w_n)|^{2\cdot 2^*} + |H^{-1}(w)|^{2\cdot 2^*} \Big| dx \to 0 \quad \text{as } n \to +\infty.$$

In particular,

$$|H^{-1}(w_n - w)|_{2\cdot 2^*}^{2\cdot 2^*} + |H^{-1}(w)|_{2\cdot 2^*}^{2\cdot 2^*} = |H^{-1}(w_n)|_{2\cdot 2^*}^{2\cdot 2^*} + o_n(1),$$

with $o_n(1) \to 0$ as $n \to +\infty$.

The proof of the above propsition is a direct consequence of Lemma 2.4 with the general Brezis-Lieb Lemma (see Theorem 2 in [5]).

2.1. (PS) **condition in the correct range.** In the sequel, we will show that $I_{\lambda,\mu}$ satisfies (PS) condition in a particular range for bounded sequences.

Proposition 2.6. *Let $(w_n) \subset H^1(\Omega)$ be a bounded $(PS)_c$ sequence for the functional $I_{\lambda,\mu}$. If $c < \frac{4^{-N/2}(Sh_\infty)^{N/2}}{N}$, then (w_n) possesses a strongly convergent subsequence.*

Proof. Let (w_n) be a bounded $(PS)_c$ sequence for functional $I_{\lambda,\mu}$. So, up to a subsequence, we may suppose that

$$\begin{gathered} w_n \rightharpoonup w \quad \text{in } H^1(\Omega), \quad w_n \to w \quad \text{in } L^2(\Omega), \\ w_n \to w \quad \text{in } L^q(\Omega), \quad w_n(x) \to w(x) \quad \text{a.e. in } \Omega. \end{gathered} \tag{2.3}$$

By the Sobolev compact embeddings, we obtain

$$\begin{aligned} I'_{\lambda,\mu}(w)v &= \int_\Omega \nabla w \nabla v dx + \int_\Omega H^{-1}(w)(H^{-1})'(w)v dx \\ &\quad + \lambda \int_\Omega |H^{-1}(w)|^{q-2} H^{-1}(w)(H^{-1})'(w)v dx \\ &\quad - \int_\Omega |H^{-1}(w)|^{2\cdot 2^*-2} H^{-1}(w)(H^{-1})'(w)v dx \\ &\quad - \mu \int_{\partial\Omega} g(x, H^{-1}(w))(H^{-1})'(w)v d\sigma_x = 0 \end{aligned} \tag{2.4}$$

for all $v \in H^1(\Omega)$. Thus, from (A5),

$$\begin{aligned} I_{\lambda,\mu}(w) &= I_{\lambda,\mu}(w) - \sigma I'_{\lambda,\mu}(w) H^{-1}(w) h(H^{-1}(w)) \\ &\ge \int_\Omega \Big(\frac{1}{2} - \sigma(1+\beta)\Big)|\nabla w|^2 dx + \big(\frac{1}{2} - \sigma\big) \int_\Omega |H^{-1}(w)|^2 dx \\ &\quad + \lambda\big(\frac{1}{q} - \sigma\big) \int_\Omega |H^{-1}(w)|^q dx + \big(\sigma - \frac{1}{2\cdot 2^*}\big) \int_\Omega |H^{-1}(w)|^{2\cdot 2^*} dx \\ &\quad + \int_{\partial\Omega} \big(\sigma g(x, H^{-1}(w)) H^{-1}(w) - G(x, H^{-1}(w))\big) d\sigma_x \ge 0. \end{aligned} \tag{2.5}$$

Let us denote $v_n := w_n - u$ and prove that $v_n \to 0$ in $H^1(\Omega)$. From (2.3), we have $v_n \to 0$ in $L^2(\Omega)$ and $L^q(\Omega)$. So,

$$\begin{aligned}
&I_{\lambda,\mu}(w) + I_{\lambda,\mu}(v_n)\\
&= \frac{1}{2}|\nabla w|_2^2 + \frac{1}{2}|H^{-1}(w)|_2^2 + \frac{\lambda}{q}|H^{-1}(w)|_q^q - \frac{\mu}{2\cdot 2^*}|H^{-1}(w)|_{2\cdot 2^*}^{2\cdot 2^*}\\
&\quad - \mu\int_{\partial\Omega} G(x, H^{-1}(w))dx + \frac{1}{2}|\nabla v_n|_2^2 + \frac{1}{2}|H^{-1}(v_n)|_2^2 + \frac{\lambda}{q}|H^{-1}(v_n)|_q^q\\
&\quad - \frac{1}{2\cdot 2^*}|H^{-1}(v_n)|_{2\cdot 2^*}^{2\cdot 2^*} - \mu\int_{\partial\Omega} G(x, H^{-1}(v_n))d\sigma_x\\
&= \frac{1}{2}|w_n|_2^2 + \frac{1}{2}|H^{-1}(w_n)|_2^2 + \frac{\lambda}{q}|H^{-1}(w_n)|_q^q\\
&\quad - \frac{1}{2\cdot 2^*}\Big(|H^{-1}(w)|_{2\cdot 2^*}^{2\cdot 2^*} + |H^{-1}(v_n)|_{2\cdot 2^*}^{2\cdot 2^*}\Big) - \mu\int_{\partial\Omega} G(x, H^{-1}(w_n))d\sigma_x + o_n(1),
\end{aligned}$$

where we used (2.3) and Lemma 2.1 to ensure the following convergences:

$$\begin{gathered}
\int_{\partial\Omega} G(x, H^{-1}(v_n))d\sigma_x = o_n(1), \quad |H^{-1}(v_n)|_q = o_n(1),\\
|H^{-1}(v_n)|_2 = o_n(1), \quad |H^{-1}(w_n)|_q = |H^{-1}(w)|_q + o_n(1),\\
\int_{\partial\Omega} G(x, H^{-1}(w_n))d\sigma_x = \int_{\partial\Omega} G(x, H^{-1}(w))d\sigma_x + o_n(1).
\end{gathered}$$

