Universidade de Brasília
Instituto de Ciências Exatas
Departamento de Matemática

Some Caffarelli-Kohn-Nirenberg's type problems in $\mathbb{R}^N$

por

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Brasília

2024

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Tese apresentada ao Departamento de Matemática da Universidade de Brasília como parte dos requisitos necessários para obtenção do grau de

DOUTOR EM MATEMÁTICA

23 de janeiro de 2024

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*O autor foi bolsista CAPES durante a elaboração deste trabalho.

À minha família.

Agradecimentos

Agradeço a Deus por ter me dado forças para chegar até aqui.

Agradeço a minha família pelo apoio pela compreensão quando eu não estava presente com eles por estar me dedicando aos estudos.

Agradeço aos meus amigos da escola pela compreensão da minha ausência durante o doutorado e pela amizade deles.

Agradeço ao meu orientador Giovany Figueiredo pela paciência, dedicação, motivação e por ser um exemplo de profissional para mim.

Agradeço aos professores Rodrigo Rodrigues, Suellen Arruda e Ricardo Ruviaro por aceitarem o convite para comporem a banca de defesa desta tese e pelas sugestões valiosas que contribuíram para a melhoria deste trabalho.

Agradeço aos professores do Departamento de Matemática da UnB que contribuíram muito para a minha formação. Também quero agradecer à professora Sandra Imaculada Moreira Neto pelas excelentes aulas de Análise Funcional e ao professor João Pablo Pinheiro da Silva pelas excelentes aulas de Métodos Variacionais.

Agradeço aos servidores do departamento de matemática da UnB pelos serviços prestados.

Agradeço ao Segundo Manuel pela ajuda com o TeX na elaboração desta tese.

Agradeço aos amigos do Departamento pela ajuda e companheirismo nos estudos das matérias e para os exames de qualificação.

Agradeço a CAPES pelo apoio financeiro.

Resumo

Título: Alguns problemas do tipo Caffarelli-Kohn-Nirenberg em $\mathbb{R}^N$.

Nesse trabalho, provamos alguns resultados referentes a problemas do tipo Caffarelli-Kohn-Nirenberg em $\mathbb{R}^N$.

No primeiro capítulo, provamos a existência de soluções não-triviais com não-linearidades do tipo Berestycki-Lions usando o Teorema do Passo da Montanha, o Princípio Variacional de Ekeland e um resultado de compacidade do tipo Strauss. Mais precisamente, estudaremos a seguinte classe de problemas

$$-\mathrm{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}|u|^{p-2}u = |x|^{-bp^*}h(u), \quad \text{em } \mathbb{R}^N, \tag{PM}$$

e

$$-\mathrm{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = |x|^{-bp^*}f(u), \quad \text{em } \mathbb{R}^N, \tag{ZM}$$

onde $1 < p < N$, $0 \leq a < \frac{N-p}{p}$, $a < b \leq a+1$, $p^* = p^*(a,b) = \frac{pN}{N-dp}$ e $d = 1 + a - b$.

No segundo capítulo, provamos a existência e concentração de soluções ground state para uma classe de problemas subcrítico, crítico ou supercrítico do tipo Caffarelli-Kohn-Nirenberg usando o Teorema do Passo da Montanha e o método de Iteração de Moser. Mais precisamente, estudaremos a seguinte classe de problemas quasilineares

$$-\mathrm{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}[1+\mu V(x)]|u|^{p-2}u = |x|^{-bp^*}[f(u) + \varrho|u|^{\sigma-2}u], \tag{$P_{\mu,\varrho,\sigma}$}$$

em $\mathbb{R}^N$, onde $1 < p < N$, $0 \leq a < \frac{N-p}{p}$, $a < b \leq a+1$, $p^* = p^*(a,b) = \frac{pN}{N-dp}$, $d = 1 + a - b$ e $\mu > 0$.

No terceiro capítulo, provamos as existências de soluções ground state positiva e nodal minimizando o funcional na variedade de Nehari e em um subconjunto da variedade de Nehari para a seguinte classe de problemas do tipo Caffarelli-Kohn-Nirenberg

$$-\mathrm{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}V(x)|u|^{p-2}u = |x|^{-bp^*}K(x)f(u), \quad \text{em } \mathbb{R}^N, \tag{P}$$

onde $1 < p < N$, $0 \leq a < \frac{N-p}{p}$, $a < b \leq a+1$, $p^* = p^*(a,b) = \frac{pN}{N-dp}$ e $d = 1 + a - b$.

Palavras-chave: Problemas do tipo Caffarelli-Kohn-Nirenberg; Problema do tipo Berestycki Lions; Existência e concentração de soluções ground state; Existência de soluções ground state positiva e nodal.

Abstract

Title: Some Caffarelli-Kohn-Nirenberg's type problems in $\mathbb{R}^N$.

In this work we prove some results concerning to Caffarelli-Kohn-Nirenberg's type problems in $\mathbb{R}^N$.

In the first chapter we prove the existence of nontrivial solutions with Berestycki-Lions type nonlinearities using the Mountain Pass Theorem, Ekeland's Variational Principle and a Strauss-type compactness result. More precisely, we study the following classes of problems

$$-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}|u|^{p-2}u = |x|^{-bp^*}h(u), \quad \text{in } \mathbb{R}^N, \qquad (PM)$$

and

$$-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = |x|^{-bp^*}f(u), \quad \text{in } \mathbb{R}^N, \qquad (ZM)$$

where $1 < p < N$, $0 \le a < \frac{N-p}{p}$, $a < b \le a+1$, $p^* = p^*(a,b) = \frac{pN}{N-dp}$ and $d = 1 + a - b$.

In the second chapter we prove the existence and concentration of ground state solutions for a class of subcritical, critical or supercritical Caffarelli-Kohn-Nirenberg type problems using the Mountain Pass Theorem and the Moser Iteration method. More precisely, we are going to study the following class of quasilinear problems

$$-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}[1+\mu V(x)]|u|^{p-2}u = |x|^{-bp^*}[f(u) + \varrho|u|^{\sigma-2}u], \quad (P_{\mu,\varrho,\sigma})$$

in $\mathbb{R}^N$, where $1 < p < N$, $0 \le a < \frac{N-p}{p}$, $a < b \le a+1$, $p^* = p^*(a,b) = \frac{pN}{N-dp}$, $d = 1 + a - b$ and $\mu > 0$.

In the third chapter we prove the existence of a positive and a nodal ground state solutions minimizing the functional in the Nehari manifold and in a subset of the Nehari manifold to the following class of Caffarelli-Kohn-Nirenberg type problems

$$-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}V(x)|u|^{p-2}u = |x|^{-bp^*}K(x)f(u), \quad \text{in } \mathbb{R}^N, \quad \text{(P)}$$

where $1 < p < N$, $0 \le a < \frac{N-p}{p}$, $a < b \le a+1$, $p^* = p^*(a,b) = \frac{pN}{N-dp}$ and $d = 1 + a - b$.

Key words: Caffarelli-Kohn-Nirenberg's type problems; Berestycki Lions' type problems; Existence and concentration of ground state solutions; Existence and concentration of positive and nodal ground state solutions.

Contents

Introduction

In this work we are going to study problems involving the operator div $\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right)$. Problems involving this kind of operator are known as Caffarelli-Kohn-Nirenberg (CKN) type problems because Caffarelli, Kohn and Nirenberg proved an important interpolation inequality in 1984 [19], which allows to work on this class of problems using a variational approach. Caffarelli-Kohn-Nirenberg type problems has some applications, for instance, in fluid mechanics, in Newtonian fluids, in flow through porous media, in glaciology (see [25]) and in problems of existence of stationary waves for anisotropic Schrödinger equation (see [48]).

Elliptic differential equations with singular terms, such as CKN type problems, are an important topic in applied mathematics that arises in various contexts such as physics, engineering, biology, and geology. These equations describe phenomena where solutions are smooth in some regions and exhibit singularities in others.

Elliptic equations are known to have smooth, continuous, and well-behaved solutions. However, when singular terms are introduced into these equations, the smoothness of solutions can be compromised.

A classic example of an elliptic equation with singular terms is the Poisson's equation with a singularity at the origin given by

$$\Delta u = \frac{1}{|x|^{\alpha}}.$$

In this equation, the term $|x|^{\alpha}$ represents a singularity at $x = 0$. The solution to this equation will be smooth everywhere except at the origin where the singularity is located.

The Poisson's equation is one of some examples of elliptic differential equations with singular terms that are essential for modeling physical and natural phenomena that exhibit singularities, such as the charge distribution in electrostatics (Coulomb's law leads to a singularity at $\frac{1}{|x|}$), wave propagation in media with discontinuities, behavior of fluids in complex geometries, elasticity problems in materials with fractures, and more.

Solving these equations is challenging due to the singularities, and advanced techniques such as regularization are needed to obtain valid solutions. Additionally, elliptic differential equations with singular terms are fundamental in the theory of distributions and the study of Sobolev spaces.

In summary, elliptic differential equations with singular terms play a crucial role in modeling a wide range of complex phenomena and are an important research topic in applied mathematics and theoretical physics. Understanding these equations and their solutions is fundamental for solving practical problems in various fields of science and engineering. More information on physical motivation for this class of problems can be seen in [29], [32], [41] and [44].

CKN type problems are a kind of elliptic differential equations with singular term. Much progress has been made for this class of problems. For example, in [46], in bounded domain, the authors consider the existence of non-trivial solutions to semi-linear Brezis-Nirenberg type problems with Hardy potential and singular coefficients. The study the eigenvalue problem for this class of problems is in [47]. Results related to problems in the $\mathbb{R}^N$ are more frequent. The best embedding constants, the existence and nonexistence of extremal functions, and their qualitative properties were studied in [19], [20] and [48]. Results of the existence of a solution for problems in the $\mathbb{R}^N$ require results of compactness. For example, in [12] the authors put special conditions about the potentials in order to overcome the lack of compactness and to show existence of solution for a problem with this class of problems. A result of compactness involving radial functions was proved in [25] to show the existence of radial solution for a problem with this class of operators.

In order to contribute to the advance of the understanding of the solutions of this class of operators, we prove some results related to CKN type problems in $\mathbb{R}^N$. Before present our results, we describe briefly what we do in each chapter of this thesis. In the first chapter we prove the existence of nontrivial solutions for a class of Caffarelli-Kohn-Nirenberg type problems adapting the ideas in [6] and [34]. Essentially, we obtain a (PS) sequence of radial functions bounded in order to prove the existence of a solution and, in sequence, we prove a Strauss-type estimate and a Strauss-type compactness result to show that the solution obtained is nontrivial. Finally, we use the Principle Symmetric Criticality to show that the critical point of the Euler-Langrange functional restricted to the subspace of the radial functions is, in fact, a critical point of the Euler-Lagrange functional in the whole space. The lemma 1.3.8 and the Principle Symmetric Criticality are proved in the Appendix A. In the second chapter we prove the existence and concentration of ground state solutions for subcritical, critical and supercritical problems, where we apply the Moser iteration method for the supercritical problem. Also, we need to prove the existence of a ground-state solution for an auxiliary problem in a bounded domain, which the proof is in Appendix B. In the third chapter we prove the existence of a positive ground-state solution and a nodal ground-state solution, which changes the sign exactly once. We prove some compactness results, some properties of functions in the Nehari set, the existence of a positive ground-state solution arguing by contradiction and the existence of a nodal ground-state solution for a minimization argument.

In order to motivate the problem of the first chapter, let us consider a classical problem in the literature. Using a constrained minimization method, Berestycki and Lions [15] show existence of positive solution of C^2 class of problem

$$-\Delta u = g(u) \quad in \quad \mathbb{R}^N \tag{0.0.1}$$

with exponential decay and spherically symmetric, where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $g(0) = 0$. The authors assume that g is odd and satisfies the following conditions.

g_1) $-\infty < \liminf_{s\to 0^+} g(s)/s \leq \limsup_{s\to 0^+} g(s)/s = -m \leq 0$;

g_2) $-\infty \leq \limsup_{s\to\infty} g(s)/s^{2^*-1} \leq 0$;

g_3) There exists $\xi > 0$ such that $G(\xi) = \int_0^\xi g(s)ds > 0$.

The constraint cause a Lagrange multiplier to appear that can be removed using the special homogeneity of the operator and a scale change in $\mathbb{R}^N$. They studied two cases: The Positive mass, that is $m > 0$ and the Zero Mass case, that is $m = 0$.

Alves, Montenegro and Souto in [4] have studied the existence of ground state solution for (0.0.1) with critical growth. By using the variational method, the authors in [4] give a unified approach in order to deal with subcritical and critical case. However, we would like to point out that a result due to Jeanjean and Tanaka [37], which say that the Mountain-Pass value gives the least energy level, was the main tool used. A similar study was made for the critical case in Zhang and Zou [51].

After this pioneering papers, many researches worked in this subject, extending or improving in several ways, see, for instance, [1], [2], [6], [8], [16], [22], [23], [24], [35] and references therein.

Motivated by this subject, we study in the first chapter the existence of nontrivial solutions for the following classes of problems

$$-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}|u|^{p-2}u = |x|^{-bp^*}h(u) \text{ in } \mathbb{R}^N, \qquad (PM)$$

and

$$-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = |x|^{-bp^*}f(u) \text{ in } \mathbb{R}^N, \qquad (ZM)$$

where $1 < p < N$, $0 \le a < \frac{N-p}{p}$, $a < b \le a+1$, $p^* = p^*(a,b) = \frac{pN}{N-dp}$ and $d = 1 + a - b$.

To present the main results of this chapter, it is necessary to put hypotheses about the nonlinearities h and f. The hypotheses on the function h are the following:

$h_1)$ There exists $q \in (p, p^*)$ such that

$$\lim_{|t|\to 0} \frac{h(t)}{|t|^{q-1}} = \lim_{|t|\to\infty} \frac{h(t)}{|t|^{p-1}} = 0;$$

$h_2)$ There exists $\xi > 0$ such that $pH(\xi) - \xi^p > 0$, where $H(t) = \displaystyle\int_0^t h(r)dr$.

Example 0.0.1. *As example of function satisfying the previous hypothesis, we have: Let $\alpha \in (p, p^*)$ and $h(s) = |s|^{\alpha-2}s$ with $\alpha > q$.*

For the Zero Mass case we use $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ that is the completion of the $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with the norm

$$\|u\|^p = \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx,$$

where $\mathcal{C}_0^\infty(\mathbb{R}^N)$ is the space of smooth functions with compact support.

For the Positive Mass case we use $E_0 = \{u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) : \displaystyle\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^p dx < \infty\}$ with the norm

$$\|u\|_0^p = \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^p dx.$$

Let

$$L_b^s(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \to \mathbb{R} : \quad u \text{ is measurable and } \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^s dx < \infty\right\}$$

with the norm defined as

$$|u|_s^s = \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^s dx.$$

The first main result is:

Theorem 0.0.2. *Assume the conditions $h_1)$ and $h_2)$. Then, problem (PM) has a nontrivial solution.*

The first class of problems is called Positive Mass because $g(t) = h(t) - t$ satisfies $g_1)$, $g_2)$ and $g_3)$ for the case $m > 0$.

In the case (ZM), the hypotheses on the function f are the following:

$f_1)$

$$\lim_{|t|\to 0} \frac{f(t)}{|t|^{p^*-1}} = \lim_{|t|\to \infty} \frac{f(t)}{|t|^{p^*-1}} = 0;$$

$f_2)$ There exists $\xi > 0$ such that $F(\xi) > 0$, where $F(t) = \int_0^t f(r)dr$.

Example 0.0.3. *As example of function satisfying the previous hypothesis, we have: Let $p < q_2 < p^* < q_1$ and*

$$f(s) = \begin{cases} |s|^{q_1-2}s, & \text{if } |s| < 1, \\ |s|^{q_2-2}s, & \text{if } |s| \geq 1. \end{cases}$$

The second main result is:

Theorem 0.0.4. *Assume the conditions $f_1)$ and $f_2)$. Then, problem (ZM) has a nontrivial solution.*

The second class of problems is called Zero Mass because f satisfies $g_1)$, $g_2)$ and $g_3)$ for the case $m = 0$.

In this chapter we adapt some arguments that can be found in [34], which was used for the first time by [36]. More precisely, we find a Palais-Smale sequence satisfying a property related to Pohozaev identity. The same approach was used in [6] for a problem involving the Grushin operator.

Finally, we would like to finish this brief introduction about the chapter 1 listing below what we believe to be the main contributions of our work.

(i) The proof of Theorems 0.0.2 and 0.0.4, we have found some difficulties to apply variational methods. For example, for this class operator there is no a result like Jeanjean and Tanaka [37], which say that the Mountain-Pass value gives the least energy level of the Pohozaev manifold, which is crucial in order to use the arguments due to Berestycki-Lions. We overcome this difficulty exploring the argument in [34].

(ii) The operator that we work is not well-behaved for translations. Thus, we have proved a Strauss-type Lemma result for this class of problems (Lemma 1.3.3 and Lemma 1.3.4) in order to show that the critical points have found in the Theorems 0.0.2 and 0.0.4 are nontrivial inspired by the argument in [6].

In the second chapter, we are interested in a class of problems with the subcritical, critical or supercritical growth on the nonlinearity. More precisely, we are going to study the following class of quasilinear problems

$$-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}[1 + \mu V(x)]|u|^{p-2}u = |x|^{-bp^*}[f(u) + \varrho|u|^{\sigma-2}u] \quad (P_{\mu,\varrho,\sigma})$$

in $\mathbb{R}^N$, where $1 < p < N$, $0 \leq a < \frac{N-p}{p}$, $a < b \leq a+1$, $p^* = p^*(a,b) = \frac{pN}{N-dp}$, $d = 1 + a - b$ and $\mu > 0$.

We are considering three cases. The first case is the subcritical growth on the nonlinearity, i.e. when $\varrho = 0$. In this case we have

$$-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}[1+\mu V(x)]|u|^{p-2}u = |x|^{-bp^*}f(u), \qquad (P_{\mu,0,\sigma})$$

in $\mathbb{R}^N$.

The second case is the critical growth on the nonlinearity, i.e. when $\varrho = 1$ and $\sigma = p^*$. In this case we have

$$\begin{aligned} -\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) &+ |x|^{-bp^*}[1+\mu V(x)]|u|^{p-2}u \\ &= |x|^{-bp^*}f(u) + |x|^{-bp^*}|u|^{p^*-2}u, \quad \text{in} \quad \mathbb{R}^N. \end{aligned} \qquad (P_{\mu,1,p^*})$$

The last case is the supercritical growth on the nonlinearity, i.e. when $\varrho = 1$ and $\sigma > p^*$. In this case we have

$$\begin{aligned} -\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) &+ |x|^{-bp^*}[1+\mu V(x)]|u|^{p-2}u \\ &= |x|^{-bp^*}f(u) + |x|^{-bp^*}|u|^{\sigma-2}u, \quad \text{in} \quad \mathbb{R}^N. \end{aligned} \qquad (P_{\mu,1,\sigma})$$

In order to state the main result, we need to introduce the hypotheses on the functions V and f. The condition in $V \in C(\mathbb{R}^N, \mathbb{R})$ are the following:

(V_1) The potential V is nonnegative, that is,

$$V(x) \geq 0, \text{ for all } x \in \mathbb{R}^N;$$

(V_2) The set $\Omega := \text{int}\left\{ x \in \mathbb{R}^N \mid V(x) = 0 \right\}$ is a non-empty bounded open set with smooth boundary $\partial\Omega$;

(V_3) There exists $V^* > 0$, such that

$$meas\left(\{x \in \mathbb{R}^N : V(x) \leq V^*\}\right) < \infty.$$

Potentials of type $1 + \mu V(x)$ satisfying (V_1), (V_2) and (V_3) are called steep potential well. Bartsch and Wang [11] considered a problem with steep potential well and Laplacian operator. They proved existence and concentration of positive ground state solution u_μ for μ large. In particular, in [3] the authors have studied the case exponential critical and in [50] the authors have studied the case polinomial critical of [11]. The existence of sign-changing solutions well was studied in [42]. In the literature, we find a lot of papers where the authors have considered elliptic problems with steep potential as [3], [9], [10], [11], [13], [26], [27], [39] and [50].

In our work, the hypotheses on the nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$ are the following:

(f_1)

$$\lim_{|s|\to 0} \frac{f(s)}{|s|^{p-1}} = 0 \quad \text{and} \quad f(s) = 0, \quad \text{for all} \quad s \leq 0;$$

(f_2) There exists $p < r < p^*$ such that

$$\lim_{|s|\to\infty} \frac{f(s)}{|s|^{r-1}} = 0;$$

(f_3) There exists $\theta \in (p, p^*)$, such that

$$0 < \theta F(s) \leq f(s)s, \quad \text{for} \quad s \neq 0,$$

where $F(s) = \displaystyle\int_0^s f(t)dt$;

(f_4) $s \mapsto \dfrac{f(s)}{s^{p-1}}$ is nondecreasing;

(f_5) There exist $\tau \in (p, p^*)$ and $\lambda^* > 1$ such that

$$f(s) \geq \lambda |s|^{\tau-1}, \quad \text{for all } s \geq 0,$$

for a fixed $\lambda > \lambda^*$ and λ^* will be fixed latter.

Example 0.0.5. *As example of functions satisfying the previous hypothesis, we have:*

- $V(x) = \begin{cases} 0, & \textit{if } |x| < 1, \\ |x|^2 - 1, & \textit{if } |x| \geq 1. \end{cases}$
- $f(s) = \begin{cases} 0, & \textit{if } s \leq 0, \\ \lambda s^{\tau-1}, & \textit{if } s \geq 0. \end{cases}$

We use $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ that is the completion of the $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with the norm

$$\|u\|^p = \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx,$$

where $\mathcal{C}_0^\infty(\mathbb{R}^N)$ is the space of smooth functions with compact support.

We use $E = \{u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^{-bp^*}[1 + \mu V(x)]|u|^p dx < \infty\}$ with the norm

$$\|u\|_\mu^p = \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx + \int_{\mathbb{R}^N} |x|^{-bp^*}[1 + \mu V(z)]|u|^p dx.$$

We also use $E_0 = \{u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^p dx < \infty\}$ with the norm

$$\|u\|_0^p = \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx + \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^p dx.$$

Let us denote by

$$L_b^s(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \quad u \text{ is measurable and } \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^s dx < \infty \right\}.$$

Using an inequality established by Caffarelli, Kohn, and Nirenberg given by [19]

$$\left(\int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p^*} dx \right)^{p/p^*} \leq S_{a,b} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx,$$

we conclude that the embedding $\mathcal{D}_a^{1,p}(\mathbb{R}^N) \hookrightarrow L_b^{p^*}(\mathbb{R}^N)$ is continuous. Moreover, by interpolation, we also conclude that $E \hookrightarrow L_b^s(\mathbb{R}^N)$ and $E_0 \hookrightarrow L_b^s(\mathbb{R}^N)$ are continuous, for $s \in [p, p^*]$.

Here is the main result of this chapter.

Theorem 0.0.6. *Assume that* $(f_1) - (f_4)$ *and* $(V_1) - (V_3)$ *are satisfied. Then,*

(i) there exists $\mu^* > 0$ *such that problem* $(P_{\mu,0,\sigma})$ *has a ground state solution* $u_\mu \in E$ *for all* $\mu > \mu^*$.

(ii) if the function f *satisfies* (f_5) *there exist positive numbers* λ^* *and* μ^{**}*, such that problem* $(P_{\mu,1,p^*})$ *or problem* $(P_{\mu,1,\sigma})$ *has a ground state solution* $u_\mu \in E$ *for all* $\mu > \mu^{**}$ *and for all* $\lambda > \lambda^*$.

(iii) Moreover, as $\mu \to +\infty$*, the sequence* (u_μ) *converges in* E *to a ground state solution* $u_\infty \in E(\Omega)$ *of the problem*

$$\begin{cases} -div\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}|u|^{p-2}u = |x|^{-bp^*}f(u) + |x|^{-bp^*}|u|^{\sigma-2}u, & in\ \Omega, \\ u = 0, & on\ \ \partial\Omega. \end{cases}$$

where $E(\Omega)$ is defined by $E(\Omega) = \{u \in \mathcal{D}^{1,p}_{0,a}(\Omega) : \int_\Omega |x|^{-bp^*}|u|^p dx < \infty\}$ with the norm

$$\|u\|^p_{0,\Omega} = \int_\Omega |x|^{-ap}|\nabla u|^p dx + \int_\Omega |x|^{-bp^*}|u|^p dx.$$

Our arguments were strongly influenced by [3], [9], [10], [11], [12], [13], [25], [26], [39] and [50].

Finally, we would like to finish this brief introduction about the chapter 2 listing below what we believe to be the main contributions of our work.

(i) The results that can be found in this chapter are complementary to the results of [12] and [25]. Furthermore, as far as we know, this is the first result of concentration of solutions for this class of problems.

(ii) Since we work with singularity not only in the nonlinearity but also in the operator, some estimates are more refined. See for example Theorem 2.2.3, which is a version of Lions's Lemma for this class of problems.

(iii) Unlike the works [3], [9], [10], [11], [13], [26], [39] and [50], we are also considering the supercritical case.

The third chapter deals mainly with the existence of a positive and a nodal solutions to the following class of Caffarelli-Kohn-Nirenberg type problems give by

$$-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}V(x)|u|^{p-2}u = |x|^{-bp^*}K(x)f(u) \quad \text{in } \mathbb{R}^N, \qquad (\text{P})$$

where $1 < p < N$, $0 \le a < \frac{N-p}{p}$, $a < b \le a+1$, $p^* = p^*(a,b) = \frac{pN}{N-dp}$ and $d = 1 + a - b$.

In order to find these solution we use $\mathcal{D}^{1,p}_a(\mathbb{R}^N)$ that is the completion of the $\mathcal{C}^\infty_0(\mathbb{R}^N)$ with the norm

$$\|u\|^p = \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx,$$

where $\mathcal{C}^\infty_0(\mathbb{R}^N)$ is the space of smooth functions with compact support.

Let us denote by

$$L^s_b(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \to \mathbb{R} : \quad u \text{ is measurable and } \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^s dx < \infty\right\}$$

and

$$L_b^\infty(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \quad u \text{ is measurable and} \sup_{\mathbb{R}^N} ess |x|^{-bp^*}|u| < \infty \right\}.$$

On functions $V, K : \mathbb{R}^N \to \mathbb{R}$ continuous on $\mathbb{R}^N$ we assume the following general conditions. We say that $(V, K) \in \mathcal{K}$ if

(VK_0) $V(x), K(x) > 0$ for all $x \in \mathbb{R}^N$ and $K \in L_b^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

(VK_1) If $\{A_n\}_n \subset \mathbb{R}^N$ is a sequence of Borel sets such that the Lebesgue measure meas$(A_n) \leq R$, for all $n \in \mathbb{N}$ and some $R > 0$, then

$$\lim_{r\to+\infty} \int_{A_n \cap B_r^c(0)} |x|^{-bp^*} K(x) = 0, \quad \text{uniformly in } n \in \mathbb{N}.$$

Furthermore, one of the below conditions occurs

(VK_2) $\frac{K}{V} \in L_b^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$

or

(VK_3) there exists $m \in (p, p^*)$ such that

$$\frac{K(x)}{V(x)^{\frac{p^*-m}{p^*-p}}} \to 0 \quad \text{as } |x| \to +\infty.$$

Moreover, we assume the following growth conditions in the origin and at infinity for the C^1 function $f : \mathbb{R} \to \mathbb{R}$:

(f_1)

$$\lim_{|t|\to0^+} \frac{f(t)}{|t|^{p-1}} = 0 \quad \text{if } (VK_2) \text{ holds}$$

or

$(\widetilde{f}_1)$

$$\lim_{|t|\to0^+} \frac{f(t)}{|t|^{m-1}} = 0 \quad \text{if } (VK_3) \text{ holds}$$

with $m \in (p, p^*)$ defined before in (VK_3);

(f_2) f has a "quasicritical growth" at infinity, namely,

$$\lim_{|t|\to+\infty} \frac{f(t)}{|t|^{p^*-1}} = 0;$$

(f_3) There exists $\theta \in (p, p^*)$ so that

$$0 < \theta F(t) = \theta \int_0^t f(s)ds \leq f(t)t, \text{ for all } |t| > 0;$$

(f_4) The map

$$t \mapsto \frac{f(t)}{|t|^{p-1}} \text{ is strictly increasing for all } |t| > 0,$$

or, equivalently,

$$f'(t) > (p-1)\frac{f(t)}{t}, \text{ for all } t \neq 0.$$

Example 0.0.7. *As example of functions satisfying the previous hypothesis, we have:*
Let

$$K(x) := \begin{cases} |x|^{bp^*}, |x| \leq 1, \\ e^{-bp^*(|x|-1)}, |x| > 1, \end{cases}$$

$$V(x) = c > 0 \text{ for all } x \in \mathbb{R}^N,$$

and

$$f(t) = |t|^{q-1}t, \text{ for all } t \in \mathbb{R} \text{ and where } q \in (m, p^*).$$

The main results of this chapter are stated in the following theorem.

Theorem 0.0.8. *Suppose that* $(V, K) \in \mathcal{K}$ *and* $f \in C^1(\mathbb{R}, \mathbb{R})$ *verifies* (f_1) *or* $(\widetilde{f}_1)$ *and* $(f_2) - (f_4)$*. Then, problem* (P) *possesses a positive ground state weak solution. Moreover,* (P) *admits a nodal ground state weak solution, which has precisely two nodal domains.*

Our arguments were strongly influenced by [12].

Finally, we would like to finish this brief introduction about the chapter 3 listing below what we believe to be the main contributions of our work.

(i) Comparing our results with [12], we observe that the singularities that appear in our work are more general. Then, the estimates are more refined.

(ii) Furthermore, the positive solution was obtained by a technique different from the technique used in [12] . Our results complete the result that can be found in [12], because we also show a solution that sign-changes. Apparently, this is the first result of a nodal solution for this class of problems.

We finish this introduction observing that the main results of this thesis as well as all hypothesis from them will stated again in each chapter for a better comprehension of the reader.