Therefore, by Proposition 2.5 and (2.5), it holds

$$\begin{aligned}
I_{\lambda,\mu}(v_n) &\le \frac{1}{2}|\nabla w_n|_2^2 + \frac{1}{2}|H^{-1}(w_n)|_2^2 + \frac{\lambda}{q}|H^{-1}(w_n)|_q^q - \frac{1}{2\cdot 2^*}|H^{-1}(w_n)|_{2\cdot 2^*}^{2\cdot 2^*}\\
&\quad - \mu\int_{\partial\Omega} G(x, H^{-1}(w_n))d\sigma_x + o_n(1)\\
&= I_{\lambda,\mu}(w_n) + o_n(1) = c.
\end{aligned} \tag{2.6}$$

Now, applying Proposition 2.5 and definition of $(PS)_c$ sequence one more time, we obtain from convergences in (2.3) and from (2.4) that

$$\begin{aligned}
o_n(1) &= I'_{\lambda,\mu}(w_n)w_n - 2\int_\Omega \nabla w_n \nabla w dx + 2|\nabla w|_2^2 - I'_{\lambda,\mu}(w)w\\
&= |\nabla w_n|_2^2 - 2\int_\Omega \nabla w_n\nabla w dx + |\nabla w|_2^2 + o_n(1)\\
&\quad - \int_\Omega |H^{-1}(w_n)|^{2\cdot 2^*-2}H^{-1}(w_n)(H^{-1})'(w_n)w_n dx\\
&\quad + \int_\Omega |H^{-1}(w)|^{2\cdot 2^*-2}H^{-1}(w)(H^{-1})'(w)w dx\\
&\quad - \mu\int_{\partial\Omega} g(x, H^{-1}(w_n))(H^{-1})'(w_n)w_n d\sigma_x\\
&\quad + \mu\int_{\partial\Omega} g(x, H^{-1}(w))(H^{-1})'(w)w d\sigma_x\\
&= |\nabla v_n|_2^2 - A_n + o_n(1),
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned} A_n := &\int_\Omega |H^{-1}(w_n)|^{2\cdot 2^*-2}H^{-1}(w_n)(H^{-1})'(w_n)w_n d\sigma_x \\ &- \int_\Omega |H^{-1}(w)|^{2\cdot 2^*-2}H^{-1}(w)(H^{-1})'(w)w d\sigma_x. \end{aligned}$$

It follows from Proposition 2.5 and property (5) that we can also prove the equality

$$A_n = \int_\Omega |H^{-1}(v_n)|^{2\cdot 2^*-2}H^{-1}(v_n)(H^{-1})'(v_n)v_n dx + o_n(1).$$

Thus, (2.7) yields

$$o_n(1) = |\nabla v_n|_2^2 - \int_\Omega |H^{-1}(v_n)|^{2\cdot 2^*-2}H^{-1}(v_n)(H^{-1})'(v_n)v_n dx.$$

Since both sequence $(|\nabla v_n|_2^2)$ and $\big(\int_\Omega |H^{-1}(v_n)|^{2\cdot 2^*-2}H^{-1}(v_n)(H^{-1})'(v_n)v_n dx\big)$ are bounded, let us suppose that

$$|\nabla v_n|_2^2 \to d \text{ and } \int_\Omega |H^{-1}(v_n)|^{2\cdot 2^*-2}H^{-1}(v_n)(H^{-1})'(v_n)v_n dx \to d$$

as $n \to +\infty$. By property (7),

$$|H^{-1}(v_n)|_{2\cdot 2^*}^2 \le (2/h_\infty)|v_n|_{2^*},$$

and from Sobolev embedding and property (10),

$$\begin{aligned} \Big(\int_\Omega |H^{-1}(v_n)|^{2\cdot 2^*-2}H^{-1}(v_n)(H^{-1})'(v_n)v_n dx\Big)^{2/2^*} &\le \Big(\int_\Omega |H^{-1}(v_n)|^{2\cdot 2^*} dx\Big)^{2/2^*} \\ &= |H^{-1}(v_n)|_{2\cdot 2^*}^4 \\ &\le (2/h_\infty)^2 |v_n|_{2^*}^2 \\ &\le \frac{4}{Sh_\infty}|\nabla v_n|_2^2. \end{aligned}$$

Then, taking $n \to +\infty$, one obtains

$$Sh_\infty d^{2/2^*} \le 4d.$$

Now, suppose by contradiction that $d \neq 0$. This implies $4^{-N/2}(Sh_\infty)^{\frac{N}{2}} \le d$. On the other hand, since $I_{\lambda,\mu}(v_n) = \frac{1}{2}|\nabla v_n|_2^2 - \frac{1}{2\cdot 2^*}|H^{-1}(v_n)|_{2\cdot 2^*}^{2\cdot 2^*} + o_n(1)$, it follows from property (10) that

$$\begin{aligned} &\frac{1}{2}|\nabla v_n|_2^2 - \frac{1}{2^*}\int_\Omega |H^{-1}(v_n)|^{2\cdot 2^*-2}H^{-1}(v_n)(H^{-1})'(v_n)v_n dx \\ &\le \frac{1}{2}|\nabla v_n|_2^2 - \frac{1}{2\cdot 2^*}|H^{-1}(v_n)|_{2\cdot 2^*}^{2\cdot 2^*} \\ &= I_{\lambda,\mu}(v_n) + o_n(1), \end{aligned}$$

which, passing to a subsequence if necessary, by (2.6), produces

$$d\Big(\frac{1}{2} - \frac{1}{2^*}\Big) \le c.$$

Hence,

$$\frac{4^{-N/2}(Sh_\infty)^{N/2}}{N} \le d\big(\frac{1}{2} - \frac{1}{2^*}\big) \le c < \frac{4^{-N/2}(Sh_\infty)^{N/2}}{N},$$

what is clearly an absurd. Necessarily, $d = 0$ and, then, $w_n \to w$ strongly in $H^1(\Omega)$, as we wished to prove. □

3. Existence of two solutions

Consider $I_\lambda^\pm : H^1(\Omega) \to \mathbb{R}$ the C^1-functional defined by

$$
\begin{aligned}
I_{\lambda,\mu}^\pm(w) &= \frac{1}{2}\int_\Omega (|\nabla w|^2 + |H^{-1}(w)^\pm|^2)dx + \frac{\lambda}{q}\int_\Omega (H^{-1}(w)^\pm)^q dx \\
&\quad - \frac{1}{2\cdot 2^*}\int_\Omega |H^{-1}(w)^\pm|^{2\cdot 2^*}(w)dx - \mu\int_{\partial\Omega} G(x, H^{-1}(w)^\pm)d\sigma_x,
\end{aligned}
$$

where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. Suppose that $w \in H^1(\Omega)$ satisfies $(I_{\lambda,\mu}^\pm)'(w) = 0$. Since $H^{-1}(s)$ has the same sign of s, we have

$$
\begin{aligned}
0 &= (I_{\lambda,\mu}^\pm)'(w)H^{-1}(w^\mp)h(H^{-1}(w^\mp)) \\
&= \int_\Omega \Big[\Big(1 + \frac{H^{-1}(w^\mp)h'(H^{-1}(w^\mp))}{h(H^{-1}(w^\mp))}\Big)|\nabla w^\mp|^2 + |H^{-1}(w^\mp)|^2\Big)\Big]dx,
\end{aligned}
$$

that is, $w^\mp = 0$. This shows that every critical point of $I_{\lambda,\mu}^+$ is non-negative and every critical point of $I_{\lambda,\mu}^-$ is non-positive. Therefore, they both are critical points of $I_{\lambda,\mu}$ as well.

To find solutions, we will consider a type of Nehari set defined by

$$
\mathcal{N}^\pm = \big\{w \in H^1(\Omega)\setminus\{0\} : (I_{\lambda,\mu}^\pm)'(w)H^{-1}(w)h(H^{-1}(w)) = 0\big\}.
$$

Every nontrivial critical point of $I_{\lambda,\mu}^\pm$ is contained in $\mathcal{N}^\pm$.

For simplicity, we prove all results taking in account the functional $I_{\lambda,\mu}$ instead of $I_{\lambda,\mu}^+$ and $I_{\lambda,\mu}^-$ because all the calculations are exactly the same in the three cases: $I_{\lambda,\mu}$, $I_{\lambda,\mu}^+$ and $I_{\lambda,\mu}^-$. We mean that, in the sequel, finding a critical point of $I_{\lambda,\mu}$, we prove simultaneously that also $I_{\lambda,\mu}^+$ and $I_{\lambda,\mu}^-$ possess critical points.

Henceforth,

$$
\mathcal{N} = \{w \in H^1(\Omega)\setminus\{0\};\ I'_{\lambda,\mu}(w)H^{-1}(w)h(H^{-1}(w)) = 0\}
$$

and

$$
\begin{aligned}
&I'_{\lambda,\mu}(w)H^{-1}(w)h(H^{-1}(w)) \\
&= \int_\Omega \Big(1 + \frac{H^{-1}(w)h'(H^{-1}(w))}{h(H^{-1}(w))}\Big)|\nabla w|^2 dx + \int_\Omega |H^{-1}(w)|^2 dx \\
&\quad + \lambda\int_\Omega |H^{-1}(w)|^q dx - \int_\Omega |H^{-1}(w)|^{2\cdot 2^*}dx \\
&\quad + \mu\int_{\partial\Omega} g(x, H^{-1}(w))H^{-1}(w)d\sigma_x.
\end{aligned} \tag{3.1}
$$

Lemma 3.1. *If $w \in H^1(\Omega)\setminus\{0\}$, with $w \ge 0$, there exists $t_w = t_{\lambda,\mu}(w) > 0$ such that $t_w w \in \mathcal{N}$. In particular, $\mathcal{N} \neq \emptyset$.*

Proof. Consider the continuous function $\xi(t) := I'_{\lambda,\mu}(tw)H^{-1}(tw)h(H^{-1}(tw))$, $t > 0$. From (3.1) we have

$$
\xi(t) \ge t^2\Big[\int_\Omega |\nabla w|^2 dx - \frac{1}{t^2}\int_\Omega |H^{-1}(tw)|^{2\cdot 2^*}dx - \frac{\mu}{t^2}\int_{\partial\Omega} g(x, H^{-1}(tw))H^{-1}(tw)d\sigma_x\Big].
$$