Notation

In this work we use the following notation:

$\mathcal{C}_0^\infty(\mathbb{R}^N)$	space of smooth functions with compact support;
$\mathcal{D}_a^{1,p}(\mathbb{R}^N)$	completion of the $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with the norm $\Vert \cdot \Vert$;
$\Vert u \Vert^p = \int\limits_{\mathbb{R}^N} \vert x\vert^{-ap} \vert \nabla u\vert^p dx$	norm of the u in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$;
$E_0 = \left\{ u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} \vert x\vert^{-bp^*} \vert u\vert^p dx < \infty \right\}$	subspace of $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$;
$\Vert u \Vert_0^p = \int\limits_{\mathbb{R}^N} \vert x\vert^{-ap} \vert \nabla u\vert^p dx + \int\limits_{\mathbb{R}^N} \vert x\vert^{-bp^*} \vert u\vert^p dx$	norm of the u in E_0;
$L_b^s(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \int\limits_{\mathbb{R}^N} \vert x\vert^{-bp^*} \vert u\vert^s dx < \infty \right\}$	Lebesgue space with weight;
$\vert u\vert_s^s = \int\limits_{\mathbb{R}^N} \vert x\vert^{-bp^*} \vert u\vert^s dx$	norm of the u in $L_b^s(\mathbb{R}^N)$;
$E_0(B_R(0))$	E_0 restrict to $B_R(0)$;
$L_b^s(B_R(0))$	$L_b^s(\mathbb{R}^N)$ restrict to $B_R(0)$;
$\vert u\vert_{s,B_R(0)}^s = \int\limits_{B_R(0)} \vert x\vert^{-bp^*} \vert u\vert^s dx$	norm of the u in $L_b^s(B_R(0))$;
$E_{0,rad}$	subspace of radial functions of E_0;
$\mathcal{C}_{0,rad}^\infty(\mathbb{R}^N)$	subspace of radial functions of $\mathcal{C}_0^\infty(\mathbb{R}^N)$;
$\mathcal{D}_{a,rad}^{1,p}(\mathbb{R}^N)$	completion of $\mathcal{C}_{0,rad}^\infty(\mathbb{R}^N)$ under the norm $\Vert \cdot \Vert$;

$E = \{u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} \|x\|^{-bp^*}[1+\mu V(x)]\|u\|^p dx < \infty\}$	subspace of $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$;
$\\|u\\|_\mu^p = \\|u\\|^p + \int\limits_{\mathbb{R}^N} \|x\|^{-bp^*}[1+\mu V(z)]\|u\|^p dx$	norm of the u in E;
$L_b^\infty(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \to \mathbb{R} : \sup\limits_{\mathbb{R}^N} ess\|x\|^{-bp^*}\|u\| < \infty\right\}$	L^∞ space with weight;
$X = \left\{u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \|x\|^{-bp^*} V(x)\|u\|^p\, dx < \infty\right\}$	subspace of $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$;
$\\|u\\|_V^p = \int\limits_{\mathbb{R}^N} \|x\|^{-ap}\|\nabla u\|^p\, dx + \int\limits_{\mathbb{R}^N} \|x\|^{-bp^*} V(x)\|u\|^p\, dx$	norm of the u in X;
$L_{b,K}^\zeta(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \to \mathbb{R} : \int\limits_{\mathbb{R}^N} \|x\|^{-bp^*} K(x)\|u\|^\zeta\, dx < \infty\right\}$	Lebesgue space with weight;
Ω	bounded domain;
$B_R(0)$	open ball of radius R centered at 0;
meas(A)	measure of a measurable set A.

Chapter 1

Caffarelli-Kohn-Nirenberg type problems with Berestycki-Lions type nonlinearities

In this chapter, we use a Mountain Pass Theorem and an Ekeland's Variational Principle developed in [34] to find weak solutions in the subspace of radial functions once that there is no a result like [37], which say that the Mountain-Pass value gives the least energy level of the Pohozaev manifold, which is crucial in order to use the arguments due to Berestycki-Lions. Next, we adapt an argument in [6] to show that the weak solutions are nontrivial because the operator that we work is not well-behaved for translations, then the Critical Symmetric Principle ensures that the weak solutions are solutions in the whole space.

1.1 Introduction

This chapter is focused to prove the existence of nontrivial solutions for the following classes of problems

$$-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}|u|^{p-2}u = |x|^{-bp^*}h(u) \ \text{ in } \mathbb{R}^N, \tag{PM}$$

and

$$-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = |x|^{-bp^*}f(u) \ \text{ in } \mathbb{R}^N, \tag{ZM}$$

where $1 < p < N$, $0 \le a < \frac{N-p}{p}$, $a < b \le a+1$, $p^* = p^*(a,b) = \frac{pN}{N-dp}$ and $d = 1 + a - b$.

Observe that the hypothesis that $a \ge 0$ is important once that the estimate (1.3.6) fails for $a < 0$.

To present the main results of this chapter, it is necessary to put hypotheses about the nonlinearities h and f. The hypotheses on the function h in this case are the following:

h_1) h is continuous and there exists $q \in (p, p^*)$ such that

$$\lim_{|t|\to 0} \frac{h(t)}{|t|^{q-1}} = \lim_{|t|\to\infty} \frac{h(t)}{|t|^{p-1}} = 0;$$

h_2) There exists $\xi > 0$ such that $pH(\xi) - \xi^p > 0$, where $H(t) = \int_0^t h(r)dr$.

The first main result is:

Theorem 1.1.1. *Assume the conditions* h_1) *and* h_2)*. Then, problem* (PM) *has a nontrivial solution.*

The first class of problems is called Positive Mass because $g(t) = h(t) - t$ satisfies g_1), g_2) and g_3) for the case $m > 0$.

For the problem (ZM), the hypotheses on the function f in this case are the following:

f_1) f is continuous and
$$\lim_{|t|\to 0} \frac{f(t)}{|t|^{p^*-1}} = \lim_{|t|\to\infty} \frac{f(t)}{|t|^{p^*-1}} = 0;$$

f_2) There exists $\xi > 0$ such that $F(\xi) > 0$, where $F(t) = \int_0^t f(r)dr$.

The second main result is:

Theorem 1.1.2. *Assume the conditions* f_1) *and* f_2)*. Then, problem* (ZM) *has a nontrivial solution.*

The second class of problems is called Zero Mass because f satisfies g_1), g_2) and g_3) for the case $m = 0$.

We would like to point out that in the proof of Theorems 0.0.2 and 0.0.4, we have found some difficulties to apply variational methods. For example, for this class operator there is no a result like Jeanjean and Tanaka [37], which say that the Mountain-Pass value gives the least energy level of the Pohozaev manifold, which is crucial in order to use the arguments due to Berestycki-Lions. Furthermore, it was necessary to prove a Straus-type Lemma result for this class of problems (Lemma 1.3.3 and Lemma 1.3.4).

Finally, it is very important to say that in the literature, we find many papers where the authors study problems involving the operator div $\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right)$, see, for example, Bastos, Miyagaki and Vieira [12], Catrina and Wang [20], Chen [25], Xuan [46] and references therein. In Chen [25] we can find a Straus-type Lemma result for this class of problems. However, the Chen's result cannot be applied for our problem, because we have another class of nonlinearities.

1.2 The variational framework

For the Zero Mass case we use $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ that is the completion of the $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with the norm
$$\|u\|^p = \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx,$$
where $\mathcal{C}_0^\infty(\mathbb{R}^N)$ is the space of smooth functions with compact support.

For the Positive Mass case we use $E_0 = \{u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^p dx < \infty\}$ with the norm
$$\|u\|_0^p = \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^p dx.$$

Let define

$$L_b^s(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \quad u \text{ is measurable and} \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^s dx < \infty \right\}$$

with the norm defined as

$$|u|_s^s = \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^s dx.$$

We also define $E_0(B_R(0)) = \{u \in \mathcal{D}_a^{1,p}(B_R(0)) : \int_{B_R(0)} |x|^{-bp^*} |u|^p dx < \infty\}$ and

$$L_b^s(B_R(0)) = \left\{ u : B_R(0) \to \mathbb{R} : \quad u \text{ is measurable and} \int_{B_R(0)} |x|^{-bp^*} |u|^s dx < \infty \right\}$$

with the norm defined as

$$|u|_{s,B_R(0)}^s = \int_{B_R(0)} |x|^{-bp^*} |u|^s dx.$$

Using an inequality established by Caffarelli, Kohn, and Nirenberg given by [19]

$$\left(\int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p^*} dx \right)^{p/p^*} \leq S_{a,b} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx, \tag{1.2.1}$$

we conclude that the embedding $\mathcal{D}_a^{1,p}(\mathbb{R}^N) \hookrightarrow L_b^{p^*}(\mathbb{R}^N)$ is continuous. Moreover, by interpolation, we also conclude that $E_0 \hookrightarrow L_b^s(\mathbb{R}^N)$ is continuous, for $s \in [p, p^*]$.

1.3 The existence of solution for Positive Mass Case

Consider the functional $I : E_0 \to \mathbb{R}$ associated given by

$$I(u) = \frac{1}{p} \|u\|_0^p - \int_{\mathbb{R}^N} |x|^{-bp^*} H(u) dx.$$

As a consequence of (h_1), we obtain that I is well-defined and of C^1 class. Moreover, note that

$$I'(u)\phi = \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla \phi dx + \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p-2} u \phi dx - \int_{\mathbb{R}^N} |x|^{-bp^*} h(u) \phi dx,$$

for all $\phi \in E_0$. Then, the critical points of I are weak solutions of (PM).

We will restrict the functional I to the space

$$E_{0,rad} = \mathcal{D}_{a,rad}^{1,p}(\mathbb{R}^N) \cap L_{b,rad}^p(\mathbb{R}^N)$$

under the norm $\|\cdot\|_0$ to overcome the loss of compactness of the space E_0, then we will use the Principle of Symmetric Criticality to obtain the solutions in the whole space.

Observe that the restriction is necessary only for prove the Lemma 1.3.3 and the Lemma 1.3.4 so that the arguments in this section can be done for the whole space with exception of these lemmas, which are important to show the nontriviality of the weak solutions.

In order to use critical point theory, we firstly derive results related to the Palais-Smale compactness condition. We say that a sequence (u_n) is a Palais-Smale sequence for the functional I if

$$I(u_n) \to c_*$$

and

$$\|I'(u_n)\| \to 0 \text{ in } (E_{0,rad})',$$

where

$$c_* = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} I(\eta(t)) > 0$$

and

$$\Gamma := \{\eta \in C([0,1], E_{0,rad}) : \eta(0) = 0,\ I(\eta(1)) < 0\}.$$

If every Palais-Smale sequence of I has a strong convergent subsequence, then one says that I satisfies the Palais-Smale condition ((PS) for short).

Lemma 1.3.1. *The functional I satisfies the following conditions:*
(i) There exist ρ_1, $\rho_2 > 0$ such that:

$$I(u) \geq \rho_2 \text{ with } \|u\|_0 = \rho_1;$$

(ii) There exists $e \in B^c_{\rho_1}(0)$ with $I(e) < 0$ and $\|e\|_0 > \rho_1$.

Proof. i) First of all, observe that

Statement 1.3.2.

$$\int_{\mathbb{R}^N} |x|^{-bp^*} H(u)dx \leq \frac{\varepsilon}{p}\|u\|_0^p + \frac{C_1C_\varepsilon}{q}\|u\|_0^q. \tag{1.3.1}$$

Proof. For h_1), given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$h(t) \leq \varepsilon|t|^{p-1} + C_\varepsilon|t|^{q-1}, \forall t \in \mathbb{R}. \tag{1.3.2}$$

Thus, (1.3.2) implies that

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^{-bp^*} H(u)dx &\leq \int_{\mathbb{R}^N} |x|^{-bp^*}|H(u)|dx \\ &\leq \int_{\mathbb{R}^N}\left(\int_0^u |x|^{-bp^*}(\varepsilon|t|^{p-1} + C_\varepsilon|t|^{q-1})dt\right)dx \\ &= \frac{\varepsilon}{p}\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^p dx + \frac{C_\varepsilon}{q}\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^q dx.\end{aligned}$$

The definition of the norm implies that

$$\frac{\varepsilon}{p}\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^p dx + \frac{C_\varepsilon}{q}\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^q dx \leq \frac{\varepsilon}{p}\|u\|_0^p + \frac{C_\varepsilon}{q}\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^q dx,$$

Finally, the continuous embedding $E_0 \hookrightarrow L^q_b(\mathbb{R}^N)$ gives

$$\frac{\varepsilon}{p}\|u\|_0^p + \frac{C_\varepsilon}{q}\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^q dx \leq \frac{\varepsilon}{p}\|u\|_0^p + \frac{C_1C_\varepsilon}{q}\|u\|_0^q,$$

where C_1 is the constant of the embedding, which proves (1.3.1). □

Using (1.3.1) and taking $\varepsilon > 0$ sufficiently small such that $\|u\|_0 = \rho_1$, we obtain

$$I(u) \geq \left(\frac{1}{p} - \frac{\varepsilon}{p}\right) \|u\|_0^p - \frac{C_1 C_\varepsilon}{q} \|u\|_0^q.$$

and the result follows because $q > p$.
ii) From h_2), there exists $\phi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |x|^{-bp^*} \left(H(\phi) - \frac{|\phi|^p}{p}\right) dx > 0.$$

For $t > 0$, setting

$$\omega_t(x) = \phi\left(\frac{x}{t}\right)$$

and deriving $\phi\left(\frac{x}{t}\right)$, we have

$$\begin{aligned} I(\omega_t) &= \frac{1}{p}\int_{\mathbb{R}^N} \left|\frac{x}{t}\right|^{-ap} \left|\nabla\left(\phi\left(\frac{x}{t}\right)\right)\right|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} \left|\frac{x}{t}\right|^{-bp^*} \left|\phi\left(\frac{x}{t}\right)\right|^p dx - \int_{\mathbb{R}^N} \left|\frac{x}{t}\right|^{-bp^*} H(\phi(x/t))dx \\ &= \frac{1}{p}\int_{\mathbb{R}^N} \left|\frac{x}{t}\right|^{-ap} t^{-p} \left|\nabla\phi\left(\frac{x}{t}\right)\right|^p dx - \int_{\mathbb{R}^N} \left|\frac{x}{t}\right|^{-bp^*} \left(H(\phi(x/t)) - \frac{|\phi(x/t)|^p}{p}\right) dx, \end{aligned}$$

doing the change of variables $x \mapsto y = x/t$, we get

$$\begin{aligned} &\frac{1}{p}\int_{\mathbb{R}^N} \left|\frac{x}{t}\right|^{-ap} t^{-p} \left|\nabla\phi\left(\frac{x}{t}\right)\right|^p dx - \int_{\mathbb{R}^N} \left|\frac{x}{t}\right|^{-bp^*} \left(H(\phi(x/t)) - \frac{|\phi(x/t)|^p}{p}\right) dx \\ &= \int_{\mathbb{R}^N} |y|^{-ap} t^{-p} |\nabla\phi(y)|^p t^N dy - \int_{\mathbb{R}^N} |y|^{-bp^*} \left(H(\phi(y)) - \frac{|\phi(y)|^p}{p}\right) t^N dy \\ &= t^{N-p}\int_{\mathbb{R}^N} |y|^{-ap} |\nabla\phi(y)|^p dy - t^N \int_{\mathbb{R}^N} |y|^{-bp^*} \left(H(\phi(y)) - \frac{|\phi(y)|^p}{p}\right) dy, \end{aligned}$$

therefore

$$I(\omega_t) = t^{N-p}\int_{\mathbb{R}^N} |y|^{-ap} |\nabla\phi(y)|^p dy - t^N \int_{\mathbb{R}^N} |y|^{-bp^*} \left(H(\phi(y)) - \frac{|\phi(y)|^p}{p}\right) dy \to -\infty,$$

as $t \to \infty$. Then, there exists $\bar{t} > 0$ large such that $e = \omega_{\bar{t}}$ satisfies $I(e) < 0$ and $\|e\|_0 > \rho_2$. Note also $c_* \geq \rho_2$. □

Next, we will prove the compactness result is crucial in our approach. We denote by $\mathcal{C}_{0,rad}^\infty(\mathbb{R}^N)$ the collection of smooth radially symmetric functions with compact support, i.e,

$$\mathcal{C}_{0,rad}^\infty(\mathbb{R}^N) = \{u \in \mathcal{C}_0^\infty(\mathbb{R}^N) : u(x) = u(|x|), \quad x \in \mathbb{R}^N\}.$$

Let $\mathcal{D}_{a,rad}^{1,p}(\mathbb{R}^N)$ be the completion of $\mathcal{C}_{0,rad}^\infty(\mathbb{R}^N)$ under the norm $\|\cdot\|$.

The next lemma is important to prove a compactness result, which will be used to show the nontriviality of the critical point.

Lemma 1.3.3. *[Radial Lemma in $E_{0,rad}$] Let $u \in E_{0,rad}$, then for almost every $x \in \mathbb{R}^N \backslash \{0\}$, then there exists $\overline{C} = \overline{C}(a, b, p) > 0$ such that*

$$|u(x)| \leq \overline{C} \frac{1}{|x|^{\frac{(N-p)-ap^*}{p}}} \|u\|_0.$$

Proof. Up to a standard density argument, we only consider $u \in \mathcal{C}^{\infty}_{0,rad}(\mathbb{R}^N)$. Denote by ω_N the volume of the unit sphere in $\mathbb{R}^N$. We have

$$-u(\Upsilon) = u(\infty) - u(\Upsilon) = \int_{\Upsilon}^{\infty} u'(s)ds.$$

Thus,

$$|u(\Upsilon)| \leq \int_{\Upsilon}^{\infty} |u'(s)|ds = \int_{\Upsilon}^{\infty} s^{-a}|u'(s)|s^{\frac{N-1}{p}} s^a s^{\frac{1-N}{p}} ds$$

From Hölder inequality, we get

$$|u(\Upsilon)| \leq \left(\int_{\Upsilon}^{\infty} s^{-ap}|u'(s)|^p s^{N-1} ds \right)^{1/p} \left(\int_{\Upsilon}^{\infty} s^{\frac{ap}{p-1}} s^{\frac{1-N}{p-1}} ds \right)^{(p-1)/p}. \tag{1.3.3}$$

Observe that

$$a < \frac{N-p}{p} < \frac{N-1}{p} \Longrightarrow ap + 1 - N < 0.$$

Thus,

$$\begin{aligned}
\left| \int_{\Upsilon}^{\infty} s^{\frac{ap+1-N}{p-1}} ds \right| &= \left| \frac{s^{\frac{ap+1-N}{p-1}+1}|_{\Upsilon}^{\infty}}{\left(\frac{ap+p-N}{p-1}\right)} \right| \\
&= \left| \frac{s^{\frac{ap+p-N}{p-1}}|_{\Upsilon}^{\infty}}{\left(\frac{ap+p-N}{p-1}\right)} \right| \\
&= \left| \left(\frac{p-1}{ap+p-N} \right) (0 - \Upsilon^{\frac{ap+p-N}{p-1}}) \right| \\
&= \left(\frac{p-1}{N-p-ap} \right) |\Upsilon|^{\frac{-(N-p-ap)}{p-1}} \\
&= \left(\frac{p-1}{N-p-ap} \right) \frac{1}{|\Upsilon|^{\frac{(N-p)-ap}{p-1}}}.
\end{aligned}$$

If $\Upsilon = |x|$, then

$$\left| \int_{\Upsilon}^{\infty} s^{\frac{ap+1-N}{p-1}} ds \right| = \left(\frac{p-1}{N-p-ap} \right) \frac{1}{|x|^{\frac{(N-p)-ap}{p-1}}}. \tag{1.3.4}$$

Proposition C.0.5 provides

$$\begin{aligned}
\left(\int_{\Upsilon}^{\infty} s^{-ap}|u'(s)|^p s^{N-1} ds \right)^{1/p} &\leq \left(\int_{0}^{\infty} s^{-ap}|u'(s)|^p s^{N-1} ds \right)^{1/p} \\
&= \omega_{N-1}^{-\frac{1}{p}} \left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx \right)^{1/p}
\end{aligned} \tag{1.3.5}$$

It follows from (1.3.3), (1.3.4) and (1.3.5) that

$$|u(\Upsilon)| \leq \omega_N^{\frac{-1}{p}} \left(\frac{p-1}{N-p-ap} \right)^{\frac{p-1}{p}} \frac{1}{|x|^{\frac{(N-p)-ap}{p}}} \left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx \right)^{1/p}.$$

□

Now we present a compactness result.

Lemma 1.3.4. *The embedding* $E_{0,rad} \hookrightarrow L_b^s(\mathbb{R}^N)$ *is compact for all* $s \in (p, p^*)$.

Proof. Let $(u_n) \subset E_{0,rad}(\mathbb{R}^N)$ be a bounded sequence and let $C > 0$ be such that

$$\|u_n\|_0 \leq C, \quad \forall n \in \mathbb{N}.$$

By Lemma 1.3.3 it follows that, for all $n \in \mathbb{N}$,

$$|u_n(x)| \leq C\overline{C}\frac{1}{|x|^{\frac{(N-p)-ap}{p}}}, \quad \text{a.e. in } \mathbb{R}^N\backslash\{0\}.$$

Since $s > 1$, given $\varepsilon > 0$, there exists $R > 0$ such that, for all $n \in \mathbb{N}$,

$$|u_n(x)|^s \leq \frac{\varepsilon}{2C\overline{C}}|u_n(x)| \quad \forall x \in B_R(0)^c.$$

This implies that

$$\int_{B_R(0)^c} |x|^{-bp^*}|u_n|^s dx \leq \frac{\varepsilon}{2C\overline{C}R^{bp^*}} \int_{B_R(0)^c} |u_n| dx \leq \frac{\varepsilon}{2R^{\frac{(N-p)-ap+bpp^*}{p}}} \leq \frac{\varepsilon}{2}, \tag{1.3.6}$$

for all $n \in \mathbb{N}$. Moreover, since $E_0(B_R(0))$ is compactly embedded into $L_b^s(B_R(0))$, there exists $u \in L_b^s(B_R(0))$ such that, up to a subsequence $u_n \to u$ in $L_b^s(B_R(0))$, as $n \to \infty$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\int_{B_R(0)} |x|^{-bp^*}|u_n - u|^s dx < \frac{\varepsilon}{2}, \quad \forall n \geq n_0. \tag{1.3.7}$$

Let us define $\overline{u} : \mathbb{R}^N \to \mathbb{R}$ as to be equal to u in $B_R(0)$ and equal to 0 in $B_R(0)^c$. Then, by (1.3.6) and (1.3.7), it follows that

$$\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n - \overline{u}|^s dx = \int_{B_R(0)} |x|^{-bp^*}|u_n - \overline{u}|^s dx + \int_{B_R(0)^c} |x|^{-bp^*}|u_n|^s dx < \varepsilon.$$

Then it is clear that $u_n \to \overline{u}$ in $L_b^s(\mathbb{R}^N)$, as $n \to \infty$. □

Following [34] and [36], we consider an auxiliary functional $\widetilde{I} \in C^1(\mathbb{R} \times E_{0,rad})$ given by

$$\begin{aligned} \widetilde{I}(\theta, u) &= \frac{\exp((N-p)\theta)}{p} \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx + \frac{\exp(N\theta)}{p} \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^p dx \\ &- \exp(N\theta) \int_{\mathbb{R}^N} |x|^{-bp^*} H(u) dx. \end{aligned} \tag{1.3.8}$$

This functional will be important to show the boundedness of a (PS) sequence that we will find.

Statement 1.3.5. *The following properties hold, for all* $(\theta, u) \in \mathbb{R} \times E_{0,rad}$,

$$\widetilde{I}(0, u) = I(u),$$

$$\widetilde{I}(\theta, u) = I(u(x/\exp(\theta))).$$

Proof. A direct computation shows that $\widetilde{I}(0,u) = I(u)$.

We proceed to prove the second property. Doing a change of variables $y \mapsto x := \exp(\theta)y$,

$$
\begin{aligned}
\widetilde{I}(\theta,u) &= \frac{\exp((N-p)\theta)}{p}\int_{\mathbb{R}^N}|y|^{-ap}|\nabla u(y)|^p dy + \frac{\exp(N\theta)}{p}\int_{\mathbb{R}^N}|y|^{-bp^*}|u(y)|^p dy \\
&- \exp(N\theta)\int_{\mathbb{R}^N}|y|^{-bp^*}H(u(y))dy \\
&= \frac{\exp((N-p)\theta)}{p}\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-ap}\left|\nabla u\left(\frac{x}{\exp(\theta)}\right)\right|^p \exp(-N\theta)dx \\
&+ \frac{\exp(N\theta)}{p}\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-bp^*}\left|u\left(\frac{x}{\exp(\theta)}\right)\right|^p \exp(-N\theta)dx \\
&- \exp(N\theta)\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-bp^*} H\left(u\left(\frac{x}{\exp(\theta)}\right)\right)\exp(-N\theta)dx.
\end{aligned}
$$

Cancelling $\exp(N\theta)$ with $\exp(-N\theta)$, we get

$$
\begin{aligned}
&\frac{\exp((N-p)\theta)}{p}\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-ap}\left|\nabla u\left(\frac{x}{\exp(\theta)}\right)\right|^p \exp(-N\theta)dx \\
&+ \frac{\exp(N\theta)}{p}\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-bp^*}\left|u\left(\frac{x}{\exp(\theta)}\right)\right|^p \exp(-N\theta)dx \\
&- \exp(N\theta)\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-bp^*} H\left(u\left(\frac{x}{\exp(\theta)}\right)\right)\exp(-N\theta)dx \\
&= \frac{\exp(-p\theta)}{p}\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-ap}\left|\nabla u\left(\frac{x}{\exp(\theta)}\right)\right|^p dx \\
&+ \frac{1}{p}\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-bp^*}\left|u\left(\frac{x}{\exp(\theta)}\right)\right|^p dx - \int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-bp^*} H\left(u\left(\frac{x}{\exp(\theta)}\right)\right)dx.
\end{aligned}
$$

Putting $\exp(-p\theta)$ inside of the integral, we have

$$
\begin{aligned}
&\frac{\exp(-p\theta)}{p}\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-ap}\left|\nabla u\left(\frac{x}{\exp(\theta)}\right)\right|^p dx \\
&+ \frac{1}{p}\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-bp^*}\left|u\left(\frac{x}{\exp(\theta)}\right)\right|^p dx - \int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-bp^*} H\left(u\left(\frac{x}{\exp(\theta)}\right)\right)dx \\
&= \frac{1}{p}\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-ap}\left|\frac{1}{\exp(\theta)}\nabla u\left(\frac{x}{\exp(\theta)}\right)\right|^p dx \\
&+ \frac{1}{p}\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-bp^*}\left|u\left(\frac{x}{\exp(\theta)}\right)\right|^p dx - \int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-bp^*} H\left(u\left(\frac{x}{\exp(\theta)}\right)\right)dx.
\end{aligned}
$$

Finally, we use the chain rule to obtain

$$\begin{aligned}
&\frac{1}{p}\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-ap}\left|\frac{1}{\exp(\theta)}\nabla u\left(\frac{x}{\exp(\theta)}\right)\right|^{p}dx\\
&+\frac{1}{p}\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-bp^*}\left|u\left(\frac{x}{\exp(\theta)}\right)\right|^{p}dx-\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-bp^*}H\left(u\left(\frac{x}{\exp(\theta)}\right)\right)dx\\
&=\frac{1}{p}\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-ap}\left|\nabla\left(u\left(\frac{x}{\exp(\theta)}\right)\right)\right|^{p}dx\\
&+\frac{1}{p}\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-bp^*}\left|u\left(\frac{x}{\exp(\theta)}\right)\right|^{p}dx-\int_{\mathbb{R}^N}\left|\frac{x}{\exp(\theta)}\right|^{-bp^*}H\left(u\left(\frac{x}{\exp(\theta)}\right)\right)dx\\
&=I(u(x/\exp(\theta))).
\end{aligned}$$

□

We equip a standard product norm

$$\|(\theta,u)\|^p_{\mathbb{R}\times E_{0,rad}}=|\theta|^p+\|u\|_0^p$$

to $\mathbb{R}\times E_{0,rad}$. Now we prove that $\widetilde{I}$ satisfies the Mountain Pass geometry.

Lemma 1.3.6. *The functional $\widetilde{I}$ satisfies the following conditions:*
(i) There exist ρ_1, $\rho_2>0$ such that:

$$\widetilde{I}(\theta,u)\geq\rho_2 \ \text{ with } \ \|(\theta,u)\|_{\mathbb{R}\times E_{0,rad}}=\rho_1;$$

(ii) There exists $\widetilde{e}\in B^c_{\rho_1}(0)$ with $\widetilde{I}(\widetilde{e})<0$ and $\|\widetilde{e}\|_{\mathbb{R}\times E_{0,rad}}>\rho_1$.

Proof. The item i) follows by using the same argument of Lemma 1.3.1 and for item ii) it is sufficient to take $\widetilde{e}=(0,e)$. Indeed,

$$\begin{aligned}
\widetilde{I}(\theta,u)&=\frac{\exp(N\theta)}{p}\left(\|u\|^p+\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^pdx\right)+\frac{\exp(-p\theta)}{p}\|u\|^p-\exp(N\theta)\int_{\mathbb{R}^N}|x|^{-bp^*}H(u)dx\\
&=\frac{\exp(N\theta)}{p}\|u\|_0^p+\frac{\exp(-p\theta)}{p}\|u\|^p-\exp(N\theta)\int_{\mathbb{R}^N}|x|^{-bp^*}H(u)dx.
\end{aligned}$$

Using that $\frac{\exp(-p\theta)}{p}\|u\|^p\geq 0$, we have

$$\begin{aligned}
&\frac{\exp(N\theta)}{p}\|u\|_0^p+\frac{\exp(-p\theta)}{p}\|u\|^p-\exp(N\theta)\int_{\mathbb{R}^N}|x|^{-bp^*}H(u)dx\\
&\geq\frac{\exp(N\theta)}{p}\|u\|_0^p-\exp(N\theta)\int_{\mathbb{R}^N}|x|^{-bp^*}H(u)dx.
\end{aligned}$$

By (1.3.1) and the continuous embedding $E_0\hookrightarrow L^q_b(\mathbb{R}^N)$,

$$\begin{aligned}
&\frac{\exp(N\theta)}{p}\|u\|_0^p-\exp(N\theta)\int_{\mathbb{R}^N}|x|^{-bp^*}H(u)dx\\
&\geq\frac{\exp(N\theta)}{p}\|u\|_0^p-\exp(N\theta)\frac{\varepsilon}{p}\|u\|_0^p-\exp(N\theta)\frac{C_1C_\varepsilon}{q}\|u\|_0^q\\
&=\frac{\exp(N\theta)}{p}\|u\|_0^p(1-\varepsilon)-\exp(N\theta)\frac{C_1C_\varepsilon}{q}\|u\|_0^q,
\end{aligned}$$

therefore

$$\widetilde{I}(\theta,u) \geq \frac{\exp(N\theta)}{p}\|u\|_0^p(1-\varepsilon) - \exp(N\theta)\frac{C_1C_\varepsilon}{q}\|u\|_0^q,$$

which proves item $i)$ if $\|u\|_0 = \varepsilon > 0$ is sufficiently small because $q > p$. Finally,

$$\widetilde{I}(\widetilde{e}) = \widetilde{I}(0,e) = I(e),$$

which is negative as we saw in the Lemma 1.3.1 and $\|\widetilde{e}\|_{\mathbb{R}\times E_{rad}} = \|e\|_0 > \rho_2$. □

In what follows, we define the Mountain Pass level $\widetilde{c}_*$ for $\widetilde{I}$ by

$$\widetilde{c}_* = \inf_{\eta\in\Gamma}\max_{t\in[0,1]}\widetilde{I}(\eta(t)) > 0$$

and

$$\widetilde{\Gamma} := \{\eta \in C([0,1],\mathbb{R}\times E_{rad}) : \eta(0)=0,\ \widetilde{I}(\eta(1)) < 0\}.$$

Note that $\widetilde{c}_* \geq \rho_2$.