Property (4) ensures that $\frac{1}{t^2}\int_\Omega |H^{-1}(tw)|^{2\cdot 2^*}dx \to 0$ as $t \to 0^+$ and hypothesis (A6) guarantees that $\frac{\mu}{t^2}\int_{\partial\Omega} g(x, H^{-1}(tw))H^{-1}(tw)d\sigma_x \to 0$ as $t \to 0^+$. Therefore,

$$
\xi(t) > 0 \quad \text{for } t > 0 \text{ small enough.} \tag{3.2}
$$

On the other hand, from (A5) and (A2),

$$
\begin{aligned}
\xi(t) = t^2\Big[&\int_\Omega\Big(1+\frac{H^{-1}(tw)h'(H^{-1}(tw))}{h(H^{-1}(tw))}\Big)|\nabla w|^2dx \\
&+\frac{1}{t^2}\int_\Omega|H^{-1}(tw)|^2dx+\frac{\lambda}{t^{2-q/2}}\int_\Omega\frac{|H^{-1}(tw)|^q}{t^{q/2}}dx \\
&-t^{2^*-2}\int_\Omega\frac{|H^{-1}(tw)|^{2\cdot 2^*}}{t^{2^*}}dx-\frac{\mu}{t^2}\int_{\partial\Omega}g(x,H^{-1}(tw))H^{-1}(tw)d\sigma_x\Big] \\
\le t^2\Big[&2\int_\Omega|\nabla w|^2dx+\frac{1}{t^2}\int_\Omega|H^{-1}(tw)|^2dx \\
&+\frac{\lambda}{t^{2-q/2}}\int_\Omega\frac{|H^{-1}(tw)|^q}{t^{q/2}}dx-t^{2^*-2}\int_\Omega\frac{|H^{-1}(tw)|^{2\cdot 2^*}}{t^{2^*}}dx\Big].
\end{aligned}
\tag{3.3}
$$

By property (8), we obtain the following three convergences:

$$
\begin{gathered}
\frac{\lambda}{t^{2-q/2}}\int_\Omega\frac{|H^{-1}(tw)|^q}{t^{q/2}}dx\to 0,\\
\int_\Omega\frac{|H^{-1}(tw)|^{2\cdot 2^*}}{t^{2^*}}dx\to(2/h_\infty)^{2^*}\int_\Omega|w|^{2^*}dx>0,\\
\frac{1}{t^2}\int_\Omega|H^{-1}(tw)|^2dx\to 0
\end{gathered}
$$

as $t\to+\infty$ since $q<4$. These convergences applied in (3.3) yield

$$
\xi(t)<0 \tag{3.4}
$$

for values of $t>0$ large enough. Since ξ is a continuous function, from (3.2) and (3.4), there exists at least one $t_w>0$ such that $\xi(t_w)=0$, that is, $t_w w\in\mathcal{N}$, and the lemma is proved. □

Remark 3.2. In the case of $I^-_{\lambda,\mu}$ in the previous lemma, we consider $w\le 0$ instead of $w\ge 0$.

Lemma 3.3. *The set $\mathcal{N}$ is a C^1 manifold.*

Proof. Define $J_{\lambda,\mu}(w):=I'_{\lambda,\mu}(w)h(H^{-1}(w))H^{-1}(w)$ and let $w\in\mathcal{N}$. A direct calculation gives us

$$
\begin{aligned}
&\frac{d}{ds}\Big(1+\frac{H^{-1}(s)h'(H^{-1}(s))}{h(H^{-1}(s))}\Big)\\
&=\frac{1}{h^2(H^{-1}(s))}\Big(h'(H^{-1}(s)+H^{-1}(s)h''(H^{-1}(s)))-\frac{H^{-1}(s)(h'(H^{-1}(s)))^2}{h(H^{-1}(s))}\Big),
\end{aligned}
$$

and then, by (A8),

$$
\begin{aligned}
&J'_{\lambda,\mu}(w)h(H^{-1}(w))H^{-1}(w)\\
&=\int_\Omega\frac{H^{-1}(w)h'(H^{-1}(w))}{h(H^{-1}(w))}|\nabla w|^2dx+\int_\Omega\frac{|H^{-1}(w)|^2h''(H^{-1}(w))}{h(H^{-1}(w))}|\nabla w|^2dx\\
&\quad-\int_\Omega\frac{|H^{-1}(w)|^2(h'(H^{-1}(w)))^2}{h^2(H^{-1}(w))}|\nabla w|^2dx\\
&\quad+2\int_\Omega\Big(1+\frac{H^{-1}(w)h'(H^{-1}(w))}{h(H^{-1}(w))}\Big)^2|\nabla w|^2dx+2\int_\Omega|H^{-1}(w)|^2dx
\end{aligned}
$$

$$
\begin{aligned}
&+\lambda q\int_\Omega |H^{-1}(w)|^q dx-2\cdot 2^*\int_\Omega |H^{-1}(w)|^{2\cdot 2^*}dx\\
&-\mu\int_{\partial\Omega}(g'(x,H^{-1}(w))H^{-1}(w)^2+g(x,H^{-1}(w))H^{-1}(w))d\sigma_x\\
=&\, 2\int_\Omega\Big(1+\frac{H^{-1}(w)h'(H^{-1}(w))}{h(H^{-1}(w))}\Big)|\nabla w|^2dx\\
&+2\int_\Omega \frac{H^{-1}(w)h'(H^{-1}(w))}{h(H^{-1}(w))}|\nabla w|^2dx+\int_\Omega\Big(\frac{H^{-1}(w)h'(H^{-1}(w))}{h(H^{-1}(w))}\\
&+\frac{|H^{-1}(w)|^2h''(H^{-1}(w))}{h(H^{-1}(w))}+\frac{[H^{-1}(w)]^2(h'(H^{-1}(w)))^2}{h^2(H^{-1}(w))}\Big)|\nabla w|^2dx\\
&+2\int_\Omega |H^{-1}(w)|^2dx+\lambda q\int_\Omega |H^{-1}(w)|^q dx-2\cdot 2^*\int_\Omega |H^{-1}(w)|^{2\cdot 2^*}dx\\
&-\mu\int_{\partial\Omega}(g'(x,H^{-1}(w))H^{-1}(w)^2+g(x,H^{-1}(w))H^{-1}(w))d\sigma_x.
\end{aligned}
$$