Lemma 1.3.7. *The Mountain Pass levels of I and $\widetilde{I}$ coincide, namely $c_* = \widetilde{c}_* > 0$.*

Proof. Note that $\Gamma \cong \{0\}\times\Gamma \subset \widetilde{\Gamma}$, which implies $\widetilde{c}_* \leq c_*$. On the other hand, consider $\widetilde{\gamma} \in \widetilde{\Gamma}$ arbitrary. Then, for each $t\in[0,1]$, we have $\widetilde{\gamma}(t) = (\theta_t,u_t)$. Define $\gamma(t) := u_t\left(\frac{x}{\exp(\theta_t)}\right)$. From the Statement 1.3.5, we conclude $\widetilde{I}(\widetilde{\gamma}_t) = \widetilde{I}(\theta_t,u_t) = I(u_t(x/\exp(\theta_t))) = I(\gamma(t))$ for each $t\in[0,1]$. Hence $\gamma\in\Gamma$, where we derive $\widetilde{c}_* \geq c_*$. □

Lemma 1.3.8. *Let $\varepsilon > 0$. Suppose that $\widetilde{\eta}\in\widetilde{\Gamma}$ satisfies*

$$\max_{t\in[0,1]}\widetilde{I}(\widetilde{\eta}) \leq c_* + \varepsilon,$$

then, there exists $(\theta,u)\in\mathbb{R}\times E_{0,rad}$ such that

- $dist_{\mathbb{R}\times E_{0,rad}}((\theta,u),\widetilde{\eta}([0,1])) \leq 2\sqrt{\varepsilon}$;
- $\widetilde{I}(\theta,u) \in [c_*-\varepsilon, c_*+\varepsilon]$;
- $\|D\widetilde{I}(\theta,u)\|_{\mathbb{R}\times E^*_{0,rad}} \leq 2\sqrt{\varepsilon}$.

Proof. See Appendix *A*. □

The proof of the next lemma is a consequence of Lemma 1.3.8.

Lemma 1.3.9. *There exists a sequence $((\theta_n,u_n)) \subset \mathbb{R}\times E_{0,rad}$ such that, as $n\to\infty$, we get*

- $\theta_n \to 0$;
- $\widetilde{I}(\theta_n,u_n) \to c_*$;
- $\partial_\theta\widetilde{I}(\theta_n,u_n) \to 0$;
- $\partial_u\widetilde{I}(\theta_n,u_n) \to 0$, *strongly in* $E^*_{0,rad}$.

Proof. For any $j \in \mathbb{N}$, we can find a $\gamma_j \in \Gamma$ such that

$$\max_{t\in[0,1]} I(\gamma_j(t)) \leq c_* + \frac{1}{j}.$$

Since $\widetilde{c}_* = c_*$ and $\widetilde{\gamma}_j(t) = (0, \gamma_j(t)) \in \widetilde{\Gamma}$ satisfies $\max_{t\in[0,1]} \widetilde{I}(\widetilde{\gamma}_j)(t) \leq \widetilde{c}_* + \frac{1}{j}$, we can find a (θ_j, u_j) by the Lemma 1.3.8 such that

- $\mathrm{dist}_{\mathbb{R}\times E_0}((\theta_j, u_j), \widetilde{\gamma}_j([0,1])) \leq 2/\sqrt{j}$;
- $\widetilde{I}(\theta_j, u_j) \in [c_* - 1/j, c_* + 1/j]$;
- $\|D\widetilde{I}(\theta_j, u_j)\|_{\mathbb{R}\times E^*_{0,rad}} \leq 2/\sqrt{j}$.

Since $\widetilde{\gamma}([0,1]) \subset \{0\} \times E_{0,rad}$, the first inequality implies $|\theta_j| \leq 2/\sqrt{j}$ and, consequently, $\theta_j \to 0$. The second item implies $\widetilde{I}(\theta_j, u_j) \to c_*$ and the last item implies the last two items of these lemma. □

1.3.1 Proof of Theorem 0.0.2

By Lemma 1.3.9, there exists a sequence $((\theta_n, u_n)) \subset \mathbb{R} \times E_{0,rad}$ such that,

$$\begin{aligned}\frac{\exp((N-p)\theta_n)}{p}\|u_n\|^p &+ \frac{\exp(N\theta_n)}{p}\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^p dx \\ &- \exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*} H(u_n)dx = c_* + o_n(1);\end{aligned} \tag{1.3.9}$$

$$\begin{aligned}(N-p)\frac{\exp((N-p)\theta_n)}{p}\|u_n\|^p &+ N\frac{\exp(N\theta_n)}{p}\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^p dx \\ &- N\exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*} H(u_n)dx = o_n(1);\end{aligned} \tag{1.3.10}$$

$$\begin{aligned}\exp((N-p)\theta_n)\|u_n\|^p &+ \exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^p dx \\ &- \exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*} h(u_n)u_n dx = o_n(1)\|u_n\|_0.\end{aligned} \tag{1.3.11}$$

From (1.3.9) and (1.3.10), we have

$$\exp((N-p)\theta_n)\|u_n\|^p = Nc_* + o_n(1). \tag{1.3.12}$$

Since $\theta_n \to 0$ and $p < N$, we have that (u_n) is bounded in $\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$ and it is bounded in $L^{p^*}_b(\mathbb{R}^N)$ by (1.2.1).

From h_1), given $\varepsilon = \frac{1}{2}$, there exist $\delta > 0$ and $A > 1$ such that

$$h(t)t \leq \frac{1}{2}|t|^p, \text{ for all } t \in (0,\delta),$$

$$h(t)t \leq \frac{1}{2}|t|^q < \frac{1}{2}|t|^{p^*}, \text{ for all } t \in (A,\infty)$$

and the continuity of h over the compact interval $[\delta, A]$ ensures that there exists $C > 0$ such that

$$h(t)t \leq C|t|^{p^*}, \text{ for all } t \in [\delta, A].$$

The last three inequalities ensures that

$$h(t)t \leq \frac{1}{2}|t|^p + C|t|^{p^*}, \quad \text{for all} \quad t \in \mathbb{R}.$$

Using the last inequality in (1.3.11), we get

$$\begin{aligned}
&\exp((N-p)\theta_n)\|u_n\|^p + \exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p dx \\
&= \exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}h(u_n)u_n dx + o_n(1)\|u_n\|_0 \\
&\leq \exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}\left(\frac{1}{2}|u_n|^p + C|u_n|^{p^*}\right)dx + o_n(1)\|u_n\|_0 \\
&= \frac{1}{2}\exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p dx + C\exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p^*}dx + o_n(1)\|u_n\|_0,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\exp((N-p)\theta_n)\|u_n\|^p + \exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p dx \\
&\leq \frac{1}{2}\exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p dx + C\exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p^*}dx + o_n(1)\|u_n\|_0.
\end{aligned}$$

Observe that $\exp((N-p)\theta_n)\|u_n\|^p \geq 0$, then

$$\frac{1}{2}\exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p dx \leq C\exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p^*}dx + o_n(1)\|u_n\|_0,$$

but $\|u_n\|_0^p = \|u_n\|^p + \int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p dx$ and (u_n) is bounded in $\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$, it follows that $o_n(1)\|u_n\|_0^p = o_n(1)\|u_n\|^p + o_n(1)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p dx$ and $\lim_{n\to\infty} o_n(1)\|u_n\|^p = 0$, then

$$(\frac{1}{2}\exp(N\theta_n) - o_n(1))\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p dx \leq C\exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p^*}dx,$$

then, up to a subsequence,

$$\frac{1}{2}\exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p dx \leq C\exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p^*}dx,$$

which implies that (u_n) is bounded in $E_{0,rad}$. Hence, there exists $u \in E_{0,rad}$ such that, up to a subsequence, $u_n \rightharpoonup u$ in $E_{0,rad}$. From Lemma 1.3.9, for all $v \in E_{0,rad}$, we have $\partial_u \widetilde{I}(\theta_n, u_n)v = o_n(1)$, that is,

$$\begin{aligned}
&\exp((N-p)\theta_n)\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla v dx \\
+ \quad &\exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p-2}u_n v dx \\
- \quad &\exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}h(u_n)v dx = o_n(1). \qquad (1.3.13)
\end{aligned}$$

Since $\theta_n \to 0$ in $\mathbb{R}$ and from weak convergence, for all $v \in E_{rad}$, Theorem D.0.6 provides

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla v dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p-2}uv dx$$
$$- \int_{\mathbb{R}^N} |x|^{-bp^*}h(u)v dx = 0,$$

showing that $I'(u)v = 0$, for all $v \in E_{rad}$, that is u is a critical point of I. We are going to show that u is not trivial. Suppose that $u = 0$. From h_1) there exist $\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$\left| \int_{\mathbb{R}^N} |x|^{-bp^*}h(u_n)u_n dx \right| \leq \varepsilon \int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^p dx + C_\varepsilon \int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^q dx.$$

Since (u_n) is bounded in $E_{0,rad}$ and since $E_{0,rad} \hookrightarrow L^q_b(\mathbb{R}^N)$ is compact from Lemma 1.3.4, there exist $M > 0$ such that

$$\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^p dx \leq M, \text{ for all } n \in \mathbb{N}$$

and

$$\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^q dx = o_n(1).$$

Then

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} |x|^{-bp^*}h(u_n)u_n dx \right| \leq \varepsilon M.$$

For $\varepsilon > 0$ small, we conclude that

$$\int_{\mathbb{R}^N} |x|^{-bp^*}h(u_n)u_n dx = o_n(1).$$

This limit combined together with the limit $\partial_u \widetilde{I}(\theta_n, u_n)u_n = o_n(1)$ allows to deduce that $u_n \to 0$ in $E_{0,rad}$. Hence, $\widetilde{I}(\theta_n, u_n) \to 0 = c_*$, which is absurd. Thus, u is a nontrivial critical point of I in $E_{0,rad}$. Finally, u is a nontrivial critical point of I in E_0 using the Principle of Symmetric Criticality (see Theorem A.2.2 in the Appendix A) if we consider the antipodal action of $G = \mathbb{Z}_2$ on E_0.

1.4 The existence of solution for Zero Mass Case

Consider the functional $I_0 : \mathcal{D}^{1,p}_a(\mathbb{R}^N) \to \mathbb{R}$ associated given by

$$I_0(u) = \frac{1}{p}\|u\|^p - \int_{\mathbb{R}^N} |x|^{-bp^*}F(u)dx.$$

Note that I_0 is well-defined and of C^1 class. Moreover, note that

$$I_0'(u)\phi = \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla \phi dx - \int_{\mathbb{R}^N} |x|^{-bp^*}f(u)\phi dx,$$

for all $\phi \in \mathcal{D}^{1,p}_a(\mathbb{R}^N)$. Then, the critical points of I_0 are weak solutions of (ZM) in $\mathcal{D}^{1,p}_a(\mathbb{R}^N)$.

We will restrict the functional I_0 to the space $\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$ under the norm $\|\cdot\|$ to overcome the loss of compactness of the space $\mathcal{D}^{1,p}_a(\mathbb{R}^N)$, then we will use the Principle of Symmetric Criticality to obtain the solutions in the whole space.

Observe that the restriction is necessary only for prove the Lemma 1.3.3 and the Lemma 1.3.4 so that the arguments in this section can be done for the whole space with exception of these lemmas, which are important to show the nontriviality of the weak solutions.

We say that a sequence (u_n) is a Palais-Smale sequence for the functional I_0 if

$$I_0(u_n) \to c_0$$

and

$$\|I_0'(u_n)\| \to 0 \text{ in } (\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N))',$$

where

$$c_0 = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} I_0(\eta(t)) > 0$$

and

$$\Gamma_0 := \{\eta \in C([0,1], \mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)) : \eta(0) = 0,\ I_0(\eta(1)) < 0\}.$$

If every Palais-Smale sequence of I_0 has a strong convergent subsequence, then one says that I_0 satisfies the Palais-Smale condition ((PS) for short).

Lemma 1.4.1. *The functional I_0 satisfies the following conditions:*
(i) There exist ρ_1, $\rho_2 > 0$ such that:

$$I_0(u) \geq \rho_2 \ \text{with} \ \|u\| = \rho_1;$$

(ii) There exists $e \in B^c_{\rho_1}(0)$ with $I_0(e) < 0$ and $\|e\| > \rho_1$.

Proof. i) First of all, observe that

Statement 1.4.2.

$$\int_{\mathbb{R}^N} |x|^{-bp^*} F(u)dx \leq \frac{\varepsilon S_{ab}^{\frac{p^*}{p}}}{p^*} \|u\|^{p^*} + \frac{C_2 C_\varepsilon}{q} \|u\|^q. \tag{1.4.1}$$

Proof. For f_1), given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$f(t) \leq \varepsilon |t|^{p^*-1} + C_\varepsilon |t|^{q-1}, \forall t \in \mathbb{R}. \tag{1.4.2}$$

Thus, (1.4.2) implies that

$$\begin{aligned}
\int_{\mathbb{R}^N} |x|^{-bp^*} F(u)dx &\leq \int_{\mathbb{R}^N} |x|^{-bp^*} |F(u)|dx \\
&\leq \int_{\mathbb{R}^N} \left(\int_0^u |x|^{-bp^*} (\varepsilon |t|^{p^*-1} + C_\varepsilon |t|^{q-1}) dt \right) dx \\
&= \frac{\varepsilon}{p^*} \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p^*} dx + \frac{C_\varepsilon}{q} \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^q dx,
\end{aligned}$$

then

$$\begin{aligned}
\frac{\varepsilon}{p} \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p^*} dx + \frac{C_\varepsilon}{q} \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^q dx &\leq \frac{\varepsilon}{p} \left(S_{a,b} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx \right)^{\frac{p^*}{p}} \\
&+ \frac{C_\varepsilon}{q} \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^q dx \\
&= \frac{\varepsilon}{p} S_{a,b}^{\frac{p^*}{p}} \|u\|^{p^*} + \frac{C_\varepsilon}{q} \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^q dx
\end{aligned}$$

by Caffarelli-Kohn-Nirenberg's inequality.

Finally, the continuous embedding $E \hookrightarrow L_b^q(\mathbb{R}^N)$ gives

$$\frac{\varepsilon}{p} S_{a,b}^{\frac{p^*}{p}} \|u\|^{p^*} + \frac{C_\varepsilon}{q} \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^q dx \leq \frac{\varepsilon}{p} S_{a,b}^{\frac{p^*}{p}} \|u\|^{p^*} + \frac{C_2 C_\varepsilon}{q} \|u\|^q,$$

where C_2 is the constant of the embedding, which proves (1.4.1). □

Using (1.4.1) and taking $\varepsilon > 0$ sufficiently small such that $\|u\|_0 = \rho_1$, we obtain

$$I_0(u) \geq \frac{1}{p}\|u\|^p - \frac{\varepsilon}{p} S_{a,b}^{\frac{p^*}{p}} \|u\|^{p^*} - \frac{C_2 C_\varepsilon}{q} \|u\|^q$$

and the result follows because $q \in (p, p^*)$.

ii) From f_2), there exists $\phi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |x|^{-bp^*} F(\phi) dx > 0.$$

For $t > 0$, setting

$$\omega_t(x) = \phi\left(\frac{x}{t}\right)$$

and deriving $\phi\left(\frac{x}{t}\right)$, we have

$$\begin{aligned} I_0(\omega_t) &= \frac{1}{p} \int_{\mathbb{R}^N} \left|\frac{x}{t}\right|^{-ap} \left|\nabla\left(\phi\left(\frac{x}{t}\right)\right)\right|^p dx - \int_{\mathbb{R}^N} \left|\frac{x}{t}\right|^{-bp^*} F(\phi(x/t)) dx \\ &= \frac{1}{p} \int_{\mathbb{R}^N} \left|\frac{x}{t}\right|^{-ap} t^{-p} \left|\nabla\phi\left(\frac{x}{t}\right)\right|^p dx - \int_{\mathbb{R}^N} \left|\frac{x}{t}\right|^{-bp^*} F(\phi(x/t)) dx, \end{aligned}$$

doing the change of variables $x \mapsto y = x/t$, we get

$$\begin{aligned} &\frac{1}{p} \int_{\mathbb{R}^N} \left|\frac{x}{t}\right|^{-ap} t^{-p} \left|\nabla\phi\left(\frac{x}{t}\right)\right|^p dx - \int_{\mathbb{R}^N} \left|\frac{x}{t}\right|^{-bp^*} F(\phi(x/t)) dx \\ &= \int_{\mathbb{R}^N} |y|^{-ap} t^{-p} |\nabla\phi(y)|^p t^N dy - \int_{\mathbb{R}^N} |y|^{-bp^*} F(\phi(y)) t^N dy \\ &= t^{N-p} \int_{\mathbb{R}^N} |y|^{-ap} |\nabla\phi(y)|^p dy - t^N \int_{\mathbb{R}^N} |y|^{-bp^*} F(\phi(y)) dy, \end{aligned}$$

therefore

$$I_0(\omega_t) = t^{N-p} \int_{\mathbb{R}^N} |y|^{-ap} |\nabla\phi(y)|^p dy - t^N \int_{\mathbb{R}^N} |y|^{-bp^*} F(\phi(y)) dy \to -\infty,$$

as $t \to \infty$. Then, there exists $\bar{t} > 0$ large such that $e = \omega_{\bar{t}}$ satisfies $I_0(e) < 0$ and $\|e\| > \rho_2$. Note also $c_* \geq \rho_2$. □

As in the previous section, we consider an auxiliary functional $\widetilde{I_0} \in C^1(\mathbb{R} \times \mathcal{D}_{a,rad}^{1,p}(\mathbb{R}^N), \mathbb{R})$ given by

$$\widetilde{I_0}(\theta, u) = \frac{\exp(N-p)\theta}{p} \|u\|_0^p - \exp(N\theta) \int_{\mathbb{R}^N} |x|^{-bp^*} F(u) dx. \tag{1.4.3}$$

This functional will be important to show the boundedness of a (PS) sequence that we will find.

Statement 1.4.3. *The following properties hold, for all* $(\theta, u) \in \mathbb{R} \times \mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$,

$$\widetilde{I}_0(0, u) = I_0(u),$$
$$\widetilde{I}_0(\theta, u) = I_0(u(x/\exp(\theta)).$$

Proof. The proof is the same of the Statement 1.3.5. □

We equip a standard product norm

$$\|(\theta, u)\|^p_{\mathbb{R}\times\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)} = |\theta|^p + \|u\|^p$$

to $\mathbb{R} \times \mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$. Now we prove that $\widetilde{I}_0$ satisfies the Mountain Pass geometry.

Lemma 1.4.4. *The functional* $\widetilde{I}_0$ *satisfies the following conditions:*
(i) There exist ρ_1, $\rho_2 > 0$ *such that:*

$$\widetilde{I}_0(\theta, u) \geq \rho_2 \ \text{ with } \ \|(\theta, u)\|_{\mathbb{R}\times\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)} = \rho_1;$$

(ii) There exists $\widetilde{e} \in B^c_{\rho_1}(0)$ *with* $\widetilde{I}(\widetilde{e}) < 0$ *and* $\|\widetilde{e}\|_{\mathbb{R}\times\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)} > \rho_1$.

Proof. The item i) follows by using the same argument of Lemma 1.4.1 and for item ii) it is sufficient to take $\widetilde{e} = (0, e)$. Indeed,

$$\widetilde{I}_0(\theta, u) = \frac{\exp((N-p)\theta)}{p}\|u\|^p - \exp(N\theta)\int_{\mathbb{R}^N} |x|^{-bp^*}F(u)dx$$

By (1.4.1),

$$\begin{aligned}
&\frac{\exp((N-p)\theta)}{p}\|u\|^p - \exp(N\theta)\int_{\mathbb{R}^N} |x|^{-bp^*}F(u)dx \\
&\geq \frac{\exp(N\theta)}{p}\|u\|^p - \exp(N\theta)\frac{\varepsilon}{p}S_{a,b}^{\frac{p^*}{p}}\|u\|^{p^*} - \exp(N\theta)\frac{C_2C_\varepsilon}{q}\|u\|^q \\
&= \frac{\exp(N\theta)}{p}\|u\|^p - \varepsilon\frac{\exp(N\theta)}{p}S_{a,b}^{\frac{p^*}{p}}\|u\|^{p^*} - \exp(N\theta)\frac{C_2C_\varepsilon}{q}\|u\|^q,
\end{aligned}$$

therefore

$$\widetilde{I}_0(\theta, u) \geq \frac{\exp(N\theta)}{p}\|u\|^p - \varepsilon\frac{\exp(N\theta)}{p}S_{a,b}^{\frac{p^*}{p}}\|u\|^{p^*} - \exp(N\theta)\frac{C_2C_\varepsilon}{q}\|u\|^q,$$

which proves item $i)$ if $\|u\|_0 = \rho_1 > 0$ is sufficiently small because $q \in (p, p^*)$. Finally,

$$\widetilde{I}_0(\widetilde{e}) = \widetilde{I}_0(0, e) = I_0(e),$$

which is negative as we saw in the Lemma 1.4.1 and $\|\widetilde{e}\|_{\mathbb{R}\times\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)} = \|e\| > \rho_1$. □

In what follows, we define the Mountain Pass level $\widetilde{c}_0$ for $\widetilde{I}_0$ by

$$\widetilde{c}_0 = \inf_{\eta\in\Gamma}\max_{t\in[0,1]} \widetilde{I}_0(\eta(t)) > 0$$

and

$$\widetilde{\Gamma} := \{\eta \in C([0,1], \mathbb{R} \times \mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)) : \eta(0) = 0, \ \widetilde{I}_0(\eta(1)) < 0\}.$$

Note that $\widetilde{c}_0 \geq \rho_2$.

Lemma 1.4.5. *The Mountain Pass levels of I_0 and $\widetilde{I}_0$ coincide, namely $c_0 = \widetilde{c}_0$.*

Proof. Note that $\Gamma \cong \{0\} \times \Gamma \subset \widetilde{\Gamma}$, which implies $\widetilde{c}_* \leq c_*$. On the other hand, consider $\widetilde{\gamma} \in \widetilde{\Gamma}$ arbitrary. Then, for each $t \in [0,1]$, we have $\widetilde{\gamma}(t) = (\theta_t, u_t)$. Define $\gamma(t) := u_t\left(\frac{x}{\exp(\theta_t)}\right)$. From the Statement 1.4.3, we conclude $\widetilde{I}(\widetilde{\gamma}_t) = \widetilde{I}(\theta_t, u_t) = I(u_t(x/\exp(\theta_t))) = I(\gamma(t))$ for each $t \in [0,1]$. Hence $\gamma \in \Gamma$, where we derive $\widetilde{c}_* \geq c_*$. □

Lemma 1.4.6. *Let $\varepsilon > 0$. Suppose that $\widetilde{\eta} \in \widetilde{\Gamma_0}$ satisfies*

$$\max_{t\in[0,1]} \widetilde{I}_0(\widetilde{\eta}) \leq c_0 + \varepsilon,$$

then, there exists $(\theta, u) \in \mathbb{R} \times \mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$ such that

- $dist_{\mathbb{R}\times\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)}((\theta,u), \widetilde{\eta}([0,1])) \leq 2\sqrt{\varepsilon}$;
- $\widetilde{I}_0(\theta, u) \in [c_0 - \varepsilon, c_0 + \varepsilon]$;
- $\|D\widetilde{I}_0(\theta, u)\|_{\mathbb{R}\times(\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)^*)} \leq 2\sqrt{\varepsilon}$.

Proof. The proof is the same proof of the Lemma 1.3.8. □

The proof of next lemma is the same proof of Lemma 1.3.9.

Lemma 1.4.7. *There exists a sequence $((\theta_n, u_n)) \subset \mathbb{R} \times \mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$ such that, as $n \to \infty$, we get*

- $\theta_n \to 0$;
- $\widetilde{I}_0(\theta_n, u_n) \to c_0$;
- $\partial_\theta \widetilde{I}_0(\theta_n, u_n) \to 0$;
- $\partial_u \widetilde{I}_0(\theta_n, u_n) \to 0$, *strongly in* $(\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N))^*$.

1.4.1 Proof of Theorem 0.0.4

By Lemma 1.4.7, there exists a sequence $((\theta_n, u_n)) \subset \mathbb{R} \times \mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$ such that,

$$\frac{(\exp(N-p)\theta_n)}{p}\|u_n\|^p - \exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*} F(u_n)dx = c_0 + o_n(1); \tag{1.4.4}$$

$$(N-p)\frac{(\exp(N-p)\theta_n)}{p}\|u_n\|^p - N\exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*} F(u_n)dx = o_n(1); \tag{1.4.5}$$

$$\exp((N-p)\theta_n)\|u_n\|^p - \exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*} f(u_n)u_n dx = o_n(1)\|u_n\|. \tag{1.4.6}$$

From (1.4.4) and (1.4.5) and since $N > p$, we have

$$(\exp(N-p)\theta_n)\|u_n\|^p = Nc_0 + o_n(1). \tag{1.4.7}$$

Since $\theta_n \to 0$, we have that (u_n) is bounded in $\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$ and $L^{p^*}_b(\mathbb{R}^N)$.

Hence, there exists $u \in \mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$ such that, up to a subsequence, $u_n \rightharpoonup u$ in $\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$. From Lemma 1.4.7, for all $v \in \mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$, we have $\partial_u \widetilde{I}_0(\theta_n, u_n)v = o_n(1)$, that is,

$$\exp((N-p)\theta_n) \int\limits_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n v dx - \exp(N\theta_n) \int\limits_{\mathbb{R}^N} |x|^{-bp^*} f(u_n) v dx = o_n(1) \quad (1.4.8)$$

Since $\theta_n \to 0$ in $\mathbb{R}$ and from weak convergence, for all $v \in \mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$, Theorem D.0.7 provides

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla v dx - \int_{\mathbb{R}^N} |x|^{-bp^*} f(u) v dx = 0,$$

showing that $I_0'(u)v = 0$, for all $v \in \mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$, that is u is a critical point of I_0. We are going to show that u is not trivial. Suppose that $u = 0$. From f_1) there exist $\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$\left| \int_{\mathbb{R}^N} |x|^{-bp^*} f(u_n) u_n dx \right| \leq \varepsilon \int_{\mathbb{R}^N} |x|^{-bp^*} |u_n|^p dx + C_\varepsilon \int_{\mathbb{R}^N} |x|^{-bp^*} |u_n|^q dx.$$

Since (u_n) is bounded in $L_b^{p^*}(\mathbb{R}^N)$ and since $\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N) \hookrightarrow L^q_b(\mathbb{R}^N)$ is compact from Lemma 1.3.4, there exist $M > 0$ such that

$$\int_{\mathbb{R}^N} |x|^{-bp^*} |u_n|^{p^*} dx \leq M, \text{ for all } n \in \mathbb{N}$$

and

$$\int_{\mathbb{R}^N} |x|^{-bp^*} |u_n|^q dx = o_n(1).$$

Then

$$\limsup_{n \to \infty} \left| \int_{\mathbb{R}^N} |x|^{-bp^*} f(u_n) u_n dx \right| \leq \varepsilon M.$$

For $\varepsilon > 0$ small, we conclude that

$$\int_{\mathbb{R}^N} |x|^{-bp^*} f(u_n) u_n dx = o_n(1).$$

This limit combined together with the limit $\partial_u \widetilde{I}_0(\theta_n, u_n)u_n = o_n(1)$ allows to deduce that $u_n \to 0$ in $\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$. Hence, $\widetilde{I}_0(\theta_n, u_n) \to 0 = c_0$, which is absurd. Thus, u is a nontrivial critical point of I_0 in $\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$. Finally, u is a nontrivial critical point of I_0 in $\mathcal{D}^{1,p}_a(\mathbb{R}^N)$ using the Principle of Symmetric Criticality (see Theorem A.2.2 in the Appendix A) if we consider the antipodal action of $G = \mathbb{Z}_2$ on $\mathcal{D}^{1,p}_a(\mathbb{R}^N)$.

Chapter 2

Existence and concentration of ground state solutions for a class of subcritical, critical or supercritical Caffarelli-Kohn-Nirenberg type problems

In this chapter, we use the Mountain Pass Theorem to prove the existence of ground state solutions for a class of problems with subcritical, critical and supercritical growth. In the critical case, we use an auxiliary problem and the hypothesis (f_5) to show that the mountain pass level is below to a specific constant, which allow us to prove the existence of ground state solution. In the supercritical case, we define a truncation function and we prove the existence of a ground state solution for an auxiliary problem defined with respect to this truncation, then we use Moser's Iteration Method to show that the norm of the solution of the auxiliary problem is below to 1, then this solution will be a solution for the problem with supercritical growth. Also, we prove a result about concentration. This chapter is based on [27]. The contributions of this chapter are the proof of a concentration result, the existence of a ground state solution for the supercritical problem for a class of Caffarelli-Kohn-Nirenberg type problems and the proof of some estimates more refined than the estimates in the work that we based on (see Theorem 2.2.3).

2.1 Introduction

This chapter is focused to prove existence and concentration of ground state solutions for a class of subcritical, critical or supercritical Caffarelli-Kohn-Nirenberg type problems. More precisely, we are going to study the following class of quasilinear problems

$$-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}[1+\mu V(x)]|u|^{p-2}u = |x|^{-bp^*}[f(u)+\varrho|u|^{\sigma-2}u], \quad (P_{\mu,\varrho,\sigma})$$

in $\mathbb{R}^N$, where $1<p<N$, $0\le a<\frac{N-p}{p}$, $a<b\le a+1$, $p^*=p^*(a,b)=\frac{pN}{N-dp}$, $d=1+a-b$ and $\mu>0$. We are considering three cases. The first case is the subcritical growth on the nonlinearity, i.e. when $\varrho=0$. In this case we have

$$-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}[1+\mu V(x)]|u|^{p-2}u = |x|^{-bp^*}f(u), \qquad (P_{\mu,0,\sigma})$$

in $\mathbb{R}^N$.