Applying firstly hypothesis (A3) (see Remark 1.1) and after using (A2), we obtain

$$
\begin{aligned}
&J'_{\lambda,\mu}(w)h(H^{-1}(w))H^{-1}(w)\\
&\le 4\int_\Omega\Big(1+\frac{H^{-1}(w)h'(H^{-1}(w))}{h(H^{-1}(w))}\Big)|\nabla w|^2dx+2\int_\Omega |H^{-1}(w)|^2dx\\
&\quad+\lambda q\int_\Omega |H^{-1}(w)|^q dx-2\cdot 2^*\int_\Omega |H^{-1}(w)|^{2\cdot 2^*}dx\\
&\quad-\mu\int_{\partial\Omega}(g'(x,H^{-1}(w))H^{-1}(w)^2+g(x,H^{-1}(w))H^{-1}(w))d\sigma_x.
\end{aligned}
$$

Since $w\in\mathcal{N}$, it follows that

$$
\begin{aligned}
&4\int_\Omega\Big(1+\frac{H^{-1}(w)h'(H^{-1}(w))}{h(H^{-1}(w))}\Big)|\nabla w|^2dx\\
&=-4\int_\Omega |H^{-1}(w)|^2dx-4\lambda\int_\Omega |H^{-1}(w)|^q dx\qquad +4\int_\Omega |H^{-1}(w)|^{2\cdot 2^*}dx,\\
&\quad+4\mu\int_{\partial\Omega} g(x,H^{-1}(w))H^{-1}(w)d\sigma_x
\end{aligned}
$$

and, once $q-4<0$, one obtains from assumption (A7) that

$$
\begin{aligned}
&J'_{\lambda,\mu}(w)h(H^{-1}(w))H^{-1}(w)\\
&\le -2\int_\Omega |H^{-1}(w)|^2dx+\lambda(q-4)\int_\Omega |H^{-1}(w)|^q dx+(4-2\cdot 2^*)\int_\Omega |H^{-1}(w)|^{2\cdot 2^*}dx\\
&\quad-\mu\int_{\partial\Omega}(g'(x,H^{-1}(w))H^{-1}(w)^2-3g(x,H^{-1}(w))H^{-1}(w))d\sigma_x\\
&<(4-2\cdot 2^*)\int_\Omega |H^{-1}(w)|^{2\cdot 2^*}dx<0.
\end{aligned}
$$

Hence, $J'_\lambda(w)\neq 0$ for all $w\in\mathcal{N}$, what proves that $\mathcal{N}$ is a C^1 manifold and completes the proof. □

Lemma 3.4. *Let (w_n) be a sequence such that $w_n\in\mathcal{N}$ and $I_{\lambda,\mu}(w_n)\to c$, as $n\to+\infty$. Then (w_n) is bounded.*

Proof. Firstly, we claim that the sequence $(H^{-1}(w_n)) \subset H^1(\Omega)$ is bounded. Indeed, consider the sequence (φ_n) defined by $\varphi_n = H^{-1}(w_n)h(H^{-1}(w_n))$, observe that by (5), we have $|\varphi_n|_2^2 \le 4|w_n|_2^2$ for all $n \ge 1$.

Since by property (2), $\frac{d}{dt}\big(H^{-1}(t)\big)h(H^{-1}(t)) = 1$ for all $t \in \mathbb{R}$, we obtain

$$\nabla\varphi_n = \frac{d}{dt}[H^{-1}(t)h(H^{-1}(t))]\Big|_{t=w_n}\nabla w_n = \Big(1 + \frac{H^{-1}(w_n)h'(H^{-1}(w_n))}{h(H^{-1}(w_n))}\Big)\nabla w_n.$$

Therefore,

$$|\nabla\varphi_n| = \Big(1 + \frac{H^{-1}(w_n)h'(H^{-1}(w_n))}{h(H^{-1}(w_n))}\Big)|\nabla w_n| \le (1+\beta)|\nabla w_n|,$$

where we used (1.5), choosing $t = H^{-1}(w_n)$. Thus, $\varphi_n \in H^1(\Omega)$ with $\|\varphi_n\| \le C\|w_n\|$ for some $C > 0$.

Recalling that $(w_n) \subset \mathcal{N}$, i.e., $I'_{\lambda,\mu}(w_n)\varphi_n = 0$, we have

$$\begin{aligned}
&c + o_n(1)\\
&\ge I_{\lambda,\mu}(w_n) - \sigma I'_{\lambda,\mu}(w_n)\varphi_n\\
&\ge \int_\Omega \big(\frac{1}{2} - \sigma(1+\beta)\big)|\nabla w_n|^2dx + \big(\frac{1}{2} - \sigma\big)\int_\Omega |H^{-1}(w_n)|^2dx\\
&\quad + \lambda\big(\frac{1}{q} - \sigma\big)\int_\Omega |H^{-1}(w_n)|^q dx + \big(\sigma - \frac{1}{2\cdot 2^*}\big)\int_\Omega |H^{-1}(w_n)|^{2\cdot 2^*}dx\\
&\quad + \int_{\partial\Omega}\big(\sigma g(x,H^{-1}(w_n))H^{-1}(w_n) - G(x,H^{-1}(w_n))\big)\,d\sigma_x,
\end{aligned} \tag{3.5}$$

where (1.5) was used. By hypothesis (A5), it follows that

$$\int_\Omega \big(\frac{1}{2} - \sigma(1+\beta)\big)|\nabla w_n|^2dx + \big(\frac{1}{2} - \sigma\big)\int_\Omega |H^{-1}(w_n)|^2dx \le c + o_n(1). \tag{3.6}$$

Suppose by contradiction that, up to a subsequence, $\|w_n\| \to +\infty$ as $n \to +\infty$ and consider $v_n := \frac{w_n}{\|w_n\|}$. Since $\|v_n\| = 1$, by the Sobolev embedding, $v_n \to v$ strongly in $L^2(\Omega)$. From (3.6) and hypothesis (A2), we have

$$\int_\Omega |\nabla v_n|^2dx \le o_n(1).$$

Since $1 = \|v_n\|^2 = \int_\Omega |\nabla v_n|^2dx + \int_\Omega v_n^2dx$, one has $\int_\Omega v^2dx = 1$ and therefore $v \neq 0$.

Dividing (3.6) by $\|w_n\|$, and using (A5) and (A2), one obtains

$$o_n(1) \ge \int_\Omega \frac{|H^{-1}(w_n)|^2}{\|w_n\|}dx = \int_\Omega \Big(\frac{H^{-1}(v_n\|w_n\|)}{|v_n|^{1/2}\|w_n\|^{1/2}}\Big)^2|v_n|dx.$$

By property (8) and noting that $v \neq 0$ and $\|w_n\| \to +\infty$ as $n \to +\infty$ in a subset Ω_0 of Ω of positive measure, we obtain

$$0 \ge \liminf_{n\to+\infty}\int_\Omega \Big(\frac{H^{-1}(v_n\|w_n\|)}{|v_n|^{1/2}\|w_n\|^{1/2}}\Big)^2|v_n|dx \ge \int_{\Omega_0}\frac{2}{h_\infty}|v|dx > 0.$$

This contradiction shows that (w_n) is bounded in $H^1(\Omega)$. □

Let us define

$$m_{\lambda,\mu} = \inf_{\mathcal{N}} I_{\lambda,\mu}(w). \tag{3.7}$$

The next result will provide a positive lower bound for the function defined by $\Psi(w) = |\nabla w|_2^2 + |H^{-1}(w)|_2^2$ for $w \in \mathcal{N}$, and consequently a positive lower bound for $m_{\lambda,\mu}$. Its proof depends on the essential role played by the linear term u in problem (1.3), which is responsible for the term $|H^{-1}(w)|_2^2$ in the definition of the functional $I_{\lambda,\mu}$.

Lemma 3.5. *There exists a positive constant $c_0 > 0$, which does not depend on λ but does on μ, such that $|\nabla w|_2^2 + |H^{-1}(w)|_2^2 \geq c_0$ for all $w \in \mathcal{N}$. Furthermore, it holds that $m_{\lambda,\mu} > c_1 > 0$ for some $c_1 > 0$.*

Proof. We have

$$\begin{aligned} &\int_\Omega \Big(1 + \frac{H^{-1}(w)h'(H^{-1}(w))}{h(H^{-1}(w))}\Big)|\nabla w|^2 dx + \int_\Omega |H^{-1}(w)|^2 dx + \lambda \int_\Omega |H^{-1}(w)|^q dx \\ &= \int_\Omega |H^{-1}(w)|^{2\cdot 2^*} dx + \mu \int_{\partial\Omega} g(x, H^{-1}(w))H^{-1}(w) d\sigma_x \end{aligned}$$

that yields (since $sh'(s) \geq 0$ for all $s \in \mathbb{R}$)

$$\Psi(w) \leq \int_\Omega |H^{-1}(w)|^{2\cdot 2^*} dx + \mu \int_{\partial\Omega} g(x, H^{-1}(w))H^{-1}(w) d\sigma_x. \tag{3.8}$$

From Lemma 2.1 with $p = 2 \cdot 2_*$, for all $\varepsilon > 0$, there is a positive constant $C_\varepsilon > 0$ such that

$$\Psi(w) \leq |H^{-1}(w)|_{2\cdot 2^*}^{2\cdot 2^*} + \mu\varepsilon \int_{\partial\Omega} |H^{-1}(w)|^4 d\sigma_x + \mu C_\varepsilon \int_{\partial\Omega} |H^{-1}(w)|^{2\cdot 2_*} d\sigma_x. \tag{3.9}$$

By the trace Sobolev embeddings $H^1(\Omega) \hookrightarrow L^4(\partial\Omega)$, for $2 \leq 4 \leq 2_* = 4$, and it follows from property (2) that

$$\begin{aligned} \int_{\partial\Omega} |H^{-1}(w)|^4 d\sigma_x &\leq C\Big(\int_\Omega (|\nabla H^{-1}(w)|^2 + |H^{-1}(w)|^2) dx\Big)^2 \\ &= C\Big[\int_\Omega \Big(\frac{1}{h^2(H^{-1}(w))}|\nabla w|^2 + |H^{-1}(w)|^2\Big) dx\Big]^2 \\ &\leq C\Big(\int_\Omega (|\nabla w|^2 + |H^{-1}(w)|^2) dx\Big)^2 \\ &= C\Psi(w)^2. \end{aligned} \tag{3.10}$$