The second case is the critical growth on the nonlinearity, i.e. when $\varrho = 1$ and $\sigma = p^*$. In this case we have

$$\begin{aligned}-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + &|x|^{-bp^*}[1+\mu V(x)]|u|^{p-2}u \\ =&|x|^{-bp^*}f(u) + |x|^{-bp^*}|u|^{p^*-2}u, \quad \text{in} \quad \mathbb{R}^N.\end{aligned} \tag{$P_{\mu,1,p^*}$}$$

The last case is the supercritical growth on the nonlinearity, i.e. when $\varrho = 1$ and $\sigma > p^*$. In this case we have

$$\begin{aligned}-\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + &|x|^{-bp^*}[1+\mu V(x)]|u|^{p-2}u \\ =&|x|^{-bp^*}f(u) + |x|^{-bp^*}|u|^{\sigma-2}u, \quad \text{in} \quad \mathbb{R}^N.\end{aligned} \tag{$P_{\mu,1,\sigma}$}$$

In order to state the main result, we need to introduce the hypotheses on the functions V and f. The condition in $V \in C(\mathbb{R}^N, \mathbb{R})$ are the following:

(V_1) The potential V is nonnegative, that is,

$$V(x) \geq 0, \text{ for all } x \in \mathbb{R}^N;$$

(V_2) The set $\Omega := \text{int}\left\{\ x \in \mathbb{R}^N \mid V(x) = 0\right\}$ is a non-empty bounded open set with smooth boundary $\partial\Omega$;

(V_3) There exists $V^* > 0$, such that

$$meas\left(\left\{x \in \mathbb{R}^N : V(x) \leq V^*\right\}\right) < \infty.$$

The hypotheses on the nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$ are the following:.

(f_1)

$$\lim_{|s|\to 0} \frac{f(s)}{|s|^{p-1}} = 0 \quad \text{and} \quad f(s) = 0, \quad \text{for all} \quad s \leq 0;$$

(f_2) There exists $p < r < p^*$ such that

$$\lim_{|s|\to\infty} \frac{f(s)}{|s|^{r-1}} = 0;$$

(f_3) There exists $\theta \in (p, p^*)$, such that

$$0 < \theta F(s) \leq f(s)s, \quad \text{for} \quad s \neq 0,$$

where $F(s) = \displaystyle\int_0^s f(t)dt$;

(f_4) $s \mapsto \dfrac{f(s)}{s^{p-1}}$ is increasing;

(f_5) There exist $\tau \in (p, p^*)$ and $\lambda^* > 1$ such that

$$f(s) \geq \lambda |s|^{\tau-1}, \quad \text{for all } s \geq 0,$$

for a fixed $\lambda > \lambda^*$ and λ^* will be fixed latter.

We use $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ that is the completion of the $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with the norm

$$\|u\|^p = \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx,$$

where $\mathcal{C}_0^\infty(\mathbb{R}^N)$ is the space of smooth functions with compact support.

We use $E = \{u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^{-bp^*}[1+\mu V(x)]|u|^p dx < \infty\}$ with the norm

$$\|u\|_\mu^p = \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx + \int_{\mathbb{R}^N} |x|^{-bp^*}[1+\mu V(z)]|u|^p dx.$$

We also use $E_0 = \{u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^p dx < \infty\}$ with the norm

$$\|u\|_0^p = \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^p dx.$$

Let us denote by

$$L_b^s(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \quad u \text{ is measurable and } \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^s dx < \infty \right\}.$$

Using an inequality established by Caffarelli, Kohn, and Nirenberg given by [19]

$$\left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*} dx \right)^{p/p^*} \leq S_{a,b} \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx,$$

we conclude that the embedding $\mathcal{D}_a^{1,p}(\mathbb{R}^N) \hookrightarrow L_b^{p^*}(\mathbb{R}^N)$ is continuous. Moreover, by interpolation, we also conclude that $E \hookrightarrow L_b^s(\mathbb{R}^N)$ and $E_0 \hookrightarrow L_b^s(\mathbb{R}^N)$ are continuous, for $s \in [p, p^*]$.

Here is the main result of this chapter.

Theorem 2.1.1. *Assume that $(f_1)-(f_4)$ and $(V_1)-(V_3)$ are satisfied. Then,*

(i) there exists $\mu^* > 0$ such that problem $(P_{\mu,0,\sigma})$ has a ground state solution $u_\mu \in E$ for all $\mu > \mu^*$.

(ii) if the function f satisfies (f_5) there exist positive numbers λ^* and μ^{**}, such that problem $(P_{\mu,1,p^*})$ or problem $(P_{\mu,1,\sigma})$ has a ground state solution $u_\mu \in E$ for all $\mu > \mu^{**}$ and for all $\lambda > \lambda^*$.

(iii) Moreover, as $\mu \to +\infty$, the sequence (u_μ) converges in E to a ground state solution $u_\infty \in E(\Omega)$ of the problem

$$\begin{cases} -div\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}|u|^{p-2}u = |x|^{-bp^*}f(u) + |x|^{-bp^*}|u|^{\sigma-2}u, & in\ \Omega, \\ u = 0, & on\ \ \partial\Omega. \end{cases} \tag{$P_{0,1,\sigma}$}$$

where $E(\Omega)$ is defined by $E(\Omega) = \{u \in \mathcal{D}_{0,a}^{1,p}(\Omega) : \int_\Omega |x|^{-bp^*}|u|^p dx < \infty\}$ with the norm

$$\|u\|_{0,\Omega}^p = \int_\Omega |x|^{-ap}|\nabla u|^p dx + \int_\Omega |x|^{-bp^*}|u|^p dx.$$

2.2 Variational framework and some preliminary results for the subcritical ($\varrho = 0$) and for the critical case ($\varrho = 1$) and ($\sigma = p^*$)

In this section, we are considering the cases $\varrho = 0$ or $\varrho = 1$ with $\sigma = p^*$. More specifically,

$$\begin{cases} -\mathrm{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) \quad +|x|^{-bp^*}[1+\mu V(x)]|u|^{p-2}u = |x|^{-bp^*}f(u) + \varrho|x|^{-bp^*}|u|^{p^*-2}u, \\ u \in E. \end{cases} \tag{P_{μ,ϱ,p^*}}$$

Since the approach is variational, let us consider the energy functional associated $I_{\mu,\varrho} : E \to \mathbb{R}$ given by

$$\begin{aligned} I_{\mu,\varrho}(u) :=& \frac{1}{p}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} |x|^{-bp^*}[1+\mu V(x)]|u|^p dx \\ & - \int_{\mathbb{R}^N} |x|^{-bp^*}F(u)dx - \frac{\varrho}{p^*}\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*}dx. \end{aligned} \tag{2.2.1}$$

By standard arguments, it is possible to prove that $I_{\mu,\varrho} \in C^1(E,\mathbb{R})$ and each critical point of $I_{\mu,\varrho}$ is a weak solution of our problem.

Note that (f_1) and (f_2) imply that for any given $\xi > 0$, there is a constant $C_\xi > 0$, such that

$$|f(s)| \le \xi|s|^{p-1} + C_\xi|s|^{r-1}, \quad \text{for all} \quad s \in \mathbb{R}. \tag{2.2.2}$$

Moreover, by (f_3), for $s > 1$ there exists a positive constant D_1 such that

$$F(s) \ge D_1|s|^\theta, \quad \text{for all} \quad s > 1. \tag{2.2.3}$$

To use the Mountain Pass Theorem [7], we define the Palais-Smale compactness condition. We say that a sequence $(u_n) \subset E$ is a Palais-Smale sequence at level $c_{\mu,\varrho}$ for the functional $I_{\mu,\varrho}$ if

$$I_{\mu,\varrho}(u_n) \to c_{\mu,\varrho}$$

and

$$\|I'_{\mu,\varrho}(u_n)\| \to 0, \text{ in } (E)',$$

where

$$c_{\mu,\varrho} = \inf_{\eta\in\Gamma}\max_{t\in[0,1]} I_{\mu,\varrho}(\eta(t)) > 0 \tag{2.2.4}$$

and

$$\Gamma := \{\eta \in C([0,1],E) : \eta(0) = 0,\ I_{\mu,\varrho}(\eta(1)) < 0\}.$$

If every Palais-Smale sequence of $I_{\mu,\varrho}$ has a strong convergent subsequence, then one says that $I_{\mu,\varrho}$ satisfies the Palais-Smale condition ((PS) for short). Now let us show that the functional $I_{\mu,\varrho}$ has the mountain pass geometry.

We say that a solution $u_{\mu,\varrho} \in E \setminus \{0\}$ of (P_{μ,ϱ,p^*}) is a ground solution if $I_{\mu,\varrho}(u_{\mu,\varrho}) = \inf\limits_{\mathcal{N}_\mu} I_{\mu,\varrho}(u)$, where $\mathcal{N}_{\mu,\varrho}$ is the Nehari manifold associated to $I_{\mu,\varrho}$ given by

$$\mathcal{N}_{\mu,\varrho} := \{u \in E : u \neq 0 : {I_{\mu,\varrho}}'(u)u = 0\}.$$

Lemma 2.2.1. *The functional $I_{\mu,\varrho} : E \to \mathbb{R}$ and the constant $c_{\mu,\varrho}$ satisfy the following conditions:*

(i) There are positive numbers α and ρ, such that

$$I_{\mu,\varrho}(u) \geq \alpha \quad \text{if} \quad \|u\|_\mu = \rho;$$

(ii) For any positive function $w \in C_0^\infty(\Omega)$, we have

$$\lim_{t\to\infty} I_{\mu,\varrho}(tw) = -\infty;$$

(iii) There exists a positive constant Υ_1 which does not depend of μ, such that $c_{\mu,\varrho} \leq \Upsilon_1$.

Proof. Using (2.2.2), we have

$$I_{\mu,\varrho}(u) \geq \frac{1}{p}\|u\|_\mu^p - \frac{\xi}{p}\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^p dx - \frac{C_\xi}{r}\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^r dx - \frac{\varrho}{p^*}\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*} dx.$$

Therefore, using the Sobolev embeddings and taking ξ and $\|u\|_\mu$ sufficiently small, there are constants $C_1,\ C_2 > 0$ such that

$$I_{\mu,\varrho}(u) \geq C_1\|u\|_\mu^p - C_2\|u\|_\mu^r - C_3\varrho\|u\|_\mu^{p^*}$$

and the item (i) is proved.

Now we are going to show that the item (ii) holds. Since for all $x \in \Omega$, we have $\mu V(x) = 0$, for a positive function $w \in C_0^\infty(\Omega)$ with $\|w\|_\infty > 1$ and $t > 0$, we can use (2.2.3) to obtain

$$I_{\mu,\varrho}(tw) \leq \frac{t^p}{p}\|w\|_p^p - D_1 t^\theta \int_{\mathbb{R}^N} |x|^{-bp^*}|w|^\theta dx.$$

Since $p < \theta$, this completes the proof of the item (ii). The proof of the item (iii) follows by the last inequality and the item (i) because

$$0 < c_{\mu,\varrho} \leq \max_{t\geq 0}\left[\frac{t^p}{p}\|w\|_p^p - D_1 t^\theta \int_{\mathbb{R}^N} |x|^{-bp^*}|w|^\theta dx\right] := \Upsilon_1,$$

where D_1 was defined in (2.2.3). □

From [49, Lemma 1.15] and Lemma 2.2.1 ensures that there exists a sequence $(PS)_{c_{\mu,\varrho}}$ for the functional $I_{\mu,\varrho}$, where $c_{\mu,\varrho}$ is set in (2.2.4).

Lemma 2.2.2. *Let (u_n) be a $(PS)_{c_{\mu,\varrho}}$ sequence of the functional $I_{\mu,\varrho}$. Then the following statements hold.*

(i) The sequence (u_n) is bounded in E.

(ii) There exists a positive constant Υ_2, which does not depend on μ, such that

$$\limsup_{\mu\to\infty} \|u_n\|_\mu \leq \Upsilon_2.$$

Consequently, $\liminf_{\mu\to+\infty} c_{\mu,\varrho} > 0$.

Proof. Since (u_n) is a $(PS)_{c_{\mu,\varrho}}$ sequence of the functional $I_{\mu,\varrho}$, then, by (f_3),

$$\begin{aligned}
o_n(1) + c_{\mu,\varrho} + o_n(1)\|u_n\|_\mu =& I_{\mu,\varrho}(u_n) - \frac{1}{\theta} I'_{\mu,\varrho}(u_n)u_n \\
=& \left(\frac{1}{p} - \frac{1}{\theta}\right) \|u_n\|_\mu^p + \frac{1}{\theta} \int\limits_{\mathbb{R}^N} |x|^{-bp^*} \left[f(u_n)(u_n) - \theta F(u_n) \right] \, dx \\
& + \varrho \left(\frac{1}{\theta} - \frac{1}{p^*}\right) \int\limits_{\mathbb{R}^N} |x|^{-bp^*} |u_n|^{p^*} dx \\
\geq& \left(\frac{1}{p} - \frac{1}{\theta}\right) \|u_n\|_\mu^p.
\end{aligned} \tag{2.2.5}$$

Then, we can concluded that (u_n) is bounded in E.

Let us show that the item (ii) holds. Using the item (i) we can consider $R_{\mu,\varrho} := \limsup\limits_{n\to\infty} \|u_n\|_\mu$. We suppose, by contradiction, that $R_{\mu,\varrho} \to +\infty$ when $\mu \to +\infty$. Hence for μ large enough we can guarantee that there exists $m_{\mu,\varrho} \in \mathbb{N}$ such that

$$\|u_{m_{\mu,\varrho}}\|_\mu \geq \frac{R_{\mu,\varrho}}{2} \to +\infty, \quad \text{when} \quad \mu \to +\infty.$$

Therefore, using (2.2.5) and the item (iii) of Proposition 2.2.1, we conclude that

$$\frac{\Upsilon_1}{\|u_{m_{\mu,\varrho}}\|_\mu} + o_\mu(1) \geq \left(\frac{1}{p} - \frac{1}{\theta}\right) \|u_{m_{\mu,\varrho}}\|_\mu^{p-1}.$$

This absurd shows the first part of item (ii). To conclude the item (ii) let us suppose by contradiction that $\liminf\limits_{\mu\to+\infty} c_{\mu,\varrho} = 0$. Then using the inequality (2.2.5), we see that

$$o_n(1) + o_n(1)\|u_n\|_\mu \geq \left(\frac{1}{p} - \frac{1}{\theta}\right) \|u_n\|_\mu^p - c_{\mu,\rho}.$$

Taking $\limsup_{\mu\to+\infty}$,

$$o_n(1) + o_n(1) \limsup_{\mu\to+\infty} \|u_n\|_\mu \geq \left(\frac{1}{p} - \frac{1}{\theta}\right) \limsup_{\mu\to+\infty} \|u_n\|_\mu^p + \limsup_{\mu\to+\infty} (-c_{\mu,\rho}),$$

then

$$o_n(1) + o_n(1) \limsup_{\mu\to+\infty} \|u_n\|_\mu \geq \left(\frac{1}{p} - \frac{1}{\theta}\right) \limsup_{\mu\to+\infty} \|u_n\|_\mu^p - \liminf_{\mu\to+\infty} c_{\mu,\rho}.$$

By hypothesis,

$$o_n(1) + o_n(1) \limsup_{\mu\to+\infty} \|u_n\|_\mu \geq \left(\frac{1}{p} - \frac{1}{\theta}\right) \limsup_{\mu\to+\infty} \|u_n\|_\mu^p - o_\mu(1).$$

$\limsup\limits_{\mu\to\infty} \|u_n\|_\mu \leq \Upsilon_2$ provides

$$o_n(1) + o_n(1)\Upsilon_2 \geq \left(\frac{1}{p} - \frac{1}{\theta}\right) \limsup_{\mu\to+\infty} \|u_n\|_\mu^p - o_\mu(1),$$

i.e.,

$$\|u_n\|_\mu = o_n(1) + o_\mu(1). \tag{2.2.6}$$

Since $I'_{\mu,\varrho}(u_n)u_n = o_n(1)$, we get

$$\|u_n\|_\mu^p = \int\limits_{\mathbb{R}^N} |x|^{-bp^*} f(u_n)u_n dx + \varrho \int\limits_{\mathbb{R}^N} |x|^{-bp^*} |u_n|^{p^*} dx + o_n(1).$$

Using Sobolev embedding and (2.2.2) there exists a constant $C > 0$ which is independent of μ such that

$$o_n(1) + (1 - \xi C)\, \|u_n\|_\mu^p \le C_\xi C \int\limits_{\mathbb{R}^N} |x|^{-bp^*}\, |u_n|^r\, dx + \varrho \int\limits_{\mathbb{R}^N} |x|^{-bp^*}\, |u_n|^{p^*}\, dx \le C[\|u_n\|_\mu^r + \varrho \|u_n\|_\mu^{p^*}].$$

Hence

$$(1 - \xi C) + o_n(1) \le C \left[\|u_n\|_\mu^{r-p} + \varrho \|u_n\|_\mu^{p^*-p} \right],$$

which is a contradiction with (2.2.6). Then, we conclude that $\liminf\limits_{\mu \to +\infty} c_{\mu,\varrho} > 0$. □

The next result is important in order to show that the solutions of our problem is not trivial.

Theorem 2.2.3. *[Lions' Lemma in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$] Suppose there exist $R > 0$, $p \le q < p^*$ and a bounded sequence (u_n) in E such that*

$$\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |x|^{-bp^*} |u_n|^q dx \to 0, \quad \text{as } n \to \infty. \tag{2.2.7}$$

Then $u_n \to 0$ in $L_b^s(\mathbb{R}^N)$ for all $s \in (p, p^*)$.

Proof. Let $q < r < p^*$ and $(u_n) \subset E$ that (2.2.7) holds. As $q < r < p^*$, it remains that $\frac{1}{p^*} < \frac{1}{r} < \frac{1}{q}$, therefore there exists $\theta \in (0,1)$ such that $\frac{1}{r} = \theta \frac{1}{q} + (1-\theta)\frac{1}{p^*}$ by the convexity of the interval $\left(\frac{1}{p^*}, \frac{1}{q}\right)$. Thus,

$$\frac{\theta r}{q} + \frac{(1-\theta) r}{p^*} = r \left(\frac{\theta}{q} + \frac{(1-\theta)}{p^*} \right) = r \frac{1}{r} = 1.$$

This allows us to use Hölder's inequality to get

$$\begin{aligned}
\int_{B_R(y)} |x|^{-bp^*} |u_n|^r dx &= \int_{B_R(y)} |x|^{-b\left(\frac{\theta r}{q}\right)p^*} |u_n|^{\theta r} |x|^{-b\left(\frac{(1-\theta)r}{p^*}\right)p^*} |u_n|^{(1-\theta)r} dx \\
&\le \left(\int_{B_R(y)} \left(|x|^{-b\left(\frac{\theta r}{q}\right)p^*} |u_n|^{\theta r} \right)^{\frac{q}{\theta r}} dx \right)^{\frac{\theta r}{q}} \\
&\quad \left(\int_{B_R(y)} \left(|x|^{-b\left(\frac{(1-\theta)r}{p^*}\right)p^*} |u_n|^{(1-\theta)r} \right)^{\frac{p^*}{(1-\theta)r}} dx \right)^{\frac{(1-\theta)r}{p^*}} \\
&= \left(\int_{B_R(y)} |x|^{-bp^*} |u_n|^q dx \right)^{\frac{\theta r}{q}} \left(\int_{B_R(y)} |x|^{-bp^*} |u_n|^{p^*} dx \right)^{\frac{(1-\theta)r}{p^*}}.
\end{aligned} \tag{2.2.8}$$

Caffarelli-Kohn-Nirenberg's inequality implies that

$$\int_{B_R(y)} |x|^{-bp^*}|u_n|^r dx \le \left(\int_{B_R(y)} |x|^{-bp^*}|u_n|^q dx\right)^{\frac{\theta r}{q}} \left(\int_{B_R(y)} |x|^{-bp^*}|u_n|^{p^*} dx\right)^{\frac{(1-\theta)r}{p^*}}$$

$$\le S_{a,b}^{\frac{(1-\theta)r}{p}} \left(\left(\int_{B_R(y)} |x|^{-bp^*}|u_n|^q dx\right)^{\frac{1}{q}}\right)^{\theta r} \|u_n\|_0^{(1-\theta)r}, \qquad (2.2.9)$$

where

$$\frac{\theta}{q} + \frac{(1-\theta)}{p^*} = \frac{1}{r}.$$

Observe that

$$\frac{\theta}{q} + \frac{(1-\theta)}{p^*} = \frac{1}{r} \iff \theta = \frac{(p^*-r)}{(p^*-q)}\frac{q}{r}.$$

Defining

$$\lambda := 1-\theta = 1 - \left(\left(\frac{p^*-r}{p^*-q}\right)\frac{q}{r}\right) = \frac{(r-q)}{(p^*-q)}\frac{p^*}{r},$$

(2.2.9) reduces to

$$\int_{B_R(y)} |x|^{-bp^*}|u_n|^r dx$$

$$\le \quad S_{a,b}^{\frac{\lambda r}{p}} \left(\left(\int_{B_R(y)} |x|^{-bp^*}|u_n|^q dx\right)^{\frac{1}{q}}\right)^{(1-\lambda)r} \left(\int_{B_R(y)} |x|^{-ap}|\nabla u|^p dx + \int_{B_R(y)} |x|^{-bp^*}|u|^p dx\right)^{\frac{\lambda r}{p}} \qquad (2.2.10)$$

Cover $\mathbb{R}^N$ by balls of radius R such that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls and consider a partition (P_m) of $\mathbb{R}^N$ such that $P_m \cap B_R(y_k) = P_m$ or $P_m \cap B_R(y_k) = \emptyset$ for every $m, k \in \mathbb{N}$ and for each $m \in \mathbb{N}$, P_m is contained in at most $N+1$ balls of the covering. Then (2.2.10) provides

$$\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^r dx \le \sum_{k=1}^{N+1} \int_{B_R(y_k)} |x|^{-bp^*}|u_n|^r dx$$

$$\le S_{a,b}^{\frac{\lambda r}{p}} \sup_{y\in\mathbb{R}^N} \left\{\left(\left(\int_{B_R(y)} |x|^{-bp^*}|u_n|^q dx\right)^{\frac{1}{q}}\right)^{(1-\lambda)r}\right\}$$

$$\sum_{k=1}^{N+1} \left(\int_{B_R(y_k)} |x|^{-ap}|\nabla u|^p dx + \int_{B_R(y_k)} |x|^{-bp^*}|u|^p dx\right)^{\frac{\lambda r}{p}}$$

$$= S_{a,b}^{\frac{\lambda r}{p}} \sup_{y\in\mathbb{R}^N} \left\{\left(\left(\int_{B_R(y)} |x|^{-bp^*}|u_n|^q dx\right)^{\frac{1}{q}}\right)^{(1-\lambda)r}\right\}$$

$$\sum_{k=1}^{N+1} \left(\sum_{m=1}^{N+1} \int_{B_R(y_k)\cap P_m} |x|^{-ap}|\nabla u_n|^p dx + \int_{B_R(y_k)\cap P_m} |x|^{-bp^*}|u_n|^p dx\right)^{\frac{\lambda r}{p}}$$

$$
\begin{aligned}
&\leq S_{a,b}^{\frac{\lambda r}{p}} \sup_{y\in\mathbb{R}^N}\left\{\left(\left(\int_{B_R(y)}|x|^{-bp^*}|u_n|^q dx\right)^{\frac{1}{q}}\right)^{(1-\lambda)r}\right\}\\
&+(N+1)\left(\sum_{m=1}^{N+1}\int_{P_m}|x|^{-ap}|\nabla u_n|^p dx+\int_{P_m}|x|^{-bp^*}|u_n|^p dx\right)^{\frac{\lambda r}{p}}\\
&= S_{a,b}^{\frac{\lambda r}{p}}(N+1) \sup_{y\in\mathbb{R}^N}\left\{\left(\left(\int_{B_R(y)}|x|^{-bp^*}|u_n|^q dx\right)^{\frac{1}{q}}\right)^{(1-\lambda)r}\right\}\\
&\left(\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^p dx+\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p dx\right)^{\frac{\lambda r}{p}}\\
&\leq S_{a,b}^{\frac{\lambda r}{p}}(N+1) \sup_{y\in\mathbb{R}^N}\left\{\left(\left(\int_{B_R(y)}|x|^{-bp^*}|u_n|^q dx\right)^{\frac{1}{q}}\right)^{(1-\lambda)r}\right\}\|u_n\|_\mu^{\lambda r},
\end{aligned}
$$

Then $u_n \to 0$ in $L_b^r(\mathbb{R}^N)$ by the hypothesis (2.2.7) and the boundedness of the sequence $(u_n) \subset E$. Now, we consider $s \in (p, r)$ and $s \in (r, p^*)$. Suppose $s \in (p, r)$ (the other case is analogous). Arguing analogously to the way that we derive the (2.2.8), we have an interpolation inequality for the weighted Sobolev space

$$
\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^s dx \leq \left(\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p dx\right)^{\frac{\gamma s}{p}}\left(\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^r dx\right)^{\frac{(1-\gamma)s}{r}},
$$

where $0 < \gamma < 1$.

$E \hookrightarrow L_b^t(\mathbb{R}^N)$ for $t \in [p, p^*]$ and boundedness of (u_n) in E imply that

$$
\begin{aligned}
\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^s dx &\leq \left(\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p dx\right)^{\frac{\gamma s}{p}}\left(\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^r dx\right)^{\frac{(1-\gamma)s}{r}}\\
&\leq C_3\|u_n\|_0^{\gamma s}\left(\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^r dx\right)^{\frac{(1-\gamma)s}{r}}\\
&\leq C_3\|u_n\|_\mu^{\gamma s}\left(\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^r dx\right)^{\frac{(1-\gamma)s}{r}}\\
&\leq C_4\left(\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^r dx\right)^{\frac{(1-\gamma)s}{r}},
\end{aligned}
$$

then $u_n \to 0$ in $L_b^s(\mathbb{R}^N)$ for $p < s < r$.

The same argument shows that $u_n \to 0$ in $L_b^s(\mathbb{R}^N)$ for $r < s < p^*$.

Now, applying again interpolation inequality on the Lebesgue spaces, we conclude that $u_n \to 0$ in $L_b^s(\mathbb{R}^N)$ for all $s \in (p, p^*)$. □

2.3 The proof of the item (i) of Theorem 2.1.1 for the subcritical case ($\varrho = 0$)

From Lemma 2.2.1 and Lemma 2.2.2 there exists a bounded $(PS)_{c_\mu,0}$ sequence (u_n) for $I_{\mu,0}$. Then, by Sobolev embedding, there exists $u_\mu \in E$ such that, up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u_\mu, \text{ in } E; \\ u_n \to u_\mu, \text{ in } L^s_{b,loc}(\Omega), \ 1 \le s < p^*; \\ u_n \to u_\mu, \ a.e \text{ in } \mathbb{R}^N. \end{cases} \tag{2.3.1}$$

Moreover, we can conclude from Theorem D.0.8 that u_μ is a critical point of $I_{\mu,0}$.

Now we prove that u_μ is a critical point of $I_{\mu,0}$ at Mountain Pass level $c_{\mu,0}$, for μ large enough. First of all, some technical lemmas.

Lemma 2.3.1. *Consider $u_\mu \in E$, then there exists a positive constant Υ_3 which does not depend on μ such that*

$$\liminf_{\mu \to +\infty} \int\limits_{\mathbb{R}^N} |x|^{-bp^*} |u_\mu|^r dx \ge \Upsilon_3.$$

Proof. Let us suppose, by contradiction, that $\liminf\limits_{\mu \to +\infty} \int\limits_{\mathbb{R}^N} |x|^{-bp^*} |u_\mu|^r dx = 0$. As u_μ is a critical point for $I_{\mu,0}$,

$$0 = I'_{\mu,0}(u_\mu)u_\mu = \|u_\mu\|^p_\mu - \int_{\mathbb{R}^N} |x|^{-bp^*} f(u_\mu)u_\mu dx.$$

Using Sobolev embeddings and (2.2.2), we obtain

$$-\|u_\mu\|^p_\mu = -\int_{\mathbb{R}^N} |x|^{-bp^*} f(u_\mu)u_\mu dx \ge -\xi C_1 \|u_\mu\|^p_\mu - C_\xi \int_{\mathbb{R}^N} |x|^{-bp^*} |u_\mu|^r dx.$$

Thus,

$$0 \ge (1 - \xi C_1)\|u_\mu\|^p_\mu - C_\xi \int_{\mathbb{R}^N} |x|^{-bp^*} |u_\mu|^r dx.$$

Taking the $\limsup\limits_{\mu \to +\infty}$ in the inequality above with ξ sufficiently small such that $1 - \xi C_1 > 0$, we have

$$\begin{aligned} 0 &\ge (1 - \xi C_1) \limsup_{\mu \to +\infty} \|u_\mu\|^p_\mu + \limsup_{\mu \to +\infty} \left\{ -C_\xi \int_{\mathbb{R}^N} |x|^{-bp^*} |u_\mu|^r dx \right\} \\ &= (1 - \xi C_1) \limsup_{\mu \to +\infty} \|u_\mu\|^p_\mu - C_\xi \liminf_{\mu \to +\infty} \left\{ \int_{\mathbb{R}^N} |x|^{-bp^*} |u_\mu|^r dx \right\}. \end{aligned}$$

By hypothesis, $\liminf\limits_{\mu \to +\infty} \int\limits_{\mathbb{R}^N} |x|^{-bp^*} |u_\mu|^r dx = 0$, then $\limsup\limits_{\mu \to +\infty} \|u_\mu\|^p_\mu = 0$. Thus,

$$\|u_\mu\|^p_\mu \le o_\mu(1). \tag{2.3.2}$$

Then, $u_\mu \equiv 0$, which implies that $u_n \rightharpoonup 0$ in E. By (2.3.1),

$$\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |x|^{-bp^*} |u_n|^q dx \to 0, \quad \text{as } n \to \infty.$$

From Theorem 2.2.3, we conclude that $u_n \to 0$ in $L^s_b(\mathbb{R}^N)$ for all $s \in (p, p^*)$. Now using (2.2.2), we obtain

$$\int_{\mathbb{R}^N} |x|^{-bp^*} f(u_n) u_n dx \to 0,$$

which implies that $u_n \to 0$ in E and $I_{\mu,0}(u_n) \to 0$. Hence, $\lim\limits_{\mu\to\infty} c_{\mu,0} = 0$ which contradicts the item (ii) of Lemma 2.2.2. □

Proposition 2.3.2. *There exists $\mu^* > 0$ such that $I_{\mu,0}$ has a critical point $u_\mu \in E$ at mountain pass level $c_{\mu,0}$, for $\mu \geq \mu^*$.*

Proof. By (2.3.1), there exists a critical point for $I_{\mu,0}$. By Lemma 2.3.1 there exists $\mu^* > 0$ such that the critical point is nontrivial, for $\mu \geq \mu^*$. On the other hand, the hypothesis (f_4) implies that

$$t \longmapsto \frac{1}{p} f(t)t - F(t), \text{ is increasing for } t \in (0, +\infty).$$

Therefore, by (2.3.1) and Fatou's Lemma, we obtain

$$\begin{aligned} I_{\mu,0}(u_\mu) = \;& I_{\mu,0}(u_\mu) - \frac{1}{p} I'_{\mu,0}(u_\mu) u_\mu \\ = \;& \int\limits_{\mathbb{R}^N} |x|^{-bp^*} \left(\frac{1}{p} f(u_\mu) u_\mu - F(u_\mu) \right) dx \\ \leq \;& \liminf_{n\to+\infty} \left[\int\limits_{\mathbb{R}^N} |x|^{-bp^*} \left(\frac{1}{p} f(u_n) u_n - F(u_n) \right) dx \right] \\ = \;& \lim_{n\to+\infty} I_{\mu,0}(u_n) = c_{\mu,0}. \end{aligned}$$

Hence, using the characterization (2.2.4) of the mountain pass level $c_{\mu,0}$, we conclude

$$c_{\mu,0} \leqslant I_{\mu,0}(u_\mu) \leq \lim_{n\to+\infty} I_{\mu,0}(u_n) = c_{\mu,0}, \quad \mu \geq \mu^*.$$

□

2.4 The proof of the item (ii) of Theorem 2.1.1 for the critical case ($\varrho = 1$ and $\sigma = p^*$)

To find a nontrivial solution for the case critical of the problem $(P_{\mu,1,p^*})$ it is necessary to control the level critical $c_{\mu,1}$. For this, we need to consider an auxiliary problem given by

$$\begin{cases} -\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}|u|^{p-2}u = |x|^{-bp^*}|u|^{\tau-2}u, & \text{in } \Omega, \\ u \in E(\Omega), \end{cases} \tag{P_Ω}$$

where τ is the constant that appeared in the hypothesis (f_5) and Ω is the bounded domain that appeared in the hypothesis (V_2). The Euler-Lagrange functional associated to (P_Ω) is given by

$$\Phi_0(u) = \frac{1}{p}\int\limits_\Omega |x|^{-ap}|\nabla u|^p dx + \frac{1}{p}\int\limits_\Omega |x|^{-bp^*}|u|^p dx - \frac{1}{\tau}\int\limits_\Omega |x|^{-bp^*}|u|^\tau dx$$

and the Nehari manifold

$$\mathcal{N}_{\Phi_0} = \{u \in E(\Omega) : \; u \neq 0 \text{ and } \Phi_0'(u)u = 0\}.$$

Then, from Appendix B, there exists $w_\tau \in E(\Omega)$ such that

$$\Phi_0(w_\tau) = c_0, \ \ \Phi_0'(w_\tau) = 0$$

and

$$c_0 \geq \left(\frac{\tau - p}{\tau p}\right) \int\limits_{\Omega} |x|^{-bp^*} |w_\tau|^\tau dx. \tag{2.4.1}$$

Lemma 2.4.1. *There exists a positive number λ^* such that the level $c_{\mu,1}$ satisfies*

$$c_{\mu,1} < \left(\frac{1}{p} - \frac{1}{p^*}\right) S_{a,b}^{N/pd}, \quad \textit{for all } \mu \geq 0 \ \ \textit{and for all } \lambda > \lambda^*.$$

Proof. Since $V(x) = 0$ for $x \in \Omega$, and the hypothesis (f_4) holds, there exists $t > 0$, such that

$$I_{\mu,1}(t_\mu w_\tau) = \sup_{t>0} I_{\mu,1}(t w_\tau).$$

Therefore, using (f_5) and that $\Phi_0'(w_\tau) w_\tau = 0$, we obtain

$$\begin{aligned} c_{\mu,1} &\leq I_{\mu,1}(t_\mu w_\tau) \leq \frac{t_\mu^p}{p} \int\limits_{\Omega} |x|^{-ap} |\nabla w_\tau|^p dx + \frac{t_\mu^p}{p} \int\limits_{\Omega} |x|^{-bp^*} |w_\tau|^p dx - \lambda \frac{t_\mu^\tau}{\tau} \int\limits_{\Omega} |x|^{-bp^*} |w_\tau|^\tau dx \\ &\leq \left[\frac{t_\mu^p}{p} - \lambda \frac{t_\mu^\tau}{\tau}\right] \int\limits_{\Omega} |x|^{-bp^*} |w_\tau|^\tau dx \leq \max_{s\geq 0} \left[\frac{s^p}{p} - \lambda \frac{s^\tau}{\tau}\right] \int\limits_{\Omega} |x|^{-bp^*} |w_\tau|^\tau dx. \end{aligned}$$

Then, using (2.4.1) and some straight forward algebric manipulations, we get

$$c_{\mu,1} \leq \max_{s \geq 0} \left[\frac{s^p}{p} - \lambda \frac{s^\tau}{\tau}\right] \frac{c_0 p \tau}{(\tau - p)} = \left[\frac{\tau - p}{p \lambda^{p/(\tau - p)}}\right] \frac{c_0 p}{(\tau - p)} = \frac{c_0}{\lambda^{p/(\tau-p)}}.$$

Hence, choosing $\lambda > \lambda^* := \left[\dfrac{c_0 p p^*}{(p^* - p) S_{a,b}^{\frac{N}{dp}}}\right]^{\frac{\tau - p}{p}}$ in (f_5), the result follows. □

Let us introduce the notation which we are going to use in the next results. From Lemma 2.2.1 and Lemma 2.2.2 there exists a bounded $(PS)_{c_{\mu,1}}$ sequence (u_n) for $I_{\mu,1}$. Then, by Sobolev embedding, there exists $u_\mu \in E$ such that, up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u_\mu, \ \text{in } E; \\ u_n \to u_\mu, \ \text{in } L^s_{b,loc}(\Omega), \ 1 \leq s < p^*; \\ u_n \to u_\mu, \ a.e \text{ in } \mathbb{R}^N. \end{cases} \tag{2.4.2}$$

Moreover, we can conclude from Theorem D.0.10 that u_μ is a critical point of $I_{\mu,1}$.

First of all, using the notation above, we are going to prove some technical result.

Lemma 2.4.2. *Let $u_\mu \in E$ be the weak limit of the sequence defined in (2.4.2). For $\lambda > \lambda^*$, there exists a positive constant Υ_4, which does not depend on μ, such that*

$$\liminf_{\mu \to +\infty} \int\limits_{\mathbb{R}^N} |x|^{-bp^*} |u_\mu|^r dx \geq \Upsilon_4.$$

Proof. Let us suppose, by contradiction, that $\liminf_{\mu\to+\infty}\int_{\mathbb{R}^N}|x|^{-bp^*}|u_\mu|^r dx = 0$. As u_μ is a critical point for $I_{\mu,1}$, it follows from Lemma 2.2.1 *(iii)* that

$$\Upsilon_1 \geq c_{\mu,\rho} = I_{\mu,\rho}(u_\mu)-\frac{1}{\theta}I'_{\mu,\rho}(u_\mu)u_\mu \geq \left(\frac{1}{p}-\frac{1}{\theta}\right)\|u_\mu\|_\mu^p \geq \left(\frac{1}{p}-\frac{1}{\theta}\right)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_\mu|^p dx \text{ for all } \mu\geq 0.$$

By (2.2.2), we obtain

$$\left|\int_{\mathbb{R}^N}|x|^{-bp^*}f(u_\mu)\,u_\mu dx\right| - C_\xi\int_{\mathbb{R}^N}|x|^{-bp^*}|u_\mu|^r dx \leq \xi\int_{\mathbb{R}^N}|x|^{-bp^*}|u_\mu|^p dx \leq \xi\frac{\Upsilon_1}{\left(\frac{1}{p}-\frac{1}{\theta}\right)}.$$

Taking the $\limsup_{\mu\to+\infty}$ followed by the limit as $\xi\to 0$,

$$\limsup_{\mu\to+\infty}\left|\int_{\mathbb{R}^N}|x|^{-bp^*}f(u_\mu)\,u_\mu dx\right| \leq o_\mu(1),$$

hence

$$\int_{\mathbb{R}^N}|x|^{-bp^*}f(u_\mu)\,u_\mu dx = o_\mu(1). \tag{2.4.3}$$

Since $I'_{\mu,1}(u_\mu)u_\mu = 0$, then

$$\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_\mu|^p dx + \int_{\mathbb{R}^N}|x|^{-bp^*}[1+\mu V(x)]|u_\mu|^p dx = \int_{\mathbb{R}^N}|x|^{-bp^*}|u_\mu|^{p^*}dx + o_\mu(1).$$

Setting

$$l := \int_{\mathbb{R}^N}|x|^{-bp^*}|u_\mu|^{p^*}dx + o_\mu(1),$$

we have that $l>0$, from Lemma 2.2.2 we have $c_{\mu,1}>0$, for all $\mu>0$. By definition of the best constant S in the embedding from $D_a^{1,p}(\mathbb{R}^N)$ into $L_b^{p^*}(\mathbb{R}^N)$, we get

$$S_{a,b} \leq \frac{\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_\mu|^p dx}{\left(\int_{\mathbb{R}^N}|x|^{-bp^*}|u_\mu|^{p^*}dx\right)^{p/p^*}} \leq l^{pd/N}. \tag{2.4.4}$$

Using (2.2.5) and (2.4.4), we obtain $c_{\mu,1}\geq\left(\frac{1}{p}-\frac{1}{p^*}\right)S^{N/pd}$, which contradicts the Lemma 2.4.1. ☐

Proposition 2.4.3. *There exist positive numbers μ^{**} and λ^*, which are independent each other, such that $I_{\mu,1}$ has a nontrivial critical point $u_\mu\in E$ at mountain pass level $c_{\mu,1}$, for $\mu\geq\mu^{**}$ and for $\lambda\geq\lambda^*$.*

Proof. The proof follows using the same reasoning that can be found in Proposition 2.3.2. ☐

2.5 Concentration Results

We are going to investigate the behavior of a sequence of ground solution (u_{μ_n}) of (P_{μ,ϱ,p^*}) when $\mu_n \to \infty$. For simplicity of notation such sequence will be denoted just by (u_n). For this goal, let us consider the limit problem

$$\begin{cases} -\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}|u|^{p-2}u = |x|^{-bp^*}f(u) + \varrho|x|^{-bp^*}|u|^{p^*-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{P_{0,ϱ,p^*}}$$

The functional associated to (P_{0,ϱ,p^*}) is

$$J_\varrho(u) = \frac{1}{p}\int\limits_\Omega |x|^{-ap}|\nabla u|^p dx + \frac{1}{p}\int\limits_\Omega |x|^{-bp^*}|u|^p dx - \int\limits_\Omega |x|^{-bp^*}F(u)dx - \frac{\varrho}{p^*}\int\limits_\Omega |x|^{-bp^*}|u|^{p^*}dx,$$

which is differentiable on $E(\Omega)$, and let $\mathcal{N}_\varrho$ be the Nehari manifold associated to J_ϱ given by

$$\mathcal{N}_\varrho = \left\{u \in E(\Omega)\backslash\{0\} : \ J_\varrho'(u)u = 0\right\}.$$

Proposition 2.5.1. *Let $(u_n) \subset E(\Omega) \setminus \{0\}$ be a sequence of ground states solutions for $(P_{\mu_n,\varrho,p^*})_{\mu_n \geq 1}$. Then, up to a subsequence, there exists $u_\infty \in E$ such that $u_n \rightharpoonup u_\infty$ in E. Furthermore,*

(i) $u_\infty = 0$ in $\mathbb{R}^N \setminus \Omega$, $u_\infty(x) \geq 0$, $u_\infty(x) \not\equiv 0$.

(ii) Setting $d_{\mu_n,\varrho} := \inf\limits_{u \in \mathcal{N}_{\mu_n}} I_{\mu_n,\varrho}(u)$, then

$$\lim_{n\to+\infty} d_{\mu_n,\varrho} = \lim_{n\to+\infty} I_{\mu_n,\varrho}(u_n) = J_\varrho(u_\infty).$$

Moreover, $u_n \to u_\infty$ in E and $J_\varrho(u_\infty) = d_\varrho := \inf\limits_{\mathcal{N}_\varrho} J_\varrho$.

Proof. Using Lemma 2.2.1 (iii), we conclude that $(\|u_n\|_{\mu_n})$ is bounded in $\mathbb{R}$ and (u_n) is bounded in E. Indeed,

$$\Upsilon_1 \geq c_{\mu,\rho} = I_{\mu,\rho}(u_n) - \frac{1}{\theta}I'_{\mu,\rho}(u_n)u_n \geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|_\mu^p \text{ for all } n \in \mathbb{N}.$$

So, up to a subsequence, there exists $u_\infty \in E$ such that

$$u_n \rightharpoonup u_\infty \text{ in } E \text{ and } u_n(x) \to u_\infty(x) \text{ for a.e. } x \in \mathbb{R}^N. \tag{2.5.1}$$

Now, for each $m \in \mathbb{N}$, we define $C_m = \left\{x \in \mathbb{R}^N\,;\, V(x) \geq \dfrac{1}{m}\right\}$. Thus

$$\int_{C_m} |x|^{-bp^*}|u_n|^p\,dx \leq \frac{m}{\mu_n}\int_{C_m} |x|^{-bp^*}\left(\mu_n V(x) + 1\right)|u_n|^p\,dx \leq \frac{C}{\mu_n}. \tag{2.5.2}$$

Taking $n \to \infty$, we have by Fatou's lemma,

$$\int_{C_m} |x|^{-bp^*}|u_\infty|^p\,dx = 0,$$

implying that $u_\infty = 0$ in C_m and consequence, $u_\infty = 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$, which implies $u_\infty \in E(\Omega)$ (see [14, Proposition 9.18]).

Next we claim that the limit u_∞ is a nontrivial solution for (P_{0,ϱ,p^*}). To prove this let us consider the following sets

$$\widetilde{A}_R = \{x \in \mathbb{R}^N \backslash B_R(0) : \ V(x) \geq V^*\} \quad \text{and} \quad A_R = \{x \in \mathbb{R}^N \backslash B_R(0) : \ V(x) < V^*\}.$$

Using Lemma 2.2.1 (iii) and (V_3) we can ensure, by Hölder's inequality and Sovolev embedding, that there exists $\Upsilon_5 > 0$ such that

$$\begin{aligned}\int_{\widetilde{A}_R} |x|^{-bp^*}|u_n|^p \, dx &\leq \frac{1}{1+\mu_n V^*}\int_{\mathbb{R}^N} |x|^{-bp^*}\left[1+\mu_n V(x)\right]|u_n|^p \, dx \\ &\leq \frac{1}{1+\mu_n V^*}\|u_n\|_\mu^p \\ &\leq \frac{\Upsilon_5}{1+\mu_n V^*}\end{aligned}$$

and

$$\int_{A_R} |x|^{-bp^*}|u_n|^p \, dx \leq \left(\int_{A_R} |x|^{-bp^*}|u_n|^{p_*} \, dx\right)^{\frac{p}{p_*}} R^{-bp^*} meas(A_R)^{\frac{p_*-p}{p^*}} \leq \Upsilon_5 o_R(1).$$

Hence, by the interpolation argument there exists $\Upsilon_6 > 0$ such that

$$\limsup_{n\to+\infty}\int_{\widetilde{A}_R} |x|^{-bp^*}|u_n|^r \, dx = 0 \quad \text{and} \quad \limsup_{n\to+\infty}\int_{A_R} |x|^{-bp^*}|u_n|^r \, dx \leq \Upsilon_6 o_R(1). \tag{2.5.3}$$

Observe that, from Lemma 2.2.1 (iii), the constants Υ_5 and Υ_6 are independent on the parameter μ. Since, up to a subsequence, $u_n \to u_\infty$ in $L^r_{loc}(\mathbb{R}^N)$ and (2.5.3) holds, we obtain that

$$\begin{aligned}\liminf_{n\to+\infty}\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^r \, dx &\leq \limsup_{n\to+\infty}\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^r \, dx \\ &\leq \limsup_{n\to\infty}\left[\int_{B_R(0)} |x|^{-bp^*}|u_n|^r \, dx + \int_{\widetilde{\Lambda}_R} |x|^{-bp^*}|u_n|^r \, dx + \int_{\Lambda_R} |x|^{-bp^*}|u_n|^r \, dx\right] \\ &\leq \int_{B_R(0)} |x|^{-bp^*}|u_\infty|^r \, dx + \Upsilon_6 o_R(1).\end{aligned} \tag{2.5.4}$$

Hence, by Lemma 2.3.1 (for $\varrho = 0$) or Lemma 2.4.2 (for $\varrho = 1$) the claim follows, for R large enough. Moreover, using (f_1) and u_∞^- a test function, we get $u_\infty \geq 0$ and $u_\infty \neq 0$.

We now prove the second item (ii). Observe that since $V = 0$ in Ω, we obtain

$$\int_{\mathbb{R}^N} |x|^{-bp^*}V(x)|u|^p dx = \int_{\mathbb{R}^N\backslash\Omega} |x|^{-bp^*}V(x)|u|^p dx + \int_{\Omega} |x|^{-bp^*}V(x)|u|^p dx = 0, \quad \text{for all } u \in E(\Omega),$$

which implies

$$I_{\mu_n,\varrho}(u) = J_\varrho(u) \quad \text{and} \quad I'_{\mu_n,\varrho}(u)u = J'_\varrho(u)u, \ \text{for all } u \in E(\Omega). \tag{2.5.5}$$

Then, from (2.5.5), we have that $u \in \mathcal{N}_{\mu_n,\varrho}$, for all $u \in \mathcal{N}_\varrho$. Hence,

$$d_{\mu_n,\varrho} \leq d_\varrho. \tag{2.5.6}$$

On the other hand, since $u_n \rightharpoonup u_\infty$ in E we have, by Fatou's Lemma,

$$\begin{aligned} 0 \le\ & \left(\frac{1}{p}-\frac{1}{\theta}\right)\int\limits_{\mathbb{R}^N}|x|^{-ap}|\nabla u_\infty|^p dx + \left(\frac{1}{p}-\frac{1}{\theta}\right)\int\limits_{\mathbb{R}^N}|x|^{-bp^*}|u_\infty|^p dx \\ \le\ & \liminf_{n\to+\infty}\left(\frac{1}{p}-\frac{1}{\theta}\right)\int\limits_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^p dx + \left(\frac{1}{p}-\frac{1}{\theta}\right)\int\limits_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^p dx. \end{aligned} \tag{2.5.7}$$

Therefore, using the fact that $u_\infty \in \mathcal{N}_\varrho$, we obtain, by (2.5.5), (2.5.6) and (2.5.7),

$$\begin{aligned} d_{\mu_n,\varrho} \le d_\varrho \le J_\varrho(u_\infty) =\ & I_{\mu_n,\varrho}(u_\infty) - \frac{1}{\theta}I'_{\mu_n,\varrho}(u_\infty)u_\infty \\ \le\ & \liminf_{n\to\infty}\left[I_{\mu_n,\varrho}(u_n) - \frac{1}{\theta}I'_{\mu_n,\varrho}(u_n)u_n\right] \\ =\ & I_{\mu_n,\varrho}(u_n) + o_n(1) = d_{\mu_n,\varrho} + o_n(1), \end{aligned} \tag{2.5.8}$$

which implies

$$\lim_{n\to+\infty} d_{\mu_n,\varrho} = \lim_{n\to+\infty} I_{\mu_n,\varrho}(u_n) = J_\varrho(u_\infty). \tag{2.5.9}$$

Assume, by contradiction, that

$$u_n \to u_\infty \quad \text{in} \quad E, \tag{2.5.10}$$

does not hold. Then, the inequality (2.5.7) is strict and hence, arguing as (2.5.8), there exists $n_0 \in \mathbb{N}$

$$d_\varrho < d_{\mu_n,\varrho} + \frac{d_\varrho}{2}, \quad n \ge n_0.$$

This contradicts (2.5.9). □

2.6 Theorem 2.1.1 (subcritical case)

Proof of Theorem 2.1.1(subcritical case). From Proposition 2.3.2, we can guarantee that there exists $\mu^* > 0$ such that $(P_{\mu,0,\sigma})$ has a positive ground state solution $u_\mu \in E$, for $\mu \ge \mu^*$. Then, using Proposition 2.5.1, we obtain, up to a subsequence, $u_\mu \to u_\infty$ in E when $\mu \to +\infty$, where u_∞ is a ground state solution to problem

$$\begin{cases} -\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}|u|^{p-2}u = |x|^{-bp^*}f(u), & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{$P_{0,0}$}$$

□

2.7 Theorem 2.1.1 (critical case)

Proof of Theorem 2.1.1(critical case). From Proposition 2.4.3, we can guarantee that there exist $\mu^{**} > 0$ and $\lambda^* > 0$ such that $(P_{\mu,1,p^*})$ has a positive ground state solution $u_\mu \in E$, for all $\mu \ge \mu^{**}$ and $\lambda \ge \lambda^*$. Then, using Proposition 2.5.1, we obtain, up to a subsequence, $u_\mu \to u_\infty$ in E when $\mu \to +\infty$, where u_∞ is a ground state solution to problem

$$\begin{cases} -\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}|u|^{p-2}u = |x|^{-bp^*}f(u) + |x|^{-bp^*}|u|^{p^*-2}u, & \text{in} \quad \Omega, \\ u = 0 & \text{, on} \quad \partial\Omega. \end{cases} \tag{$P_{0,1,p^*}$}$$

□

2.8 Case supercritical

In this section we are going to study the supercritical case of the problem ($P_{\mu,1,\sigma}$), that is, when $\varrho = 1$ and $\sigma > p^*$, observe that in this case $\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^\sigma \, dx$ is not well defined in E. Then, inspired by [21] and [30], we are going to consider in this section the function $\psi : \mathbb{R} \to \mathbb{R}$ given by

$$\psi(s) := \begin{cases} 0 & \text{, if } \quad s < 0, \\ s^{\sigma-1} & \text{, if } \quad 0 \leq s \leq 1, \\ s^{p^*-1} & \text{, if } \quad s > 1. \end{cases}$$

It follows immediately that

$$\psi(s) \leq |s|^{p^*-1}, \text{ for all } s \in \mathbb{R}, \tag{2.8.1}$$

and

$$\begin{aligned} &\frac{1}{\theta} \int\limits_{\mathbb{R}^N} |x|^{-bp^*} \left[\psi(u)u - \theta\Psi(u)\right] \, dx \\ \geq \; &\left(\frac{1}{\theta} - \frac{1}{p^*}\right) \left[\int\limits_{\{|u|\leq 1\}} |x|^{-bp^*}|u|^\sigma \, dx + \int\limits_{\{|u|>1\}} |x|^{-bp^*}|u|^{p^*} \, dx \right] > 0, \end{aligned} \tag{2.8.2}$$

where $\Psi(s) := \int_0^s \psi(t)dt$. We also consider the auxiliary problem

$$\begin{cases} -\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}[1+\mu V(x)]|u|^{p-2}u = |x|^{-bp^*}f(u) + |x|^{-bp^*}, \psi(u) \;\; \text{in } \mathbb{R}^N, \\ u \in E. \end{cases} \tag{$P_{\mu,\sigma}$}$$

Remark 1. *If u_μ is a nonnegative solution of ($P_{\mu,\sigma}$) with $\|u_\mu\|_\infty \leq 1$, then u_μ is also a nonnegative solution of ($P_{\mu,1,\sigma}$).*

2.8.1 Existence of positive solution for problem ($P_{\mu,\sigma}$)

The nonnegative weak solutions for the problem ($P_{\mu,\sigma}$) are the critical points of the functional $I_{\mu,\sigma} : E \to \mathbb{R}$ given by

$$\begin{aligned} I_{\mu,\sigma}(v) &= \frac{1}{p}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla v|^p dx + \frac{1}{p}\int_{\mathbb{R}^N} |x|^{-bp^*}[1+\mu V(x)]|v|^p dx \\ &\quad - \int_{\mathbb{R}^N} |x|^{-bp^*} F(v)dx - \int_{\mathbb{R}^N} |x|^{-bp^*}\Psi(v)dx, \end{aligned}$$

where $\Psi(s) := \int_0^s \psi(t)dt$. Now we are going to find a nontrivial and nonnegative solution for ($P_{\mu,\sigma}$).

Using the same arguments of Lemma 2.4.2 and Proposition 2.4.3 with short modifications we can prove the following results.

Proposition 2.8.1. *There exist $\mu^{**} > 0$ and $\lambda^* > 0$ such that the functional $I_{\mu,\sigma}$ has a nontrivial critical point $u_\mu \in E$ at the mountain pass level $c_{\mu,\sigma}$, for all $\mu \geq \mu^{**}$ and $\lambda \geq \lambda^*$.*

The next result relates the critical points of the functional $I_{\mu,\sigma}$ with solutions to the problem ($P_{\mu,1,\sigma}$), the arguments used here are inspired by [5, Lemma 5.5] and [33, Theorem 3].

Lemma 2.8.2. *Let $u_\mu \in E$ be a nonnegative solution for problem ($P_{\mu,\sigma}$). Then,*

$$\|u_\mu\|_{L^\infty(\mathbb{R}^N)} \leq 1, \text{ for all } \lambda > \lambda^*.$$

Moreover, the function u_μ is a solution of ($P_{\mu,1,\sigma}$).

Proof. For each $L > 0$, let

$$u_L(x) = \begin{cases} u_\mu(x), & u_\mu(x) \leq L, \\ L, & u_\mu(x) > L, \end{cases} \tag{2.8.3}$$

and

$$z_L := u_L^{p(\gamma-1)} u_\mu$$

with $\gamma > 1$ will be determined later.

Taking z_L as a test function, we obtain that $I'_{\mu,\sigma}(u_\mu) z_L = 0$. That is,

$$\begin{aligned}
&\int_{\mathbb{R}^N} |x|^{-ap} u_L^{p(\gamma-1)} |\nabla u_\mu|^p dx + p(\gamma-1) \int_{\mathbb{R}^N} |x|^{-ap} u_L^{p(\gamma-1)-1} u_\mu |\nabla u_\mu|^{p-2} \nabla u_\mu \nabla u_L dx \\
&+ \int_{\mathbb{R}^N} |x|^{-bp^*} [1 + \mu V(x)] |u_\mu|^p u_L^{p(\gamma-1)} dx = \int_{\mathbb{R}^N} |x|^{-bp^*} f(u_\mu) u_\mu u_L^{p(\gamma-1)} dx \\
&+ \int_{\mathbb{R}^N} |x|^{-bp^*} \psi(u_\mu) u_\mu u_L^{p(\gamma-1)} dx.
\end{aligned}$$

Using (f_1), (f_2) and (2.8.1) we obtain that given $\xi > 0$ there exists $C_\xi > 0$, such that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |x|^{-ap} u_L^{p(\gamma-1)} |\nabla u_\mu|^p dx + p(\gamma-1) \int_{\mathbb{R}^N} |x|^{-ap} u_L^{p(\gamma-1)} |\nabla u_L|^p dx + \int_{\mathbb{R}^N} |x|^{-bp^*} u_L^{p(\gamma-1)} |u_\mu|^p \\
\leq& \xi \int_{\mathbb{R}^N} |x|^{-bp^*} u_L^{p(\gamma-1)} |u_\mu|^p dx + (C_\xi + 1) \int_{\mathbb{R}^N} |x|^{-bp^*} u_L^{p(\gamma-1)} |u_\mu|^{p^*} dx.
\end{aligned}$$

Let us now consider the function $w_L := u_\mu u_L^{\gamma-1}$. Hence, by inequality above,

$$\begin{aligned}
\int_{\mathbb{R}^N} |x|^{-ap} |\nabla w_L|^p dx \leq& 2^p \int_{\mathbb{R}^N} |x|^{-ap} u_L^{p(\gamma-1)} |\nabla u_\mu|^p dx \\
&+ 2^p (\gamma-1)^p \int_{\mathbb{R}^N} |x|^{-ap} u_L^{p(\gamma-1)} |\nabla u_L|^p dx \\
\leq& 4^p \gamma^p \xi \int_{\mathbb{R}^N} |x|^{-bp^*} u_L^{p(\gamma-1)} |u_\mu|^p dx \\
&+ 4^p \gamma^p (C_\xi + 1) \int_{\mathbb{R}^N} |x|^{-bp^*} u_L^{p(\gamma-1)} |u_\mu|^{p^*} dx.
\end{aligned} \tag{2.8.4}$$

Therefore, since $u_L \leq u_\mu$,

$$\begin{aligned}
\|w_L\|^p_{L_b^{p^*}(\mathbb{R}^N)} \leq& S_{a,b} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla w_L|^p dx \\
\leq& 4^p \gamma^p S_{a,b} \xi \int_{\mathbb{R}^N} |x|^{-bp^*} |u_\mu|^{p\gamma} dx \\
&+ 4^p \gamma^p S_{a,b} (C_\xi + 1) \int_{\mathbb{R}^N} |x|^{-bp^*} |u_\mu|^{\gamma p} |u_\mu|^{p^*-p} dx,
\end{aligned} \tag{2.8.5}$$

where $S_{a,b}$ is the best Sobolev constant of the embedding $D_a^{1,p}(\mathbb{R}^N) \hookrightarrow L_b^{p^*}(\mathbb{R}^N)$.

Also, observe that

$$\int\limits_{\mathbb{R}^N} |x|^{-bp^*}|u_L|^{p(\gamma-1)}|u_\mu|^{p^*}dx = \int\limits_{\mathbb{R}^N} |x|^{-bp^*}|w_L|^p|u_\mu|^{p^*-p}dx.$$

This and (2.8.5) ensure that

$$\begin{aligned} \|w_L\|^p_{L_b^{p^*}(\mathbb{R}^N)} \leq & S_{a,b}\int\limits_{\mathbb{R}^N} |x|^{-ap}|\nabla w_L|^p dx \\ \leq & 4^p\gamma^p S_{a,b}\xi \int\limits_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{p\gamma}dx \\ & + 4^p\gamma^p S_{a,b}(C_\xi+1)\int\limits_{\mathbb{R}^N} |x|^{-bp^*}|w_L|^p|u_\mu|^{p^*-p}dx, \end{aligned} \tag{2.8.6}$$

The next step is to show that $u_\mu \in L_b^{\frac{(p^*)^2}{p}}(\mathbb{R}^N)$.

Statement 2.8.3. $u_\mu \in L_b^{\frac{(p^*)^2}{p}}(\mathbb{R}^N)$.