Finally, the trace Sobolev embeddings one more time, now applied to $(H^{-1}(w))^2$, together with property (H_2), produce

$$\begin{aligned} \int_{\partial\Omega} |H^{-1}(w)|^{2\cdot 2_*} d\sigma_x &\leq C\Big(\int_\Omega (|\nabla (H^{-1}(w))^2|^2 + |H^{-1}(w)|^4) dx\Big)^{2_*/2} \\ &\leq C\Big(\int_\Omega \frac{4|H^{-1}(w)|^2}{h^2(H^{-1}(w))}|\nabla w|^2 + |H^{-1}(w)|^4) dx\Big)^{2_*/2} \\ &\leq C\Big(\int_\Omega (|\nabla w|^2 + |H^{-1}(w)|^4) dx\Big)^{2_*/2} \\ &\leq C\Psi(w)^{2_*/2} + C\Big(\int_\Omega |H^{-1}(w)|^4) dx\Big)^{2_*/2} \\ &\leq C\Psi(w)^{2_*/2} + C\Psi(w)^{2_*}, \end{aligned} \tag{3.11}$$

where, in the last inequality, we used the same calculations as in (3.10), and applied the Sobolev embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ for $2 \leq 4 \leq 2^* = 6$. The same arguments also show that

$$\int_\Omega |H^{-1}(w)|^{2\cdot 2^*}dx \le C\Psi(w)^{2^*/2} + C\Psi(w)^{2^*}. \tag{3.12}$$

Now, using (3.10), (3.11) and (3.12) in (3.9), it follows that

$$\Psi(w) \le C\Big(\Psi(w)^{2^*/2} + \Psi(w)^{2^*} + \Psi(w)^2 + \Psi(w)^{2_*/2} + \Psi(w)^{2_*}\Big).$$

Since $\Psi(w) > 0$ and $2^*/2$, $2_*/2$, 2^* and 2_* are bigger than 1, necessarily, there exists a positive constant $c_0 > 0$ such that

$$\Psi(w) \ge c_0 > 0$$

and we prove the first part of this result. The second part may be obtained following the calculation in (3.5) of Lemma 3.4 and by using the first part of this lemma. □

Remark 3.6. Here is the point that we highlight the reason for having fixed the dimension of the Euclidean space in $N = 3$. In the previous result, we need to relate the term $|H^{-1}(w)|_4^4$ with the gradient norm $|\nabla w|_2^2$, for $w \in H^1(\Omega)$, to get the positive lower bound for $\Psi(w)$. However, since Ω is a bounded domain in $\mathbb{R}^N$ and the space $H^1(\Omega)$ contains functions that are not zero on $\partial\Omega$, every embedding theorem brings up the norm of w in $L^2(\Omega)$, which does not compare with the gradient norm. Thus, we use the trace Sobolev embedding to deal with $\int_{\partial\Omega} |H^{-1}(w)|^4 d\sigma_x$ and then we need that $2 \le 4 \le 2\cdot 2_*$, what implies $N = 3$.

Lemma 3.7. *Let (w_n) be a $(PS)_c$ sequence for $I_{\lambda,\mu}|_{\mathcal{N}}$ restrict to the set $\mathcal{N}$. Then $I'_{\lambda,\mu}(w_n) \to 0$ as $n \to +\infty$ in the dual space $\big(H^1(\Omega)\big)^*$.*

Proof. Let $\varphi_n = H^{-1}(w_n)h(H^{-1}(w_n))$ as in (3.5). We claim that the sequence $\big(\int_\Omega |H^{-1}(w_n)|^{2\cdot 2^*}dx\big)$ does not converge to zero as $n \to +\infty$. Otherwise, once we have $I'_{\lambda,\mu}(w_n)\varphi_n = 0$ and the growth of g is subcritical, by Hölder inequality and since $|\Omega| < \infty$, one obtains

$$\int_\Omega \Big(1 + \frac{H^{-1}(w_n)h'(H^{-1}(w_n))}{h(H^{-1}(w_n))}\Big)|\nabla w_n|^2 dx + o_n(1) = \int_\Omega |H^{-1}(w_n)|^{2\cdot 2^*}dx = o_n(1).$$

Therefore, by Lemma 3.5,

$$\begin{aligned}
0 < c_0 &\le \Psi(w_n)\\
&= \int_\Omega |\nabla w_n|^2 dx + o_n(1)\\
&\le \int_\Omega \Big(1 + \frac{H^{-1}(w_n)h'(H^{-1}(w_n))}{h(H^{-1}(w_n))}\Big)|\nabla w_n|^2 dx = o_n(1),
\end{aligned}$$

which is a contradiction. Hence, there exists $C > 0$ such that

$$\int_\Omega |H^{-1}(w_n)|^{2\cdot 2^*}dx \ge C > 0.$$

This and Lemma 3.3 imply that the sequence $(J'_\lambda, \mu(w_n)\varphi_n)$ does not converge to zero as $n \to +\infty$. The next arguments are standard and the lemma follows. □

Despite being a minimizing sequence in $\mathcal{N}$ for functional $I_{\lambda,\mu}$, it may not be a sequence that converges weakly to a solution of problem (1.8). In the next result, we will show the existence of an appropriate minimizing sequence for our purpose.

Proposition 3.8. *Let $m_{\lambda,\mu}$ as in* (3.7)*. There exists a bounded $(PS)_{m_{\lambda,\mu}}$ sequence $(w_n) \subset \mathcal{N}$ for functional $I_{\lambda,\mu}$.*

Proof. Note that the functional $J_{\lambda,\mu}(w) := I'_{\lambda,\mu}(w)H^{-1}(w)h(H^{-1}(w))$ belongs to the space $C(H^1(\Omega), \mathbb{R})$, whence $\mathcal{N}$ is a complete metric subspace. By Lemma 3.5, $I_{\lambda,\mu}$ is bounded from below on $\mathcal{N}$ and $I_{\lambda,\mu}$ is a C^1-functional, hence we may apply Ekeland's Principle to ensure the existence of a $(PS)_{m_{\lambda,\mu}}$ sequence $(w_n) \subset \mathcal{N}$ for functional $I_{\lambda,\mu}|_{\mathcal{N}}$. Finally, by Lemma 3.4, it is bounded and, by Lemma 3.7, it is a $(PS)_{m_{\lambda,\mu}}$ sequence in the whole space $H^1(\Omega)$. □