Proof. We choose $\gamma = \dfrac{p^*}{p}$ in (2.8.6) then, by Hölder's inequality,

$$\|w_L\|^p_{L_b^{p^*}(\mathbb{R}^N)} \leq \left(\frac{4p^*}{p}\right)^p S_{a,b}\xi\|u_\mu\|^{p^*}_{L_b^{p^*}(\mathbb{R}^N)} + \left(\frac{4p^*}{p}\right)^p S_{a,b}(C_\xi+1)\|u_\mu\|^{p^*-p}_{L_b^{p^*}(\mathbb{R}^N)}\|w_L\|^p_{L_b^{p^*}(\mathbb{R}^N)}.$$

Using (2.8.2) and Lemma 2.4.1 and that the function u_μ is a critical point of $I_{\mu,\sigma}$, we have that

$$\begin{aligned} \frac{c_0}{\lambda^{\frac{p}{\tau-p}}} \geq c_{\mu,\sigma} = & \; I_{\mu,\sigma}(u_\mu) - \tfrac{1}{\theta}I'_{\mu,\sigma}(u_\mu)u_\mu \\ \geq & \left(\tfrac{1}{p}-\tfrac{1}{\theta}\right)\|u_\mu\|^p_\mu \\ \geq & \left(\tfrac{1}{p}-\tfrac{1}{\theta}\right)S_{a,b}^{-1}\|u_\mu\|^p_{L_b^{p^*}(\mathbb{R}^N)}. \end{aligned} \tag{2.8.7}$$

Remember that $\gamma = \frac{p^*}{p}$. From the definition of u_L and w_L,

$$u_L(x) \overset{L\to\infty}{\longrightarrow} u_\mu(x) \text{ a.e. } x \in \mathbb{R}^N$$

and

$$w_L(x) \overset{L\to\infty}{\longrightarrow} (u_\mu(x))^{\gamma-1} \text{ a.e. } x \in \mathbb{R}^N. \tag{2.8.8}$$

Observe that

$$\begin{aligned} \|u_\mu\|^{p\gamma}_{L_b^{p^*\gamma}(\mathbb{R}^N)} &= \left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{p^*\gamma}dx\right)^{\frac{p\gamma}{p^*\gamma}} \\ &= \left(\int_{\mathbb{R}^N} |x|^{-bp^*}(|u_\mu|^\gamma)^{p^*}dx\right)^{\frac{p}{p^*}} \\ &= \|u_\mu^\gamma\|_{L_b^{p^*}(\mathbb{R}^N)}. \end{aligned}$$

This, (2.8.8) and Fatou's lemma ensure that

$$
\begin{aligned}
\|u_\mu\|^{p\gamma}_{L_b^{p^*\gamma}(\mathbb{R}^N)} &= \|u_\mu^\gamma\|^p_{L_b^{p^*}(\mathbb{R}^N)} \\
&= \left(\int_{\mathbb{R}^N} |x|^{-bp^*}(|u_\mu|^\gamma)^{p^*}dx\right)^{\frac{p}{p^*}} \\
&\le \liminf_{L\to\infty}\left(\int_{\mathbb{R}^N} |x|^{-bp^*}(w_L)^{p^*}dx\right)^{\frac{p}{p^*}} \\
&= \liminf_{L\to\infty}\|w_L\|^p_{L_b^{p^*}(\mathbb{R}^N)} \\
&\le \|w_L\|^p_{L_b^{p^*}(\mathbb{R}^N)}.
\end{aligned}
$$

From this and (2.8.5), it follows that

$$
\begin{aligned}
\left[\int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{\frac{(p^*)^2}{p}}\,dx\right]^{\frac{p}{p^*}} =&\|u_\mu\|^{p\gamma}_{L_b^{p^*\gamma}(\mathbb{R}^N)} \\
\le&\|w_L\|^p_{L_b^{p^*}(\mathbb{R}^N)} \\
\le&S_{a,b}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla w_L|^p dx \\
\le&4^p\gamma^p S_{a,b}\xi\int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{p\gamma}dx \\
&+4^p\gamma^p S_{a,b}(C_\xi+1)\int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{\gamma p}|u_\mu|^{p^*-p}dx.
\end{aligned}
$$

By Hölder with exponents $\frac{p^*}{p}$ and $\frac{p^*}{p^*-p}$,

$$
\int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{\gamma p}|u_\mu|^{p^*-p}dx \le \left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{\frac{(p^*)^2}{p}}\,dx\right)^{\frac{p}{p^*}}\left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{p^*}\,dx\right)^{\frac{p^*-p}{p^*}}.
$$

Thus,

$$
\begin{aligned}
\left[\int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{\frac{(p^*)^2}{p}}\,dx\right]^{\frac{p}{p^*}} =&\|u_\mu\|^{p\gamma}_{L_b^{p^*\gamma}(\mathbb{R}^N)} \\
\le&\|w_L\|^p_{L_b^{p^*}(\mathbb{R}^N)} \\
\le&S_{a,b}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla w_L|^p dx \\
\le&4^p\gamma^p S_{a,b}\xi\int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{p\gamma}dx \\
&+4^p\gamma^p S_{a,b}(C_\xi+1)\int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{\gamma p}|u_\mu|^{p^*-p}dx
\end{aligned}
$$

$$\leq 4^p\gamma^p S_{a,b}\xi \int\limits_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{p^*}dx$$

$$+ 4^p\gamma^p S_{a,b}(C_\xi+1)\left(\int\limits_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{\frac{(p^*)^2}{p}}\ dx\right)^{\frac{p}{p^*}}\left(\int\limits_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{p^*}\ dx\right)^{\frac{p^*-p}{p^*}}$$

By (2.8.7),

$$\int\limits_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{p^*}dx \leq \left(\frac{S_{a,b}c_0}{\left(\frac{1}{p}-\frac{1}{\theta}\right)\lambda^{\frac{p}{\tau-p}}}\right)^{\frac{p^*}{p}}.$$

Let $A_\lambda := \frac{S_{a,b}c_0}{\left(\frac{1}{p}-\frac{1}{\theta}\right)\lambda^{\frac{p}{\tau-p}}}$. Then

$$\left[\int\limits_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{\frac{(p^*)^2}{p}}\ dx\right]^{\frac{p}{p^*}} \leq 4^p\gamma^p S_{a,b}\xi A_\lambda^{\frac{p^*}{p}}$$

$$+ 4^p\gamma^p S_{a,b}(C_\xi+1)\left(\int\limits_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{\frac{(p^*)^2}{p}}\ dx\right)^{\frac{p}{p^*}}\left(A_\lambda^{\frac{p^*}{p}}\right)^{\frac{p^*-p}{p^*}},$$

hence

$$\left[1-4^p\gamma^p S_{a,b}(C_\xi+1)\left(A_\lambda^{\frac{p^*}{p}}\right)^{\frac{p^*-p}{p^*}}\right]\left[\int\limits_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{\frac{(p^*)^2}{p}}\ dx\right]^{\frac{p}{p^*}} \leq 4^p\gamma^p S_{a,b}\xi A_\lambda^{\frac{p^*}{p}}$$

Observe that $\lim\limits_{\lambda\to+\infty} A_\lambda = 0$. This and the last estimate shows that $u_\mu \in L_b^{\frac{(p^*)^2}{p}}(\mathbb{R}^N)$ for λ large. □

Note that from (2.8.4) and previous arguments there exists a positive constant K, such that

$$\|w_L\|^p_{L_b^{p^*}(\mathbb{R}^N)} \leq 4^p\gamma^p S_{a,b}(K+1)\int\limits_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{\gamma p}|u_\mu|^{p^*-p}dx. \tag{2.8.9}$$

We are now going to consider $\gamma = \gamma_0 := \dfrac{p^*}{p}\dfrac{(t-1)}{t}$ in (2.8.9), where $t := \dfrac{(p^*)^2}{p(p^*-p)} > 1$. Then, use Hölder inequality with exponents t and $\frac{t}{t-1}$ in the integral in (2.8.9) and Fatou's Lemma,

$$\begin{aligned}\|u_\mu\|^{p\gamma_0}_{L_b^{p^*\gamma_0}(\mathbb{R}^N)} &\leq \liminf_{L\to+\infty}\|w_L\|^p_{L_b^{p^*}(\mathbb{R}^N)}\\ &\leq \|w_L\|^p_{L_b^{p^*}(\mathbb{R}^N)} \leq 4^p\gamma_0^p S_{a,b}(K+1)\|u_\mu\|^{p^*-p}_{L_b^{\frac{(p^*)^2}{p}}(\mathbb{R}^N)}\|u_\mu\|^{\gamma_0 p}_{L_b^{p^*}(\mathbb{R}^N)}.\end{aligned}$$

Hence,

$$\|u_\mu\|_{L_b^{p^*\gamma_0}(\mathbb{R}^N)} \leq \left[4S_{a,b}^{\frac{1}{p}}(K+1)^{\frac{1}{p}}\|u_\mu\|_{L_b^{\frac{(p^*)^2}{p}}(\mathbb{R}^N)}^{\frac{p^*-p}{p}}\right]^{\frac{1}{\gamma_0}} \gamma_0^{\frac{1}{\gamma_0}}\|u_\mu\|_{L_b^{p^*}(\mathbb{R}^N)}. \tag{2.8.10}$$

Considering $\gamma = \gamma_0^2$, Hölder's inequality with exponents t and $\frac{t}{t-1}$ in the integral in (2.8.5) ensures that

$$\int_{\mathbb{R}^N}|x|^{-bp^*}|u_\mu|^{\gamma_0^2 p}|u_\mu|^{p^*-p}dx \leq \|u_\mu\|_{L_b^{p^*\gamma_0}(\mathbb{R}^N)}^{\gamma_0^2 p}\|u_\mu\|_{L_b^{\frac{(p^*)^2}{p}}(\mathbb{R}^N)}^{p^*-p}.$$

Repeat previous arguments to see that

$$\begin{aligned}
\|u_\mu\|_{L_b^{p^*\gamma_0^2}(\mathbb{R}^N)}^{p\gamma_0^2} &= \|u_\mu^{\gamma_0^2}\|_{L_b^{p^*}(\mathbb{R}^N)}^{p} \\
&= \left(\int_{\mathbb{R}^N}|x|^{-bp^*}(|u_\mu|^{\gamma_0^2})^{p^*}dx\right)^{\frac{p}{p^*}} \\
&\leq \liminf_{L\to\infty}\left(\int_{\mathbb{R}^N}|x|^{-bp^*}(w_L)^{p^*}dx\right)^{\frac{p}{p^*}} \\
&= \liminf_{L\to\infty}\|w_L\|_{L_b^{p^*}(\mathbb{R}^N)}^{p} \\
&\leq \|w_L\|_{L_b^{p^*}(\mathbb{R}^N)}^{p} \\
&\leq 4^p(\gamma_0^2)^p S_{a,b}(K+1)\int\limits_{\mathbb{R}^N}|x|^{-bp^*}|u_\mu|^{\gamma_0^2 p}|u_\mu|^{p^*-p}dx.
\end{aligned}$$

The last two estimates provides that

$$\|u_\mu\|_{L_b^{p^*\gamma_0^2}(\mathbb{R}^N)}^{p\gamma_0^2} \leq 4^p(\gamma_0^2)^p S_{a,b}(K+1)\|u_\mu\|_{L_b^{p^*\gamma_0}(\mathbb{R}^N)}^{\gamma_0^2 p}\|u_\mu\|_{L_b^{\frac{(p^*)^2}{p}}(\mathbb{R}^N)}^{p^*-p},$$

i.e.,

$$\|u_\mu\|_{L_b^{p^*\gamma_0^2}(\mathbb{R}^N)} \leq 4^{\frac{1}{\gamma_0^2}}\gamma_0^{\frac{2}{\gamma_0^2}}S_{a,b}^{\frac{1}{\gamma_0^2 p}}(K+1)^{\frac{1}{\gamma_0^2 p}}\|u_\mu\|_{L_b^{p^*\gamma_0}(\mathbb{R}^N)}\|u_\mu\|_{L_b^{\frac{(p^*)^2}{p}}(\mathbb{R}^N)}^{\frac{p^*-p}{p\gamma_0^2}}.$$

By (2.8.10),

$$\|u_\mu\|_{L_b^{p^*\gamma_0^2}(\mathbb{R}^N)} \leq \left[4S_{a,b}^{\frac{1}{p}}(K+1)^{\frac{1}{p}}\|u_\mu\|_{L_b^{\frac{(p^*)^2}{p}}(\mathbb{R}^N)}^{\frac{p^*-p}{p}}\right]^{\sum\limits_{i=1}^{2}\frac{1}{\gamma_0^i}} \gamma_0^{\sum\limits_{i=1}^{2}\frac{i}{\gamma_0^i}}\|u_\mu\|_{L_b^{p^*}(\mathbb{R}^N)}.$$

Repeating the arguments above for $\gamma_0^3,\ \gamma_0^4,\cdots$ we can concluded that

$$\|u_\mu\|_{L_b^{p^*\gamma_0^m}(\mathbb{R}^N)} \leq \left[4S_{a,b}^{\frac{1}{p}}(K+1)^{\frac{1}{p}}\|u_\mu\|_{L_b^{\frac{(p^*)^2}{p}}(\mathbb{R}^N)}^{\frac{p^*-p}{p}}\right]^{\sum\limits_{i=1}^{m}\frac{1}{\gamma_0^i}} \gamma_0^{\sum\limits_{i=1}^{m}\frac{i}{\gamma_0^i}}\|u_\mu\|_{L_b^{p^*}(\mathbb{R}^N)}. \tag{2.8.11}$$

Once that

$$\sum_{i=1}^{\infty}\frac{1}{\gamma_0^i} \quad \text{and} \quad \sum_{i=1}^{\infty}\frac{i}{\gamma_0^i},$$

are convergent series it follows from (2.8.11) that

$$\begin{aligned}\|u_\mu\|_{L^\infty(\mathbb{R}^N)} &\leq \left[4S_{a,b}^{\frac{1}{p}}(K+1)^{\frac{1}{p}}\|u_\mu\|_{L_b^{\frac{(p^*)^2}{p}}(\mathbb{R}^N)}^{\frac{p^*-p}{p}}\right]^{\sum_{i=1}^{\infty}\frac{1}{\gamma_0^i}} \gamma_0^{\sum_{i=1}^{\infty}\frac{i}{\gamma_0^i}}\|u_\mu\|_{L_b^{p^*}(\mathbb{R}^N)} \qquad (2.8.12)\\ &= \|u_\mu\|_{L_b^{\frac{(p^*)^2}{p}}(\mathbb{R}^N)}^{\frac{(p^*-p)}{p}\sum_{i=1}^{\infty}\frac{1}{\gamma_0^i}}\left[4S_{a,b}^{\frac{1}{p}}(K+1)^{\frac{1}{p}}\right]^{\sum_{i=1}^{\infty}\frac{1}{\gamma_0^i}}\gamma_0^{\sum_{i=1}^{\infty}\frac{i}{\gamma_0^i}}\|u_\mu\|_{L_b^{p^*}(\mathbb{R}^N)}.\end{aligned}$$

Finally there exists $\lambda^* > 1$ such that, by Statement 2.8.3 and (2.8.12), we can conclude that

$$\|u_\mu\|_\infty \leq 1, \quad \text{for all} \quad \lambda > \lambda^*.$$

Hence, $\psi(u_\mu) = |u_\mu|^{\sigma-2}u_\mu$ which implies that the function u_μ is a solution of the problem $(P_{\mu,1,\sigma})$. □

2.9 Theorem 2.1.1 (supercritical case)

Proof of Theorem 2.1.1(supercritical case). From Proposition 2.8.1, we can guarantee that there exists $\mu^{**} > 0$ such that $(P_{\mu,1,\sigma})$ has a positive ground state solution $u_\mu \in E$, for all $\mu \geq \mu^{**}$ and $\lambda \geq \lambda^*$. Then, using Proposition 2.5.1 with short modifications, we obtain, up to a subsequence, $u_\mu \to u_\infty$ in E when $\mu \to +\infty$, where u_∞ is a ground state solution to problem $(P_{0,1,\sigma})$. □

Chapter 3

Existence of least energy positive and nodal solutions for a class of Caffarelli-Kohn-Nirenberg type problems

In this chapter, we look for positive and nodal ground state solutions minimizing the Euler-Lagrange functional over the Nehari manifold and over its subset. This chapter is based on [12]. It is important to observe that the method used here to find positive ground state solution is different from [12], we work with more general singularities so that some estimates are more refined and we prove the existence of nodal solution.

3.1 Introduction

This chapter is focused to prove the existence of a positive and a nodal solutions to the following class of Caffarelli-Kohn-Nirenberg type problems give by

$$-\mathrm{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}V(x)|u|^{p-2}u = |x|^{-bp^*}K(x)f(u) \quad \text{in } \mathbb{R}^N, \qquad \text{(P)}$$

where $1 < p < N$, $0 \leq a < \frac{N-p}{p}$, $a < b \leq a+1$, $p^* = p^*(a,b) = \frac{pN}{N-dp}$ and $d = 1+a-b$.

In order to find these solutions we use $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ that is the completion of the $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with the norm

$$\|u\|^p = \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx,$$

where $\mathcal{C}_0^\infty(\mathbb{R}^N)$ is the space of smooth functions with compact support.

Let us denote by

$$L_b^s(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \quad u \text{ is measurable and } \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^s dx < \infty \right\}$$

and

$$L_b^\infty(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \quad u \text{ is measurable and} \sup_{\mathbb{R}^N} ess|x|^{-bp^*}|u| < \infty \right\}.$$

On functions $V, K : \mathbb{R}^N \to \mathbb{R}$ continuous on $\mathbb{R}^N$ we assume the following general conditions. Indeed, we say that $(V, K) \in \mathcal{K}$ if

(VK_0) $V(x), K(x) > 0$ for all $x \in \mathbb{R}^N$ and $K \in L_b^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

(VK_1) If $\{A_n\}_n \subset \mathbb{R}^N$ is a sequence of Borel sets such that the Lebesgue measure meas$(A_n) \leq R$, for all $n \in \mathbb{N}$ and some $R > 0$, then

$$\lim_{r \to +\infty} \int_{A_n \cap B_r^c(0)} |x|^{-bp^*} K(x) = 0, \quad \text{uniformly in } n \in \mathbb{N}.$$

Furthermore, one of the below conditions occurs

(VK_2) $\frac{K}{V} \in L_b^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$

or

(VK_3) there exists $m \in (p, p^*)$ such that

$$\frac{K(x)}{V(x)^{\frac{p^*-m}{p^*-p}}} \to 0, \quad \text{as } |x| \to +\infty.$$

Moreover, we assume the following growth conditions in the origin and at infinity for the C^1 function $f : \mathbb{R} \to \mathbb{R}$:

(f_1)
$$\lim_{|t| \to 0^+} \frac{f(t)}{|t|^{p-1}} = 0, \quad \text{if } (VK_2) \text{ holds}$$

or

($\widetilde{f}_1$)
$$\lim_{|t| \to 0^+} \frac{f(t)}{|t|^{m-1}} = 0, \quad \text{if } (VK_3) \text{ holds}$$

with $m \in (p, p^*)$ defined before in (VK_3);

(f_2) f has a "quasicritical growth" at infinity, namely,

$$\lim_{|t| \to +\infty} \frac{f(t)}{|t|^{p^*-1}} = 0;$$

(f_3) There exists $\theta \in (p, p^*)$ so that

$$0 < \theta F(t) = \theta \int_0^t f(s) ds \leq f(t)t, \text{ for all } |t| > 0;$$

(f_4) The map

$$t \mapsto \frac{f(t)}{|t|^{p-1}} \text{ is strictly increasing for all } |t| > 0,$$

or, equivalently,

$$f'(t) > (p-1)\frac{f(t)}{t}, \text{ for all } t \neq 0.$$

The main results of this chapter are stated in the following theorem.

Theorem 3.1.1. *Suppose that $(V, K) \in \mathcal{K}$ and $f \in C^1(\mathbb{R}, \mathbb{R})$ verifies (f_1) or ($\widetilde{f}_1$) and (f_2) − (f_4). Then, problem (P) possesses a positive ground state weak solution. Moreover, (P) admits a nodal ground state weak solution, which has precisely two nodal domains.*

3.2 Variational framework and Compactness results

In order to prove that problem (P) has a variational structure, let us consider the space

$$X = \left\{ u \in \mathcal{D}_a^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^{-bp^*} V(x)|u|^p \, dx < +\infty \right\}$$

endowed with the norm

$$\|u\|_V^p = \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p \, dx + \int_{\mathbb{R}^N} |x|^{-bp^*} V(x)|u|^p \, dx.$$

Let X' the dual space of X endowed with the norm $\|\cdot\|_{X'}$. Recall that a weak solution of problem (P) is a function $u \in X$ such that

$$\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^N} |x|^{-bp^*} V(x)|u|^{p-2} uv \, dx \\ - \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) f(u) v \, dx = 0, \text{ for all } v \in X.$$

Note that the weak solutions of (P) are the critical points of the energy functional J defined on X by

$$J(u) := \frac{1}{p} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p \, dx + \frac{1}{p} \int_{\mathbb{R}^N} |x|^{-bp^*} V(x)|u|^p \, dx - \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) F(u) \, dx.$$

More precisely, $J \in C^1(X, \mathbb{R})$ and its differential $J' : X \to X'$ is defined as

$$\begin{aligned} \langle J'(u), v \rangle &= \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx \\ &+ \int_{\mathbb{R}^N} |x|^{-bp^*} V(x)|u|^{p-2} uv \, dx - \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) f(u) v \, dx, \end{aligned}$$

for every $u, v \in X$.

In order to prove the compactness result, first assume that (VK_2) holds. By (f_1) and (f_2) and then, by integration, it follows that, fixing any $\varepsilon > 0$ there exists a positive constant $C_\varepsilon > 0$ such that

$$|f(t)| \leq \varepsilon |t|^{p-1} + C_\varepsilon |t|^{p^*-1}, \quad |F(t)| \leq \frac{\varepsilon}{p} |t|^p + \frac{C_\varepsilon}{p^*} |t|^{p^*}, \quad \text{for all } t \in \mathbb{R}. \tag{3.2.1}$$

Instead if (VK_3) holds, by $(\widetilde{f_1})$ and (f_2) and then by integration for any $\varepsilon > 0$ a positive constant $C_\varepsilon > 0$ exists such that

$$|f(t)| \leq \varepsilon |t|^{m-1} + C_\varepsilon |t|^{p^*-1}, \quad |F(t)| \leq \frac{\varepsilon}{m} |t|^m + \frac{C_\varepsilon}{p^*} |t|^{p^*}, \quad \text{for all } t \in \mathbb{R}, \tag{3.2.2}$$

with $m \in (p, p^*)$.

At this point, in order to recover compactness, we prove the following Hardy-type inequality. First, for every $\zeta \in \mathbb{R}$, $\zeta \geq 1$, let us define the Lebesgue space

$$L^\zeta_{b,K}(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} : \quad u \text{ is measurable and } \int_{\mathbb{R}^N} |x|^{-bp^*} K(x)|u|^\zeta \, dx < +\infty \right\}.$$

Proposition 3.2.1. *Assume $(V, K) \in \mathcal{K}$. Then, if (VK_2) holds, X is compactly embedded in $L^\zeta_{b,K}(\mathbb{R}^N)$ for every $\zeta \in (p, p^*)$. If (VK_3) holds, X is compactly embedded in $L^m_{b,K}(\mathbb{R}^N)$.*

Proof. First, assume that (VK_2) holds.Let $\zeta \in (p, p^*)$. Observe that (VK_0) implies that

$$\lim_{|t|\to 0} \frac{K(x)}{V(x)} \frac{|t|^\zeta}{|t|^p} = 0 \text{ a.e. on } x \in \mathbb{R}^N.$$

Fixed $\varepsilon > 0$, there exists $t_0 > 0$ such that

$$K(x)|t|^\zeta < \varepsilon V(x)|t|^p \text{ for all } |t| < t_0 \text{ and a.e. on } x \in \mathbb{R}^N. \tag{3.2.3}$$

(VK_2) implies that

$$\lim_{|t|\to +\infty} K(x) \frac{|t|^\zeta}{|t|^{p^*}} = 0 \text{ a.e. on } x \in \mathbb{R}^N.$$

Then there exists $t_1 > 0$ such that

$$K(x)|t|^\zeta < \varepsilon |t|^{p^*} \text{ for all } |t| > t_1 \text{ and a.e. on } x \in \mathbb{R}^N. \tag{3.2.4}$$

Continuity of the function $|t|^{\zeta - p^*}$ over the compact interval $[t_0, t_1]$ implies the existence of $C > 0$ such that

$$|t|^\zeta \leq C|t|^{p^*} \text{ for all } t \in [t_0, t_1]. \tag{3.2.5}$$

Thus,

$$K(x)|t|^\zeta \leq \varepsilon\, C\, (V(x)|t|^p + |t|^{p^*}) + C\, K(x)\, \chi_{[t_0,t_1]}(|t|)|t|^{p^*}, \qquad \text{for all } t \in \mathbb{R}.$$

Fix $u \in X$ and let $r_1 > 0$, the last estimate provides

$$\begin{aligned} \int_{B_{r_1}^c} |x|^{-bp^*} K(x)|u|^q dx &\leq \varepsilon C \left(\int_{B_{r_1}^c} |x|^{-bp^*} V(x)|u|^p dx + \int_{B_{r_1}^c} |x|^{-bp^*} |u|^{p^*} \right) \\ &\quad + C \int_{B_{r_1}^c} K(x)\chi_{[t_0,t_1]}(|u|)|u|^{p^*} \\ &\leq \varepsilon C Q(u) + C \int_{A \cap B_{r_1}^c} |x|^{-bp^*} K(x) dx, \end{aligned} \tag{3.2.6}$$

where

$$Q(u) = \int_{B_{r_1}^c} |x|^{-bp^*} V(x)|u|^p dx + \int_{B_{r_1}^c} |x|^{-bp^*} |u|^{p^*} dx$$

and $A = \{x \in \mathbb{R}^N; t_0 \leq |u(x)| \leq t_1\}$.

Since $v_n \rightharpoonup v$ in X, (v_n) is bounded in X. By (1.2.1), there exists $c_1 > 0$ such that

$$Q(v_n) \leq c_1. \tag{3.2.7}$$

From (VK_1), choose $r_2 > r_1$ such that

$$\int_{A \cap B_{r_2}^c} |x|^{-bp^*} K(x) dx \leq \varepsilon. \tag{3.2.8}$$

From (3.2.6), (3.2.7) and (3.2.8),

$$\int_{B_{r_2}^c} |x|^{-bp^*} K(x)|v_n|^q dx \leq \varepsilon. \tag{3.2.9}$$

Consider $v_n = v$ for all n. We see that (3.2.9) allow us to choose $r_3 > r_2$ such that

$$\int_{B^c_{r_3}} |x|^{-bp^*} K(x)|v|^q dx \leq \varepsilon. \tag{3.2.10}$$

From (3.2.9) and (3.2.10),

$$\int_{B^c_{r_3}} |x|^{-bp^*} K(x)|v_n|^q dx \to \int_{B^c_{r_3}} |x|^{-bp^*} K(x)|v|^q dx. \tag{3.2.11}$$

Observe that $X(B_{r_3}) \hookrightarrow D^{1,p}_a(B_{r_3}) \overset{c}{\hookrightarrow} L^{\zeta}_b(B_{r_3}) \hookrightarrow L^{\zeta}_{b,K}(B_{r_3})$ from Theorem C.0.7 and the hypothesis (VK_0), then

$$int_{B_{r_3}} |x|^{-bp^*} K(x)|v_n|^q dx \to \int_{B_{r_3}} |x|^{-bp^*} K(x)|v|^q dx. \tag{3.2.12}$$

It follows that

$$\int_{\mathbb{R}^N} |x|^{-bp^*} K(x)|v_n|^q dx \to \int_{\mathbb{R}^N} |x|^{-bp^*} K(x)|v|^q dx. \tag{3.2.13}$$

If instead (VK_3) holds, we consider, for every $x \in \mathbb{R}^N$ fixed, the function

$$g(t) = V(x)t^{p-m} + t^{p^*-m}, \quad \text{for every } t > 0.$$

Since its minimum value is $C_m V(x)^{\frac{p^*-m}{p^*-p}}$ with $C_m = \left(\frac{p^*-p}{p^*-m}\right)\left(\frac{m-p}{p^*-m}\right)^{\frac{p-m}{p^*-p}}$, it is

$$C_m V(x)^{\frac{p^*-m}{p^*-p}} \leq V(x)t^{p-m} + t^{p^*-m}, \quad \text{for every } x \in \mathbb{R}^N \text{ and } t > 0.$$

Combining this inequality with (VK_3), for any $\varepsilon > 0$ there exists a positive radius $r > 0$ sufficiently large such that

$$K(x)|t|^m \leq \varepsilon\, C'_m(V(x)|t|^p + |t|^{p^*}), \quad \text{for every } t \in \mathbb{R} \text{ and } |x| > r$$

where $C'_m = C_m^{-1}$, from which it follows

$$\int_{B^c_r(0)} |x|^{-bp^*} K(x)|u|^m\, dx \leq \varepsilon C'_m \int_{B^c_r(0)} |x|^{-bp^*} (V(x)|u|^p + |u|^{p^*}) dx, \quad \text{for all } u \in X.$$

If $\{u_n\}_n$ is a sequence such that $u_n \rightharpoonup u$ in X, there exists $C' > 0$ such that

$$\int_{\mathbb{R}^N} |x|^{-bp^*} V(x)|u_n|^p \leq C' \quad \text{and} \quad \int_{\mathbb{R}^N} |x|^{-bp^*} |u_n|^{p^*}\, dx \leq C', \quad \text{for all } n \in \mathbb{N},$$

and then

$$\int_{B^c_r(0)} |x|^{-bp^*} K(x)|u_n|^m\, dx \leq 2C'C'_m\varepsilon, \quad \forall n \in \mathbb{N}. \tag{3.2.14}$$

Since $m \in (p, p^*)$ and K is a continuous function, from Sobolev imbeddings on bounded domains it is

$$\lim_{n\to+\infty} \int_{B_r(0)} |x|^{-bp^*} K(x)|u_n|^m\, dx = \int_{B_r(0)} |x|^{-bp^*} K(x)|u|^m\, dx. \tag{3.2.15}$$

Then, from for $\varepsilon > 0$ small enough such that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|x|^{-bp^*}K(x)|u_n|^m\,dx = \int_{\mathbb{R}^N}|x|^{-bp^*}K(x)|u|^m\,dx$$

and this implies

$$u_n \to u, \quad \text{in } L^m_{b,K}(\mathbb{R}^N).$$

□

Therefore, we can prove the following compactness result related to the nonlinear term.

Lemma 3.2.2. *Suppose that f satisfies $(f_1)-(f_2)$ or $(\widetilde{f_1})-(f_2)$ and $(V,K)\in\mathcal{K}$. If $\{u_n\}_n$ is a sequence such that $u_n \rightharpoonup u$ in X, then*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|x|^{-bp^*}K(x)F(u_n)\,dx = \int_{\mathbb{R}^N}|x|^{-bp^*}K(x)F(u)\,dx$$

and

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|x|^{-bp^*}K(x)f(u_n)u_n\,dx = \int_{\mathbb{R}^N}|x|^{-bp^*}K(x)f(u)u\,dx.$$

Proof. We are going to prove only the second limit because the proof of the first limit is similar to the proof of the second one. Assume that (VK_2) holds. From $(f_1)-(f_2)$, fixing $\zeta\in(p,p^*)$ and taking $\varepsilon>0$, there exists $C>0$ such that

$$|K(x)f(t)t| \le \varepsilon C(V(x)|t|^p + |t|^{p^*}) + K(x)|t|^\zeta, \quad \text{for all } t\in\mathbb{R}. \tag{3.2.16}$$

From Proposition 3.2.1 since

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}|x|^{-bp^*}K(x)|u_n|^\zeta\,dx = \int_{\mathbb{R}^N}|x|^{-bp^*}K(x)|u|^\zeta\,dx,$$

there exists a positive radius $r>0$ such that

$$\int_{B_r^c(0)}|x|^{-bp^*}K(x)|u_n|^\zeta\,dx < \varepsilon, \quad \text{for all } n\in\mathbb{N}. \tag{3.2.17}$$

Since $\{u_n\}_n$ is bounded in X, there exists a positive constant C' such that

$$\int_{\mathbb{R}^N}|x|^{-bp^*}V(x)|u_n|^p \le C' \quad \text{and} \quad \int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p^*}\,dx \le C', \quad \text{for all } n\in\mathbb{N}.$$

From this inequality together with (3.2.16) and (3.2.17) it is

$$\left|\int_{B_r^c(0)}|x|^{-bp^*}K(x)|u_n|^\zeta\,dx\right| < (2CC'+1)\varepsilon, \quad \text{for all } n\in\mathbb{N}.$$

Now assume (VK_3) and following the same arguments in the second part of proof of Proposition 3.2.1, given $\varepsilon>0$ sufficiently small, there exists $r>0$ large enough such that

$$K(x)\le \varepsilon C'_m(V(x)|t|^{p-m} + |t|^{p^*-m}), \quad \text{for every } |t|>0 \text{ and } |x|>r.$$

Consequently, for all $|t|>0$ and $|x|>r$

$$K(x)|f(t)t| \le \varepsilon\, C'_m(V(x)|f(t)t||t|^{p-m} + |f(t)t||t|^{p^*-m}).$$

From $(\widetilde{f_1})$ and (f_2), there exist $C, t_0, t_1 > 0$ satisfying

$$K(x)|f(t)t| \leq \varepsilon\, C(V(x)|t|^p + |t|^{p^*}), \quad \text{for every } t \in I \text{ and } |x| > r$$

where $I = \{t \in \mathbb{R} : |t| < t_0 \quad \text{or} \quad |t| > t_1\}$. Therefore, for every $u \in X$ the following estimate holds

$$\int_{B_r^c(0)} |x|^{-bp^*} K(x) f(u) u\, dx \quad \leq \quad \varepsilon C Q(u) + C \int_{A \cap B_r^c(0)} |x|^{-bp^*} K(x)\, dx$$

with

$$Q(u) = \int_{\mathbb{R}^N} |x|^{-bp^*} V(x) |u|^p\, dx + \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p^*}\, dx$$

and

$$A = \left\{ x \in \mathbb{R}^N : t_0 \leq |u(x)| \leq t_1 \right\}.$$

Since $\{u_n\}_n$ is bounded in X, there exists $C' > 0$ such that

$$\int_{\mathbb{R}^N} |x|^{-bp^*} V(x) |u_n|^p \leq C' \quad \text{and} \quad \int_{\mathbb{R}^N} |x|^{-bp^*} |u_n|^{p^*}\, dx \leq C', \quad \text{for all } n \in \mathbb{N}.$$

Therefore,

$$\int_{B_r^c(0)} |x|^{-bp^*} K(x) f(u_n) u_n\, dx \leq C''\varepsilon + C \int_{A_n \cap B_r^c(0)} |x|^{-bp^*} K(x)\, dx,$$

where

$$A_n = \left\{ x \in \mathbb{R}^N : t_0 \leq |u_n(x)| \leq t_1 \right\}.$$

Following the same arguments in the proof of Proposition 3.2.1 and by (VK_1) we deduce that

$$\int_{A_n \cap B_r^c(0)} |x|^{-bp^*} K(x)\, dx \to 0, \quad \text{as } r \to +\infty,$$

uniformly in $n \in \mathbb{N}$ and, for $\varepsilon > 0$ small enough

$$\left| \int_{B_r^c(0)} |x|^{-bp^*} K(x) f(u_n) u_n\, dx \right| < (C'' + 1)\varepsilon.$$

In order to complete the proof, we have to prove that

$$\lim_{n \to +\infty} \int_{B_r(0)} |x|^{-bp^*} K(x) f(u_n) u_n\, dx = \int_{B_r(0)} |x|^{-bp^*} K(x) f(u) u\, dx.$$

Since $\{u_n\}_n$ is bounded in X, $\{u_n\}_n$ is bounded in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$, then it is bounded in $L_b^{p^*}(\mathbb{R}^N)$ by (1.2.1). Furthermore, there exists $u \in L_b^{p^*}(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup u \text{ in } L_b^{p^*}(\mathbb{R}^N),$$

then

$$u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N.$$

Let $P(x, s) = |x|^{-bp^*} K(x) f(s) s$ and $Q(x, s) = |x|^{-bp^*} K(x) |s|^{p^*}$.
If (VK_2) holds, then (f_1) and (f_2) provide

$$f(t)t \leq \varepsilon(|t|^p + |t|^{p^*}) + C|t|^p \text{ for all } t \in \mathbb{R}.$$

If (VK_3) holds, then $(\tilde{f}_1)$ and (f_2) provide

$$f(t)t \leq \varepsilon(|t|^m + |t|^{p^*}) + C|t|^p \text{ for all } t \in \mathbb{R}$$

with $m \in (p, p^*)$.

In both cases, holds

$$\lim_{|s| \to +\infty} \frac{P(x,s)}{Q(x,s)} = 0 \text{ uniformly in } x \in \mathbb{R}^N.$$

Thus, the result follows from Theorem C.0.9. □

3.3 Existence of a least energy positive solution

Since we intend to find a positive solution, we will assume that

(f_+) $f(t) = 0$ for all $t \in (-\infty, 0]$.

Now we define the Nehari set associated to the functional J given by

$$\mathcal{N} = \{u \in X \backslash \{0\} : J'(u)u = 0\}.$$

In the next result we show that for each $u \in X$ with $u \neq 0$, there is an unique projection in $\mathcal{N}$.

Lemma 3.3.1. *If (f_1)- (f_4) hold, then, for each $u \in X$ with $u \neq 0$, there exists an unique $t_0 = t_0(u) > 0$ such that $t_0 u \in \mathcal{N}$ and $J(t_0 u) = \max_{t \geq 0} J(tu)$.*

Proof. Let $u \in X$ be a function with $u \neq 0$ and $h(t) = J(tu)$, i.e.,

$$h(t) = \frac{t^p}{p} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx + \frac{t^p}{p} \int_{\mathbb{R}^N} |x|^{-bp^*} V(x)|u|^p dx - \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) F(tu) dx.$$

Let us start by assuming that (VK_2) holds. By (f_1) and (f_2), fixing $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} h(t) \geq\ & \frac{t^p}{p} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx + (1-\varepsilon)\frac{t^p}{p} \int_{\mathbb{R}^N} |x|^{-bp^*} V(x)|u|^p dx \\ & - C_\varepsilon \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^N} |x|^{-bp^*} K(x)|u|^{p^*} dx. \end{aligned}$$

Then, there exists $t_1 > 0$ sufficient small such that $h(t) > 0$, for all $0 < t < t_1$.

Now suppose that (VK_3) holds. By $(\widetilde{f_1})$ and (f_2), fixing $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} h(t) \geq\ & \frac{t^p}{p} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx + \frac{t^p}{p} \int_{\mathbb{R}^N} |x|^{-bp^*} V(x)|u|^p dx \\ & - \varepsilon \frac{t^m}{m} \int_{\mathbb{R}^N} |x|^{-bp^*} K(x)|u|^m dx - C_\varepsilon \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^N} |x|^{-bp^*} K(x)|u|^{p^*} dx. \end{aligned}$$

Then, there exists $t_1 > 0$ sufficient small such that $h(t) > 0$, for all $0 < t < t_1$.

Let us recall that, from (f_3), there exist two positive constants $D, D' > 0$ such that $F(t) \geq Dt^\theta - D'$, for all $t > 0$. Then, choosing $\varphi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$ with $\varphi \geq 0$ in $\mathbb{R}^N$, we have

$$\begin{aligned} J(t\varphi) \leq & \frac{t^p}{p} \int_{\operatorname{supp}\varphi} |x|^{-ap} |\nabla \varphi|^p \, dx + \frac{t^p}{p} \int_{\operatorname{supp}\varphi} |x|^{-bp^*} V(x) |\varphi|^p \, dx \\ & - Dt^\theta \int_{\operatorname{supp}\varphi} |x|^{-bp^*} K(x) |\varphi|^\theta \, dx + D' \int_{\operatorname{supp}\varphi} |x|^{-bp^*} K(x) \, dx. \end{aligned}$$

Since $p < \theta < p^*$, there exists $\bar{t} > 1$ such that $h(t) < 0$, for every $t \geq \bar{t}$.

Hence, there exists $t_0 > 0$ such that

$$h(t_0) = \max_{t \geq 0} h(t) = \max_{t \geq 0} J(tu),$$

which implies $h'(t_0) = 0$, i.e.,

$$t_0^{p-1} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx + t_0^{p-1} \int_{\mathbb{R}^N} |x|^{-bp^*} V(x) |u|^p dx = \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) f(t_0 u) u dx$$

that implies $t_0 u \in \mathcal{N}$.

We show that t_0 is unique. Suppose, by contradiction, there exists $s > 0$ such that $su \in \mathcal{N}$. Then,

$$\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx + \int_{\mathbb{R}^N} |x|^{-bp^*} V(x) |u|^p dx = \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) \frac{f(t_0 u)}{t_0^{p-1}} u dx$$

and

$$\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p dx + \int_{\mathbb{R}^N} |x|^{-bp^*} V(x) |u|^p dx = \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) \frac{f(s_0 u)}{s_0^{p-1}} u dx.$$

from (f_4), follows by $t_0 = s_0$. □

In the next lemma we show that the minimizing sequences cannot converge to zero. Moreover, there exists a real number $c = \inf_{\mathcal{N}} J > 0$.

Lemma 3.3.2. *For all $u \in \mathcal{N}$, there exists a positive constant C independent on u such that $0 < C \leq \|u\|$ and $J(u) \geq 0$.*

Proof. Note that using (VK_2), (f_1), (f_2) and for all $u \in \mathcal{N}$, we have

$$\|u\|_V^p = \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) f(u) u \, dx \leq \varepsilon \left\| \frac{K}{V} \right\|_\infty \int_{\mathbb{R}^N} |x|^{-bp^*} V(x) |u|^p \, dx + C_\varepsilon \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) |u|^{p^*} \, dx.$$

Then, choosing $\varepsilon > 0$ we get

$$0 < \left[\frac{(1 - \varepsilon C_1 \left\| \frac{K}{V} \right\|_\infty)}{C_\varepsilon C_2 \|K\|_\infty} \right]^{1/(p^* - p)} \leq \|u\|.$$

Note that using (VK_3), $(\widetilde{f_1})$, (f_2) and for all $u \in \mathcal{N}$, we have

$$\begin{aligned} \|u\|_V^p &= \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) f(u) u \, dx \\ &\leq \varepsilon \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) |u|^m \, dx + C_\varepsilon \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) |u|^{p^*} \, dx. \end{aligned}$$

Arguing as Lemma 3.2.2 we get

$$\|u\|_V^p \le \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) f(u) u \; dx \le \varepsilon \int_{\mathbb{R}^N} |x|^{-bp^*} V(x) u^p \; dx + C \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) |u|^{p^*} \; dx,$$

using that K is a continuous function and $m < p^*$ we have

$$\|u\|_V^p \le \varepsilon C_1 \|u\|^p + C_2 \|K\|_\infty \|u\|^m$$

and choosing $\varepsilon > 0$ we get

$$0 < \left[\frac{1 - \varepsilon C_1}{C_2 \|K\|_\infty}\right]^{1/(m-p)} \le \|u\|.$$

Note that, from (f_4), we obtain

$$f'(t)t - (p-1) f(t) > 0, \tag{3.3.1}$$

for all $t > 0$. But this inequality implies that

$$\frac{1}{p} f(t)t - F(t) \;\; \text{is increasing for} \;\; t > 0. \tag{3.3.2}$$

Using (3.3.2) and (VK_0) we derive

$$J(u) = J(u) - \frac{1}{p} J'(u)u \ge \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) [\frac{1}{p} f(u)u - F(u)] dx \ge 0.$$

□

In the next result we prove that the minimizing sequence is bounded.

Lemma 3.3.3. *If $(u_n) \subset \mathcal{N}$ is a minimizing sequence for J, then (u_n) is bounded in E.*

Proof. From (f_+), we can consider $u_n \ge 0$, for all $n \in \mathbb{N}$. Then,

$$c + o_n(1) = J(u_n) - \frac{1}{p} J'(u_n) u_n \ge \frac{1}{p} \|u_n\|_V^p + \int_{\mathbb{R}^N} K(x) [\frac{1}{p} f(u_n) u_n - F(u_n)] dx.$$

Using (3.3.2) and (VK_0) the proof is over. □

In the next result we prove that c is achieved.

Lemma 3.3.4. *There exists $u \in \mathcal{N}$ such that $J(u) = c$.*

Proof. Consider $(u_n) \subset \mathcal{N}$ a minimizing sequence. Then, it is bounded in X and, up to a subsequence, we have $u_n \rightharpoonup u_0$ in X. Note that $u_0 \neq 0$, because otherwise, using Lemma 3.2.2, we obtain $\|u_n\|_V^p = \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) f(u_n) u_n \; dx = o_n(1)$, which is a contradiction with Lemma 3.3.2.

Consider $t_0 > 0$ such that $u = t_0 u_0 \in \mathcal{N}$. Since $\|\cdot\|_V$ is weak lower semicontinuous and $\int_{\mathbb{R}^N} |x|^{-bp^*} K(x) F(t_0 u_n) dx = \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) F(t_0 u_0) dx + o_n(1)$ and we get

$$c \le J(u) = J(t_0 u_0) \le \liminf_{n \to \infty} J(t_0 u_n) \le \liminf_{n \to \infty} J(u_n) = c.$$

□

3.3.1 Proof of Theorem 3.1.1

Lemma 3.3.5. *Suppose that $u \in \mathcal{N}$, $c = \inf_{v \in \mathcal{N}} J(v)$ and $J(u) = c$. Then u is a weak solution of the problem (P).*

Proof. Suppose, by contradiction, that u is not a weak solution of (P). Then we find a function $\phi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\begin{aligned} J'(u)\phi &= \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla \phi dx + \int_{\mathbb{R}^N} |x|^{-bp^*} V(x)|u|^{p-2}u\phi dx \\ &- \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) f(u)\phi dx \leq -1. \end{aligned}$$

Choose $\varepsilon > 0$ small such that

$$J'(tu + \sigma\phi)\phi \leq -\frac{1}{2}, \quad \text{for } |t-1| + |\sigma| \leq \varepsilon. \tag{3.3.3}$$

Let η be a cut-off function that $\eta(t) = 1$ for $|t-1| \leq \varepsilon/2$ and $\eta(t) = 0$ for $|t-1| \geq \varepsilon$.

Now we estimate $\sup_{t \geq 0} J(tu + \varepsilon\eta\phi)$. Observe that for all (t,σ) we have $J(tu+\varepsilon\eta\phi) < J(u)$. In fact, for $|t-1| \geq \varepsilon$, we have $J(tu+\varepsilon\eta\phi) = J(tu) < J(u)$. For $0 < |t-1| \leq \varepsilon$, from (3.3.3) we have

$$\begin{aligned} J(tu + \varepsilon\eta\phi) &= J(tu) + \int_0^1 J'(tu + \sigma\varepsilon\eta(t)\phi)\varepsilon\eta(t)\phi d\sigma \leq J(tu) - \frac{1}{2}\varepsilon\eta(t) \\ &\leq J(tu) < J(u). \end{aligned}$$

Now for $t = 1$, $J(tu + \varepsilon\eta(t)\phi) = J(u + \varepsilon\eta(1)\phi) \leq J(u) - \frac{1}{2}\varepsilon < J(u)$. We concluded $\sup_{t \geq 0} J(tu + \varepsilon\eta\phi) < c = \inf_{u \in \mathcal{N}} J(u)$. Now it is sufficient to find $\bar{t} > 0$ such that $\bar{t}u + \varepsilon\eta(\bar{t})\phi \in \mathcal{N}$, which is a contradiction by definition of c. For this, consider the function $h : [1-\varepsilon, 1+\varepsilon] \to X$ given by $h(t) = tu + \varepsilon\eta(t)\phi$ and $\Upsilon : [1-\varepsilon, 1+\varepsilon] \to \mathbb{R}$ given by $\Upsilon(t) = J'(tu + \varepsilon\eta(t)\phi)(tu + \varepsilon\eta(t)\phi)$. Note that $\Upsilon(t) = P(t) - Q(t)$ where P is a polynomial and $Q(t) = \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) f(tu + \varepsilon\eta(t)\phi)(tu + \varepsilon\eta(t)\phi)dx$ arguing as Lemma 3.2.2, we get that Υ is a continuous function.

Observe that $\Upsilon(1-\varepsilon) = J'((1-\varepsilon)u)(1-\varepsilon)u > 0$ and $\Upsilon(1+\varepsilon) = J'((1+\varepsilon)u)(1+\varepsilon)u < 0$. Indeed, $u \in \mathcal{N}$, then $J(u) = \max_{t \geq 0} J(tu)$ from the Lemma 3.3.1, i.e., 1 is the maximum point of the function $w : \mathbb{R} \to \mathbb{R}$, $w(t) = J(tu)$, then

$$w'(1-\varepsilon) = J'((1-\varepsilon)u)u > 0$$

and

$$w'(1+\varepsilon) = J'((1+\varepsilon)u)u < 0.$$

Since $\varepsilon > 0$ is small, then

$$J'((1-\varepsilon)u)(1-\varepsilon)u > 0$$

and

$$J'((1+\varepsilon)u)(1+\varepsilon)u < 0,$$

hence

$$\Upsilon(1-\varepsilon) > 0$$

and

$$\Upsilon(1+\varepsilon) < 0.$$

Thus, Intermediate Value Theorem ensures that there exists $\bar{t} \in (1-\varepsilon, 1+\varepsilon)$ such that $\Upsilon(\bar{t}) = 0$. □

3.4 Existence of a least Energy nodal solution

In the following we search a nodal or sign-changing weak solution of problem (P), i.e., a function $u \in X$ such that $u^+ := \max\{u, 0\} \neq 0$, $u^- := \min\{u, 0\} \neq 0$ in $\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla v\, dx + \int_{\mathbb{R}^N} |x|^{-bp^*}V(x)|u|^{p-2}uv\, dx - \int_{\mathbb{R}^N} |x|^{-bp^*}K(x)f(u)v\, dx = 0, \quad \text{for all } v \in X.$$

In particular, we look for $u \in X$ which has exactly two nodal domains or equivalently changes sign exactly once. Since the Nehari manifold associated to the functional J

$$\mathcal{N} := \Big\{ u \in X \backslash \{0\} : \big\langle J'(u), u \big\rangle = 0 \Big\}$$

is a natural constraint for J, we can look for critical points of J constrained on $\mathcal{N}$ still denoting, for simplicity of notations, $J_{|\mathcal{N}}$ by J.
Recall that a non zero critical point w of J is a least energy weak solution of (P) if $J(w) = \min\limits_{v \in \mathcal{N}} J(v)$ and, since our purpose is to prove the existence of a least energy sign-changing weak solution of (P), in particular, we look for $w \in \mathcal{M}$ such that $J(w) = \min\limits_{v \in \mathcal{M}} J(v)$, where $\mathcal{M}$ is the subset of $\mathcal{N}$ containing all sign-changing weak solutions of (P), i.e.,

$$\mathcal{M} \quad = \quad \Big\{ w \in \mathcal{N} : w^+ \neq 0, w^- \neq 0, \big\langle J'(w^+), w^+ \big\rangle = 0 = \big\langle J'(w^-), w^- \big\rangle \Big\}.$$

For sake of simplicity, in the following we often denote

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla w^\pm|^p\, dx + \int_{\mathbb{R}^N} |x|^{-bp^*}V(x)|w^\pm|^p\, dx = \int_{\mathbb{R}^N} |x|^{-bp^*}K(x)f(w^\pm)w^\pm\, dx.$$

So, let us begin by establishing some preliminary results which will be exploited in the last section for a minimization argument.
In particular, in this first lemma, we prove that J is strictly positive on $\mathcal{N}$ then on $\mathcal{M}$, $\|\cdot\|$ is uniformly bounded from below by a strictly positive radius on $\mathcal{N}$ and then on $\mathcal{M}$ and the same applies to the positive and negative part $w^\pm$ of every $w \in \mathcal{M}$. It follows that J is coercive on $\mathcal{N}$ and in particular on $\mathcal{M}$ since $J(u) \to +\infty$ as $\|u\| \to +\infty$, for every $u \in \mathcal{N}$.

Lemma 3.4.1. **(i)** *For all $u \in \mathcal{N}$ such that $\|u\|_V \to +\infty$, then $J(u) \to +\infty$.*

(ii) *There exists $\rho > 0$ such that $\|u\|_V \geq \rho$ for all $u \in \mathcal{N}$ and $\|w^\pm\|_V \geq \rho$ for all $w \in \mathcal{M}$.*

Proof. Consider $(u_n) \subset \mathcal{N}$ such that $\|u_n\|_V^p \to \infty$. By (f_3) and (VK_0), we have that

$$J(u_n) - \frac{1}{\theta}J'(u_n)u_n = \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|_V^p + \int_{\mathbb{R}^N} |x|^{-bp^*}K(x)[\frac{1}{\theta}f(u_n)u_n - F(u_n)]dx,$$

which proves (i).

In order to prove (ii), let us observe that, by assumptions $(f_1) - (f_2)$ or $(\widetilde{f}_1) - (f_2)$, respectively we have that, for any $\varepsilon > 0$, a positive constant $C_\varepsilon > 0$ exists such that

$$|f(t)t| \leq \varepsilon|t|^p + C_\varepsilon|t|^{p^*}, \text{ for all } t \in \mathbb{R}. \tag{3.4.1}$$

$$|f(t)t| \leq \varepsilon|t|^m + C_\varepsilon|t|^{p^*}, \text{ for all } t \in \mathbb{R}. \tag{3.4.2}$$

Since for every $u \in \mathcal{N}$ it holds

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx + \int_{\mathbb{R}^N} |x|^{-bp^*}V(x)|u|^p dx = \int_{\mathbb{R}^N} |x|^{-bp^*}K(x)f(u)udx$$

if (VK_2) holds, considering (3.4.1), by Sobolev embeddings it is

$$C_0'\|u\|_V^p \leq \widetilde{C}_\varepsilon\|u\|_V^{p^*}, \tag{3.4.3}$$

where C_0' and $\widetilde{C}_\varepsilon$ are positive constants. Instead if (VK_3) holds, using (3.4.2) and by continuous Sobolev imbeddings we get

$$\|u\|_V^p \leq \varepsilon\widetilde{C}\|u\|_V^m + \widetilde{C}_\varepsilon\|u\|_V^{p^*}. \tag{3.4.4}$$

Hence, in both cases, there exists a positive radius $\rho_1 > 0$ such that $\|u\|_V \geq \rho_1$.

Now, if $w \in \mathcal{M}$, we have that $\langle J'(w^\pm), w^\pm\rangle = 0$ namely $w^\pm \in \mathcal{N}$, hence by the previous estimate we obtain $0 < \rho \leq \|w^\pm\|_V$. □

From previous lemma we deduce a result valid for every sequence in $\mathcal{M}$ that we apply in the last section to every bounded minimizing sequence of J on $\mathcal{M}$ so that the candidate minimizer is different from zero.

Remark 2. *If (w_n) is a sequence in $\mathcal{M}$, we have that*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |x|^{-bp^*}|w_n^\pm|^m\, dx > 0.$$

Lemma 3.4.2. *If $v \in X$ with $v^\pm \neq 0$, then there exist $t, s > 0$ such that*

$$\left\langle J'(tv^+ + sv^-), v^+\right\rangle = 0 \text{ and } \left\langle J'(tv^+ + sv^-), v^-\right\rangle = 0.$$

Consequently, $tv^+ + sv^- \in \mathcal{M}$.

Proof. Since the support of positive part of v and negative part of v are disjoint, the proof is similar of Lemma 3.3.1. □

3.4.1 Proof of the existence of nodal solution

At this point, we can finally prove the existence of $w \in \mathcal{M}$ in which the infimum of J is attained on $\mathcal{M}$. We find that w is a critical point of J and then a least energy nodal solution of (P). In order to complete the proof of Theorem 3.1.1, we conclude by showing that w has exactly two nodal domains.

First, let us start with the existence of a minimizer $w \in \mathcal{M}$ of J. In what follows, we denote c_0 the infimun of J in $\mathcal{M}$

$$c_0 = \inf_{v \in \mathcal{M}} J(v).$$

By Lemma 3.4.1, we deduce that $c_0 > 0$. Thus, there exists a bounded minimizing sequence (w_n) in $\mathcal{M}$ and J is coercive on $\mathcal{M}$ from Lemma 3.4.1. If (VK_3) holds, then, arguing as Proposition 3.2.1, we can assume up to a subsequence that there exist $w, w_1, w_2 \in X$ such that

$$\begin{aligned} &w_n \rightharpoonup w, \quad w_n^+ \rightharpoonup w_1, \quad w_n^- \rightharpoonup w_2 \quad \text{in } X \\ &w_n \to w, \quad w_n^+ \to w_1, \quad w_n^- \to w_2 \quad \text{in } L^m_{b,K}(\mathbb{R}^N),\ m \in (p, p^*). \end{aligned}$$

Since the transformations $w \to w^+$ and $w \to w^-$ are continuous from $L^m_{b,K}(\mathbb{R}^N)$ in $L^m_{b,K}(\mathbb{R}^N)$ (see Lemma 2.3 in [17] with suitable adaptations), we have that $w^+ = w_1 \geq 0$ and $w^- = w_2 \leq 0$. At this point, we can prove that $w \in \mathcal{M}$. Indeed, by $w_n^+ \to w^+$ and $w_n^- \to w^-$ in $L^m_b(\mathbb{R}^N)$ it is, as $n \to +\infty$

$$\int_{\mathbb{R}^N} |x|^{-bp^*} |(w_n)^{\pm}|^m \, dx \to \int_{\mathbb{R}^N} |x|^{-bp^*} |w^{\pm}|^m \, dx.$$

Then, by Remark 2, we conclude that $w^{\pm} \neq 0$ and consequently $w = w^+ + w^-$ is sign-changing.

By Lemma 3.4.2 and the fact that w^+ and w^- have disjoint supports, there exist $\bar{t}, \bar{s} > 0$ such that

$$\left\langle J'(\bar{t}w^+ + \bar{s}w^-), w^+ \right\rangle = \left\langle J'(\bar{t}w^+), w^+ \right\rangle = \left\langle J'(\bar{t}w^+), \bar{t}w^+ \right\rangle = 0, \tag{3.4.5}$$

$$\left\langle J'(\bar{t}w^+ + \bar{s}w^-), w^- \right\rangle = \left\langle J'(\bar{s}w^-), w^- \right\rangle = \left\langle J'(\bar{s}w^-), \bar{s}w^- \right\rangle = 0, \tag{3.4.6}$$

then $\bar{t}w^+ + sw^- \in \mathcal{M}$.

Now, let us prove that $\bar{t}, \bar{s} \leq 1$.

Proposition 3.4.3. *Let $\bar{t}, \bar{s} > 0$ be the values of the projections of w^+ and w^- in $\mathcal{M}$. Then $\bar{t}, \bar{s} \leq 1$.*

Proof. Since $\|.\|_V$ is weak lower semicontinuous and

$$\int_{\mathbb{R}^N} |x|^{-bp^*} K(x) f((w_n)^{\pm})(w_n)^{\pm} \, dx \to \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) f(w^{\pm}) w^{\pm} \, dx$$

from the Lemma 3.2.2, we get

$$\left\langle J'(w^+), w^+ \right\rangle \leq 0 \quad \text{and} \quad \left\langle J'(w^-), w^- \right\rangle \leq 0. \tag{3.4.7}$$

Note that if $\bar{t} > 1$, then $\langle J'(w^+), w^+ \rangle \neq 0$ once that $\bar{t}w^+ \in \mathcal{M}$ and the projection in $\mathcal{M}$ is unique, we conclude that

$$\left\langle J'(w^+), w^+ \right\rangle < 0. \tag{3.4.8}$$

Let $h : \mathbb{R}^+ \to \mathbb{R}$, $h(t) = J(tw^+)$. Observe that

$$h'(\bar{t}) = \langle J'(\bar{t}w^+), w^+\rangle = \frac{1}{\bar{t}}\langle J'(\bar{t}w^+), \bar{t}w^+\rangle = 0,$$

hence h has exactly one critical point once that the projection in $\mathcal{M}$ is unique and it is a maximum point from Lemma 3.3.1. Thus, for $\varepsilon > 0$ small so that $(1-\varepsilon)\bar{t} > 1$ and

$$\left\langle J'((1-\varepsilon)\bar{t}w^+), (1-\varepsilon)\bar{t}w^+\right\rangle > 0. \tag{3.4.9}$$

(3.4.8), (3.4.9) and the Intermediate Value Theorem provide the existence of $\xi \in (1, (1-\varepsilon)\bar{t})$ with

$$\left\langle J'(\xi w^+), \xi w^+\right\rangle = 0,$$

then $h'(\xi) = \frac{1}{\xi}\left\langle J'(\xi w^+), \xi w^+\right\rangle = 0$, which is a contradiction with the uniqueness of the critical point of the function h and this shows that $\bar{t} \leq 1$. The same argument shows that $\bar{s} \leq 1$. □

In the next step we show that $J(\bar{t}w^+ + \bar{s}w^-) = c_0$ and $\bar{t} = \bar{s} = 1$ or better $J(w) = c_0$. Indeed, since $\bar{t}, \bar{s} \leq 1$ and $w_n \rightharpoonup w$ in X as $n \to +\infty$, the weak lower semicontinuity of J on X described above and

$$t \longmapsto \frac{1}{p}f(t)t - F(t), \text{ is increasing for } t \in \mathbb{R}.$$

by the hypothesis (f_4), we get

$$\begin{aligned}
c_0 &\leq J(\bar{t}w^+ + \bar{s}w^-) = J(\bar{t}w^+ + \bar{s}w^-) - \frac{1}{p}\left\langle J'(\bar{t}w^+ + \bar{s}w^-), (\bar{t}w^+ + \bar{s}w^-)\right\rangle \\
&= int_{\mathbb{R}^N}|x|^{-bp^*}K(x)\frac{1}{p}f(\bar{t}w^+)tw^+ - F(\bar{t}w^+)dx \\
&+ \int_{\mathbb{R}^N}|x|^{-bp^*}K(x)\frac{1}{p}f(\bar{s}w^-w^-)\bar{s}w^-w^- - F(\bar{s}w^-w^-)dx \\
&\leq \int_{\mathbb{R}^N}|x|^{-bp^*}K(x)\frac{1}{p}f(w^+)w^+ - F(w^+)dx \\
&+ \int_{\mathbb{R}^N}|x|^{-bp^*}K(x)\frac{1}{p}f(w^-)w^- - F(w^-)dx \\
&= J(w^+ + w^-) - \frac{1}{p}\left\langle J'(w^+ + w^-), (w^+ + w^-)\right\rangle \\
&\leq \liminf_{n\to+\infty}\left(J(w_n^+ + w_n^-) - \frac{1}{p}\left\langle J'(w_n^+ + w_n^-), (w_n^+ + w_n^-)\right\rangle\right) \\
&= \lim_{n\to+\infty} J(w_n) = c_0.
\end{aligned}$$

Then we have found that $J(\bar{t}w^+ + \bar{s}w^-) = c_0$ or, equivalently, that there exist $0 < \bar{t}, \bar{s} \leq 1$ such that $\bar{t}w^+ + \bar{s}w^- \in \mathcal{M}$ and $J(tw^+ + \bar{s}w^-) = c_0$. Let us observe that, if $\bar{t} \neq 1$ or $\bar{s} \neq 1$ by above calculations we would obtain a contradiction. Thus, $\bar{t} = \bar{s} = 1$, $w^+ + w^- \in \mathcal{M}$ and $J(w) = c_0$.