Lemma 3.9. *There exists $\mu^* > 0$ such that, for all $\mu \in [\mu^*, +\infty)$ and for λ in a bounded set, it holds that $m_{\lambda,\mu} < 4^{-N/2}(Sh_\infty)^{N/2}/N$.*

Proof. Let $w \in H^1(\Omega) \setminus \{0\}$, $w \geq 0$ (in the case of $I^-_{\lambda,\mu}$, we choose $w \leq 0$), and consider $t_{\lambda,\mu} > 0$ given by Lemma 3.1, which shows that $t_{\lambda,\mu}w \in \mathcal{N}$. Then, from hypothesis (A2), we have

$$\begin{aligned} &2t^2_{\lambda,\mu}\int_\Omega |\nabla w|^2 dx + \int_\Omega |H^{-1}(t_\lambda, \mu w)|^2 dx + \lambda \int_\Omega |H^{-1}(t_{\lambda,\mu}w)|^q dx \\ &\geq \int_\Omega |H^{-1}(t_{\lambda,\mu}w)|^{2\cdot 2^*} dx + \mu \int_{\partial\Omega} g(x, H^{-1}(t_{\lambda,\mu}w))H^{-1}(t_{\lambda,\mu}w) d\sigma_x, \end{aligned} \tag{3.13}$$

which implies from property (3), and by assumption (A5) that

$$C(t^2_{\lambda,\mu} + t^{q/2}_{\lambda,\mu}) \geq \int_\Omega |H^{-1}(t_{\lambda,\mu}w)|^{2\cdot 2^*} dx,$$

for some $C > 0$ which does not depend on μ and λ in a bounded set. We claim that $(t_{\lambda,\mu})_{\mu\geq 1}$ is bounded as $\mu \to +\infty$. Otherwise, it follows that

$$\begin{aligned} C\Big(1 + \frac{1}{t^{2-q/2}_{\lambda,\mu}}\Big) &\geq \int_\Omega \frac{1}{t^2_{\lambda,\mu}} |H^{-1}(t_{\lambda,\mu}w)|^{2\cdot 2^*} dx \\ &= \int_\Omega \frac{|H^{-1}(t_{\lambda,\mu}w)|^4}{t^2_{\lambda,\mu}} |H^{-1}(t_{\lambda,\mu}w)|^{2\cdot 2^*-4} dx. \end{aligned}$$

But this is an absurd in view of properties $(H_8) - (H_9)$ and $q/2 < 2$. So, let $t_0 \geq 0$ be such that $t_{\lambda,\mu} \to t_0$ as $\mu \to +\infty$ uniformly on λ in a bounded set. This implies the following boundedness

$$2t^2_{\lambda,\mu}\int_\Omega |\nabla w|^2 dx + \int_\Omega |H^{-1}(t_{\lambda,\mu}w)|^2 dx + \lambda \int_\Omega |H^{-1}(t_{\lambda,\mu}w)|^q dx \leq C$$

for all $\mu > 0$ and for some constant $C > 0$, whence by (3.13)

$$\mu \int_{\partial\Omega} g(x, H^{-1}(t_{\lambda,\mu}w))H^{-1}(t_{\lambda,\mu}w) d\sigma_x \leq C,$$

for all $\mu > 0$ and λ in a bounded set. By assumptions (A4) and (A5), this implies, necessarily, that $t_0 = 0$. Therefore

$$m_{\lambda,\mu} \leq I_{\lambda,\mu}(t_{\lambda,\mu}w) \leq \frac{t^2_{\lambda,\mu}}{2}|\nabla w|^2_2 + \frac{1}{2}|H^{-1}(t_\lambda, \mu w)|^2_2 + \frac{\lambda}{q}|H^{-1}(t_{\lambda,\mu}w)|^q_q \to 0 \tag{3.14}$$

as $\mu \to +\infty$ and λ is in a bounded set. The lemma follows choosing μ sufficiently large. □

Proof of Theorem 1.5. By Lemma 3.9, we take $\mu > 0$ sufficiently large such that $m_{\lambda,\mu} < \frac{4^{-N/2}(Sh_\infty)^{N/2}}{N}$ and from Proposition 3.8, there is a $(PS)_{m_{\lambda,\mu}}$ sequence (w_n) for functional $I_{\lambda,\mu}$, which converges strongly to $w_{\lambda,\mu} \in H^1(\Omega)$ in view of Lemma 2.6. By Lemma 3.5, $I_{\lambda,\mu}(w) = \lim_{n\to+\infty} I_{\lambda,\mu}(w_n) \geq m_{\lambda,\mu} > 0$ and, consequently, $w_{\lambda,\mu} \neq 0$. Since $I'_\lambda, \mu(w_\lambda, \mu) = 0$, then $w_{\lambda,\mu} \in \mathcal{N}$ is a nontrivial ground state solution of problem (1.8). □

Remark 3.10. We observe that any ground state solution w_λ, μ obtained in Theorem 1.5 as a minimum on this new natural constraint $\mathcal{N}$ is always a signed solution since $w^+_{\lambda,\mu}$ and $w^-_{\lambda,\mu}$ belong to $\mathcal{N}$.

Proof of Corollary 1.6. This is a direct consequence of the (3.14) and the fact that both solutions are ground state (for the respective functional). □

Proof of Theorem 1.7. It is an application of [13, Theorem 6.31] and of the remark there subsequent to the theorem. □

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