At this point, we state that w is a critical point of J, i.e. $J'(w) = 0$. Then w is a weak solution of the problem (P). Suppose, by contradiction, that w is not a weak solution of (P). Then we find a function $\phi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\begin{aligned}
J'(u)\phi &= \int_{\mathbb{R}^N}|x|^{-ap}|\nabla w|^{p-2}\nabla w\nabla\phi dx + \int_{\mathbb{R}^N}|x|^{-bp^*}V(x)|w|^{p-2}w\phi dx \\
&- \int_{\mathbb{R}^N}|x|^{-bp^*}K(x)f(w)\phi dx \leq -1.
\end{aligned}$$

Choose $\varepsilon > 0$ small such that

$$J'(tw^+ + sw^- + \sigma\phi)\phi \le -\frac{1}{2} \text{ for } |t-1| + |s-1| + |\sigma| \le \varepsilon. \tag{3.4.10}$$

Let η be a cut-off function that $\eta(t,s) = 1$ for $|t-1| \le \varepsilon/2$ and $|s-1| \le \varepsilon/2$ and $\eta(t,s) = 0$ for $|t-1| \ge \varepsilon$ or for $|s-1| \ge \varepsilon$.

Now we estimate $\sup\limits_{t,s\ge 0} J(tw^+ + sw^- + \varepsilon\eta\phi)$. Observe that for all (t,s,σ) we have $J(tw^+ + sw^- + \varepsilon\eta\phi) < J(w)$. In fact, for $|t-1| \ge \varepsilon$ or for $|s-1| \ge \varepsilon$, we have $J(tw^+ + sw^- + \varepsilon\eta\phi) = J(tw^+ + sw^-) < J(w)$. For $0 < |t-1| \le \varepsilon$ and for $0 < |s-1| \le \varepsilon$, from (3.4.10) we have

$$\begin{aligned} J(tw^+ + sw^- + \varepsilon\eta\phi) &= J(tw^+ + sw^-) + \int_0^1 J'(tw^+ + sw^- + \sigma\varepsilon\eta(t)\phi)\varepsilon\eta(t)\phi d\sigma \\ &\le \quad J(tw^+ + sw^-) - \frac{1}{2}\varepsilon\eta(t) \le J(tw^+ + sw^-) < J(w). \end{aligned}$$

Now for $t = 1$ and $s = 1$, $J(tw^+ + sw^- + \varepsilon\eta(t)\phi) = J(w + \varepsilon\eta(1)\phi) \le J(w) - \frac{1}{2}\varepsilon < J(w)$. We concluded

$$\sup_{t,s\ge 0} J(tw^+ + sw^- + \varepsilon\eta\phi) < c_0 = \inf_{v\in\mathcal{M}} J(v).$$

Now it is sufficient to find $\bar{t}, \bar{s} > 0$ such that $\bar{t}w^+ + \bar{s}w^- + \varepsilon\eta(\bar{t})\phi \in \mathcal{M}$, which is a contradiction by definition of c_0. For this, consider the function $h : [1-\varepsilon, 1+\varepsilon] \times [1-\varepsilon, 1+\varepsilon] \to X$ given by $h(t,s) = tw^+ + sw^- + \varepsilon\eta(t)\phi$ and $\Upsilon : [1-\varepsilon, 1+\varepsilon] \times [1-\varepsilon, 1+\varepsilon] \to \mathbb{R}$ given by $\Upsilon(t,s) = J'(tw^+ + sw^- + \varepsilon\eta(t)\phi)(tw^+ + sw^- + \varepsilon\eta(t)\phi)$. Note that $\Upsilon(t,s) = P(t,s) - Q(t,s)$ where P is a polynomial and $Q(t,s) = \int_{\mathbb{R}^N} |x|^{-bp^*} K(x) f(tw^+ + sw^- + \varepsilon\eta(t)\phi)(tw^+ + sw^- + \varepsilon\eta(t)\phi)dx$ arguing as Lemma 3.2.2, we get that Υ is a continuous function.

Observe that $\Upsilon((1-\varepsilon),(1-\varepsilon)) = J'((1-\varepsilon)w^+ + (1-\varepsilon)w^-)((1-\varepsilon)w^+ + (1-\varepsilon)w^-) > 0$ and $\Upsilon((1+\varepsilon),(1+\varepsilon)) = J'((1+\varepsilon)w^+ + (1+\varepsilon)w^-)(1+\varepsilon)w^+ + (1+\varepsilon)w^-) < 0$. Indeed, $w^+, w^- \in \mathcal{N}$, then $J(w^+) = \max_{t\ge 0} J(tw^+)$ and $J(w^-) = \max_{s\ge 0} J(sw^-)$ from the Lemma 3.3.1, i.e., $\bar{t} = 1$ is the maximum point of the function $u_+ : \mathbb{R} \to \mathbb{R}$, $u_+(t) = J(tw^+)$ and $\bar{s} = 1$ is the maximum point of the function $u_- : \mathbb{R} \to \mathbb{R}$, $u_-(s) = J(sw^-)$, then

$$(u_+)'(1-\varepsilon) = J'((1-\varepsilon)w^+)w^+ > 0,$$

$$(u_+)'(1+\varepsilon) = J'((1+\varepsilon)w^+)w^+ < 0,$$

$$(u_-)'(1-\varepsilon) = J'((1-\varepsilon)w^-)w^- > 0$$

and

$$(u_-)'(1+\varepsilon) = J'((1+\varepsilon)w^-)w^- < 0.$$

As a consequence,
$\Upsilon((1-\varepsilon),(1-\varepsilon)) = J'((1-\varepsilon)w^+ + (1-\varepsilon)w^-)((1-\varepsilon)w^+ + (1-\varepsilon)w^-) > 0$ and $\Upsilon((1+\varepsilon),(1+\varepsilon)) = J'((1+\varepsilon)w^+ + (1+\varepsilon)w^-)(1+\varepsilon)w^+ + (1+\varepsilon)w^-) < 0$.

Since $\varepsilon > 0$ is small, the Intermediate Value Theorem ensures that there exist $\bar{t}, \bar{s} \in (1-\varepsilon, 1+\varepsilon)$ such that

$$J'(\bar{t}w^+)w^+ = 0$$

and

$$J'(\bar{s}w^-)w^- = 0,$$

hence

$$J'(\bar{t}w^+)\bar{t}w^+ = 0$$

and

$$J'(\bar{s}w^-)\bar{s}w^- = 0.$$

The supports of w^+ and w^- are disjoints, then $\Upsilon(\bar{t}, \bar{s}) = 0$.

Similar argument shows that there exists $w \in \mathcal{M}$ such that w is a critical point of J if (VK_2) holds.

Finally, we prove that w has exactly two nodal domains or equivalently it changes sign exactly once. Let us observe that assumptions (f_1) and (f_2) or $(\widetilde{f_1})$ and (f_2) ensure that w is continuous and then $\widetilde{\mathbb{R}}^N = \{x \in \mathbb{R}^N : w(x) \neq 0\}$ is open. Suppose by contradiction that $\widetilde{\mathbb{R}}^N$ has more than two components or w has more than two nodal domains and, since w changes sign, without loss of generality, we can assume that $w = w_1 + w_2 + w_3$, where $w_1 \geq 0$, $w_2 \leq 0$, $w_3 \neq 0$, and $\mathrm{supp}(w_i) \cap \mathrm{supp}(w_j) = \emptyset$, for $i \neq j$, $i, j = 1, 2, 3$. Clearly it is understood that $w_i = 0$ on $\mathbb{R}^N \setminus \mathrm{supp}(w_i)$ for $i = 1, 2, 3$. So the disjointness of the supports combined with $J'(w) = 0$ implies $\langle J'(w_1 + w_2), w_1 \rangle = 0 = \langle J'(w_1 + w_2), w_2 \rangle$. Since $0 \neq w_1 = (w_1 + w_2)^+$ and $0 \neq w_2 = (w_1 + w_2)^-$, by previous arguments, there exist $t, s \in (0, 1]$ such that $t(w_1 + w_2)^+ + s(w_1 + w_2)^- \in \mathcal{M}$ namely $tw_1 + sw_2 \in \mathcal{M}$ and then $J(tw_1 + sw_2) \geq c_0$.
On the other side, $0 \neq w_3 \in \mathcal{N}$, Lemma 3.4.1 (ii) and

$$c_0 \leq J(tw_1 + sw_2) \leq J(w_1 + w_2) < J(w_1 + w_2) + J(w_3) = J(w) = c_0$$

then a contradiction and we conclude that $w_3 = 0$.

Thus, the proof of Theorem 3.1.1 is complete.

Appendix A

Proof of the Lemma 1.3.8 and Principle of Symmetric Criticality

A.1 Proof of the Lemma 1.3.8

Recall the Lemma 1.3.8.

Lemma A.1.1. *Let $\varepsilon > 0$. Suppose that $\widetilde{\eta} \in \widetilde{\Gamma}$ satisfies*

$$\max_{t\in[0,1]} \widetilde{I}(\widetilde{\eta}) \leq c_* + \varepsilon,$$

then, there exists $(\theta, u) \in \mathbb{R} \times E_{0,rad}$ such that

- $dist_{\mathbb{R}\times E_{0,rad}}((\theta, u), \widetilde{\eta}([0, 1])) \leq 2\sqrt{\varepsilon}$;
- $\widetilde{I}(\theta, u) \in [c_* - \varepsilon, c_* + \varepsilon]$;
- $\|D\widetilde{I}(\theta, u)\|_{\mathbb{R}\times E^*_{0,rad}} \leq 2\sqrt{\varepsilon}$.

Proof. Observe that

$$\widetilde{I}(\widetilde{u}) = \widetilde{I}(0, u) = I(u)$$

and

$$-I(u) \leq |I(u)| \leq \frac{1}{p}\|u\|^p + \frac{\varepsilon}{p}\|u\|^p + \frac{C_1 C_\varepsilon}{q}\|u\|^q \leq C,$$

for all $u \in \overline{B}(0, \rho) \subset E_{0,rad}$ by (1.3.1), then I is bounded below on $\overline{B}(0, \rho)$.

As $X = \big(\overline{B}(0, \rho), \|\cdot\|\big)$ is a complete metric space, I is a lower semicontinuous functional and bounded below on X, the result follows from the hypothesis and by the Ekeland's Variational Principle (see Theorem 1.1 in [28]). □

A.2 Principle of Symmetric Criticality

We prove the Principle of Symmetric Criticality following the proofs of Proposition 2.1 and Theorem 2.7 in [40].

Let X be a real Banach space and let X^* be its dual. The norms of X and X^* will be denoted by $\|\cdot\|$ and $\|\cdot\|_*$; respectively. We shall denote by ${}_{X^*}\langle\cdot,\cdot\rangle_X$ the duality pairing between X and X^*; which will be simply denoted by $\langle\cdot,\cdot\rangle$ if no confusion arises.

Let G be a group and let π be a representation of G over X; that is, $\pi(g)$ is a bounded linear operator in X for each $g \in G$ and

$$\pi(e)u = u, \forall u \in X$$
$$\pi(g_1 g_2)u = \pi(g_1)(\pi(g_2))u, \forall g_1, g_2 \in G, \forall u \in X,$$

where e is the identity element of G: the representation π_x of G over X^* is naturally induced by π through the relation:

$$\langle \pi_*(g)v^*, u\rangle = \langle v^*, \pi(g^{-1})u\rangle, g \in G, v^* \in X^*, u \in X. \tag{A.2.1}$$

For simplicity, we shall often write gu or gv^* instead of $\pi(g)u$ or $\pi_*(g)v^*$, respectively. A function h on X (or X^*) is called G-invariant if

$$h(gu) = h(u), \forall u \in X (\text{or } h(gu^*) = h(u^*), \forall u^* \in X^*), \forall g \in G,$$

and a subset M of X (or M^* of X^*) is called G-invariant if

$$gM = \{gu; u \in M\} \subset M (\text{or } gM^* \subset M^*), \forall g \in G.$$

The linear subspaces of G-symmetric points of X and X^* are defined as the common fixed points of G:

$$\Sigma = \{u \in X; gu = u, \forall g \in G\},$$
$$\Sigma_* = \{v^* \in X^*; gv^* = v^*, \forall g \in G\},$$

Hence, by (A.2.1), $v^* \in X^*$ is symmetric if and only if it is a G-invariant functional. Σ and Σ_* form closed linear subspaces of X and X^*, respectively, so Σ and Σ_* are regarded as Banach spaces with their induced topologies.

Let $C^1_G(X)$ be the set of all G-invariant C^1-functional on X: we consider the following principle:

(P) For all $J \in C^1_G(X)$, it holds that $(J|_\Sigma)'(u) = 0$ assures $J'(u) = 0$ and $u \in \Sigma$.

Here $(J|_\Sigma)'(u)$ and $J'(u)$ denote the Fréchet derivatives of $J|_\Sigma$ and J at u in Σ and X, respectively.

Proposition A.2.1 ([40], Proposition 2.1)**.** *The principle* (P) *is valid if and only if* $\Sigma_* \cap \Sigma^\perp = \{0\}$, *where* $\Sigma^\perp = \{v^* \in X^*; \langle v^*, u\rangle = 0, \forall u \in \Sigma\}$.

Proof. Suppose $\Sigma_* \cap \Sigma^\perp = \{0\}$ and let u_0 be a critical point of $J|_\Sigma$. We must show $J'(u_0) = 0$. Since $J(u_0) = J|_\Sigma(u_0)$ and $J(u_0 + v) = J|_\Sigma(u_0 + v)$ for all $v \in \Sigma$, we get

$${}_{X^*}\langle J'(u_0), v\rangle_X =_{\Sigma^*} \langle (J|_\Sigma)'(u_0), v\rangle_\Sigma = 0, \forall v \in \Sigma,$$

where ${}_{\Sigma^*}\langle \cdot, \cdot\rangle_\Sigma$ denotes the duality pairing between Σ and its dual Σ^*. This implies $J'(u_0) \in \Sigma^\perp$. On the other hand, it follows from the G-invariance of J that

$$\begin{aligned}\langle J'(gu), v\rangle &= \lim_{t\to 0} \frac{J(gu + tv) - J(gu)}{t} \\ &= \lim_{t\to 0} \frac{J(u + tg^{-1}v) - J(u)}{t} \\ &= \langle J'(u), g^{-1}v\rangle \\ &= \langle gJ'(u), v\rangle,\end{aligned}$$

for all $g \in G$ and $u, v \in X$. This means J' is G-equivariant, i.e.,

$$J'(gu) = gJ'(u), \forall g \in G, \forall u \in X. \tag{A.2.2}$$

Especially, since $u_0 \in \Sigma$, we obtain $gJ'(u_0) = J'(u_0)$ for all $g \in G$, that is, $J'(u_0) \in \Sigma_*$. Thus, we conclude $J'(u_0) \in \Sigma_* \cap \Sigma^\perp = \{0\}$, i.e., $J'(u_0) = 0$.

Reciprocally, suppose that there exists a non-zero element $v^* \in \Sigma_* \cap \Sigma^\perp$ and define $J_*(\cdot)$ by $J_*(u) = \langle v^*, u\rangle$. Then $J_* \in C^1_G(X)$ and $(J_*)'(\cdot) = v^* \neq 0$ has no critical point in X. On the other hand, the assumption $v^* \in \Sigma^\perp$ implies $v^*|_\Sigma \equiv 0$, whence follows $(J_*|_\Sigma)'(u) = 0$ for all $u \in \Sigma$. This violates the principle (P). Therefore the condition $\Sigma_* \cap \Sigma^\perp = \{0\}$ is necessary for the principle (P). □

We introduce an assumption on G:

(A) G is a compact topological group and the representation π of G over X is continuous, i.e., $(g, u) \mapsto gu$ is a continuous mapping from $G \times X$ into X.

By Rudin [[43], Theorem 3.27], for each $u \in X$, there exists a unique element $Au \in X$ such that

$$\langle v^*, Au\rangle = \int_G \langle v^*, gu\rangle d\mu(g), \forall v^* \in X^*, \tag{A.2.3}$$

where μ is the normalized Haar measure on G: The mapping A is called the averaging over G and has the following properties:

- A is a continuous linear projection from X onto Σ.
- If K is a G-invariant closed convex subset of X; then $A(K) \subset K$.

Theorem A.2.2 ([40], Theorem 2.7)**.** *If (A) is satisfied, then the (P) is valid.*

Proof. We check the condition $\Sigma_* \cap \Sigma^\perp = \{0\}$ again. Let $v^* \in \Sigma_* \cap \Sigma^\perp$ and suppose $v^* \neq 0$. Since $v^* \in \Sigma_*$, the hyperplane $H = \{u; \langle v^*, u\rangle = 1\}$ becomes a non-empty G-invariant closed convex subset of X: Then, for any $u \in H$, we have $Au \in H \cap \Sigma$ and hence $\langle v^*, u\rangle = 0$ since $v^* \in \Sigma^\perp$. This contradicts the fact that $Au \in H$. □

Appendix B

Existence of a ground state solution for an auxiliary problem

Consider the problem

$$\begin{cases} -\text{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) + |x|^{-bp^*}|u|^{p-2}u = |x|^{-bp^*}|u|^{\tau-2}u \ \text{ in } \Omega, \\ u \in E(\Omega), \end{cases} \tag{P_Ω}$$

where τ is the constant that appeared in the hypothesis (f_5) and Ω is a bounded domain that appeared in the hypothesis (V_2). The Euler-Lagrange functional associated to (P_Ω) is given by

$$\Phi_0(u) = \frac{1}{p}\int\limits_\Omega |x|^{-ap}|\nabla u|^p dx + \frac{1}{p}\int\limits_\Omega |x|^{-bp^*}|u|^p dx - \frac{1}{\tau}\int\limits_\Omega |x|^{-bp^*}|u|^\tau dx.$$

Arguing as Lemma 2.2.1, from [49, Lemma 1.15], there exists $(u_n) \subset E(\Omega)$, a sequence $(PS)_{c_0}$ for the functional Φ_0. Arguing as Lemma 2.2.2, we can prove that $(u_n) \subset E(\Omega)$ is bounded.

Then, by Sobolev embedding, there exists $u \in E(\Omega)$ such that, up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u, \ \text{in } E(\Omega); \\ u_n \to u, \ \text{in } L^s_{b,loc}(\Omega), \ 1 \le s < p^*; \\ u_n \to u, \ a.e \text{ in } \Omega. \end{cases} \tag{B.0.1}$$

Then,

$$C_p\|u_n - u\|^p_{0,\Omega} = C_p\left[\int_\Omega |x|^{-ap}|\nabla u_n - \nabla u|^p + \int_\Omega |x|^{-bp^*}|u_n - u|^p\right],$$

which implies

$$C_p\|u_n - u\|^p \le \Phi_0'(u_n)u_n - \Phi_0'(u_n)u + o_n(1).$$

Appendix C

Basic Results

Lemma C.0.1 (Fatou's lemma). *If (f_n) is a sequence of nonnegative measurable functions, then*

$$\int_{\mathbb{R}^{\mathbb{N}}} \liminf_{n\to+\infty} f_n dx \leq \liminf_{n\to+\infty} \int_{\mathbb{R}^{\mathbb{N}}} f_n dx.$$

Theorem C.0.2 (Brezis Lieb's Theorem). *Let Ω be a domain of $\mathbb{R}^N$ and $(h_n) \subset L^s(\Omega)$ with $s > 1$ and*

$$h_n(x) \to h(x) \ a.e. \ in \ \Omega.$$

If there exists $C > 0$ such that

$$\int_{\Omega} |h_n|^s dx \leq C,$$

then

$$h_n \rightharpoonup h \ in \ L^s(\Omega).$$

Theorem C.0.3 (Lebesgue's Dominated Convergence Theorem). *Let A be a measurable set of $\mathbb{R}^N$ and (f_j) a sequence of measurable functions such that*

$$f_j(x) \to f(x) \ a.e. \ in \ A,$$

where f is a measurable function. If there exists a function $g \in L^1(A)$ such that

$$|f_j(x)| \leq g(x) \ a.e. \ in \ A,$$

then

$$\lim_{j\to\infty} \int_A f_j(x) dx = \int_A f(x) dx.$$

Theorem C.0.4 (Vainberg's Theorem). *Let (f_j) be a sequence of functions in $L^q(\Omega)$ and $f \in L^q(\Omega)$ such that*

$$f_j \to f \ in \ L^q(\Omega).$$

Then there exist $(f_{j_k}) \subset (f_j)$ and a function $g \in L^q(\Omega)$ such that

$$|f_{j_k}(x)| \leq g(x) \ a.e. \ in \ \Omega$$

and

$$f_{j_k}(x) \to f(x) \text{ a.e. in } \Omega.$$

Proposition C.0.5. *[[38], p.250] If $f : \mathbb{R}^N \to \mathbb{R}$ is a radial function, then*

$$\int_{\mathbb{R}^N} f(|x|)dx = \omega_{N-1} \int_0^\infty f(r)r^{N-1}dr,$$

where ω_{N-1} is the area of the unit sphere $S^{N-1} \subset \mathbb{R}^N$.

Proposition C.0.6. *Let $x, y \in \mathbb{R}^N$ and $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{R}^N$. Then*

$$\langle |x|^{p-2}x - |y|^{p-2}y \rangle \geq \begin{cases} c_p \frac{|x-y|^2}{(|x|+|y|)}^{2-p}, 1 < p < 2, \\ c_p|x-y|^p, p \geq 2, \end{cases} \tag{C.0.1}$$

where c_p is a positive constant.

Theorem C.0.7. *[[46], p. 706] Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary and $0 \in \Omega$, $1 < p < N$, $-\infty < a < \frac{N-p}{p}$. The imbedding $\mathcal{D}_a^{1,p}(\Omega) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is compact if $1 \leq r < \frac{Np}{N-p}$, $\alpha < (1+a)r + N\left(1 - \frac{r}{p}\right)$.*

Theorem C.0.8. *[Concentration Compactness Principle [46], p.709] Let $1 < p < N$, $-\infty < a < \frac{N-p}{p}, a \leq b \leq a+1, q = p^*(a,b) = \frac{Np}{N-dp}, d = 1 + a - b \in [0,1]$ and $\mathcal{M}(\mathbb{R}^N)$ be the space of bounded measures on $\mathbb{R}^N$. Suppose that $\{u_m\} \subset \mathcal{D}_a^{1,p}(\mathbb{R}^N)$ be a sequence such that:*

$$\begin{aligned} u_m &\rightharpoonup u \text{ in } \mathcal{D}_a^{1,p}(\mathbb{R}^N), \\ \mu_m := ||x|^a \nabla u_m|^p dx &\rightharpoonup \mu \text{ in } \mathcal{M}(\mathbb{R}^N), \\ \nu_m := ||x|^b u_m|^q dx &\rightharpoonup \nu \text{ in } \mathcal{M}(\mathbb{R}^N), \\ u_m &\to u \text{ a.e. on } \mathbb{R}^N. \end{aligned}$$

then there are the following statements:

(1) There exists some at most countable set J, a family $\{x_j; j \in J\}$ of distinct points in $\mathbb{R}^N$ and a family $\{\nu_j; j \in J\}$ of positive numbers such that

$$\nu = ||x|^{-b}u|^q dx + \sum_{j \in J} \nu_j \delta_{x_j},$$

where δ_x is the Dirac-mass of mass 1 concentrated at $x \in \mathbb{R}^N$.

(2) The following inequality holds

$$\mu \geq ||x|^{-a}\nabla u|^p dx + \sum_{j \in J} \mu_j \delta_{x_j}$$

for some family $\{\mu_j; j \in J\}$ satisfying

$$S_{a,b}(\nu_j)^{\frac{p}{q}} \leq \mu_j \text{ for all } j \in J.$$

In particular, $\sum_{j \in J} (\nu_j)^{\frac{p}{q}} < \infty$.

Theorem C.0.9. *[Compactness lemma of Strauss [15], p.338] Let* Ω *be a bounded domain and let* P *and* $Q : \Omega \times \mathbb{R} \to \mathbb{R}$ *be two continuous functions satisfying*

$$\lim_{|s| \to +\infty} \frac{P(x,s)}{Q(x,s)} = 0 \text{ uniformly in } x \in \mathbb{R}^N.$$

Let (u_n) *be a sequence of measurable functions from* $\mathbb{R}^N$ *to* $\mathbb{R}$ *such that*

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} |Q(x, u_n(x))| dx < +\infty$$

and

$$P(u_n(x,s)) \to v(x,s) \text{ a.e. in } \mathbb{R}^N \text{ and uniformly in } x \in \mathbb{R}^N, \text{ as } n \to +\infty.$$

Then for any bounded Borel set B *one has*

$$\int_B |P(x, u_n(x)) - v(x)| dx \to 0 \text{ as } n \to +\infty.$$

Appendix D

Technical results

We show the existence of critical points in this appendix.

Lemma D.0.1 (convergence a.e. of the gradient)**.** *Let $\widetilde{I}$ the functional defined in (1.3.8) and (u_n) a bounded sequence in E_0 such that $u_n \rightharpoonup u$ in E_0 and $\widetilde{I}'(u_n) \to 0$, then $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$ up to a subsequence.*

Proof. Along this proof, we denote all positive constants by c.

Let (u_n) a bounded sequence in E_0 such that $u_n \rightharpoonup u$ in E_0 and $\widetilde{I}'(u_n) \to 0$. As $u_n \rightharpoonup u$ in E_0, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$. Define $e_n := |x|^{-ap}\langle |\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u, \nabla(u_n - u)\rangle \geq 0$. Let $\varepsilon > 0$, define the function $\tau_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$\tau_\varepsilon(s) = \begin{cases} s, & |s| \leq \varepsilon, \\ \frac{\varepsilon s}{|s|}, & |s| > \varepsilon. \end{cases} \tag{D.0.1}$$

Observe that $|\tau_\varepsilon(s)| \leq |s|$, then $\tau_\varepsilon \in E_0$.

Hölder's inequality provides

$$\begin{aligned} \left| \int_{\mathbb{R}^N} e_n dx \right| &\leq \int_{\mathbb{R}^N} ||x|^{-ap+a}(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)|||x|^{-a}\nabla(u_n - u)|dx \\ &\leq \left(\int_{\mathbb{R}^N} \left||x|^{-ap+a}|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &\times \left(\int_{\mathbb{R}^N} \left||x|^{-a}\nabla(u_n - u)\right|^p dx \right)^{\frac{1}{p}} \\ &\leq \left(\int_{\mathbb{R}^N} |x|^{-ap} \left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\ &\times \left(\int_{\mathbb{R}^N} |x|^{-ap} \left|\nabla(u_n - u)\right|^p dx \right)^{\frac{1}{p}} \end{aligned} \tag{D.0.2}$$

Use the inequality $(a+b)^r \leq 2^{r-1}(a^r + b^r)$ in the last two integrals to have

$$\int_{\mathbb{R}^N} |x|^{-ap} \left||\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right|^{\frac{p}{p-1}} dx \leq 2^{\frac{1}{p-1}} \int_{\mathbb{R}^N} |x|^{-ap}(|\nabla u_n|^p + |\nabla u|^p)dx.$$

and

$$\int_{\mathbb{R}^N} |x|^{-ap} \left|\nabla(u_n - u)\right|^p dx \leq 2^{p-1} \int_{\mathbb{R}^N} |x|^{-ap}(|\nabla u_n|^p + |\nabla u|^p)dx.$$

These two estimates and the boundedness of the sequence (u_n) in E_0 ensure that (e_n) is bounded in $L^1(\mathbb{R}^N)$.

Let $\phi \in C_0^\infty(\mathbb{R}^N)$ such that supp $\phi \subset B_{m+1}(0)$, $0 \le \phi \le 1$ and $\phi|_{B_m(0)} \equiv 1$. Let $l > 0$ and define

$$\Omega_l = \{x \in \mathbb{R}^N; |u(x)| > l\} \text{ and } \omega_l = \{x \in \mathbb{R}^N; |u(x)| \le l\},$$

then

$$\int_{\mathbb{R}^N} \phi e_n^{\frac{1}{p}} dx = \int_{\Omega_l} \phi e_n^{\frac{1}{p}} dx + \int_{\omega_l} \phi e_n^{\frac{1}{p}} dx. \tag{D.0.3}$$

We proceed to estimate the integral in the right-hand side of the equation above.

Step 1. Estimate over Ω_l.

Hölder inequality and the boundedness of (e_n) provide

$$\begin{aligned}
\int_{\Omega_l} \phi e_n^{\frac{1}{p}} dx &\le \left(\int_{\Omega_l} e_n dx\right)^{\frac{1}{p}} \left(\int_{\Omega_l} \phi^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \\
&\le \left(\int_{\mathbb{R}^N} e_n dx\right)^{\frac{1}{p}} \left(\int_{\Omega_l \cap B_{m+1}(0)} \phi^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \\
&\le c \left(\int_{\Omega_l \cap B_{m+1}(0)} dx\right)^{\frac{p-1}{p}} \\
&\le c \left(\int_{\Omega_l \cap B_{m+1}(0)} \frac{|u|}{l} dx\right)^{\frac{p-1}{p}} \\
&\le c \left(\left(\int_{\mathbb{R}^N} |x|^{-bp^*} \frac{|u|^{p^*}}{l^{p^*}} dx\right)^{\frac{1}{p^*}} \left(\int_{B_{m+1}(0)} |x|^{bp^*} dx\right)^{\frac{p^*-1}{p^*}}\right)^{\frac{p-1}{p}} \\
&\le c \frac{1}{l^{\frac{p-1}{p}}}
\end{aligned} \tag{D.0.4}$$

with c independent of l and n.

Step 2. Estimate over ω_l.

Define

$$\Omega_{n,\varepsilon} = \{x \in \mathbb{R}^N; |u_n(x) - u(x)| \ge \varepsilon\} \text{ and } \omega_{n,\varepsilon} = \{x \in \mathbb{R}^N; |u_n(x) - u(x)| < \varepsilon\},$$

then

$$\int_{\omega_l} \phi e_n^{\frac{1}{p}} dx = \int_{\omega_l \cap \omega_{n,\varepsilon}} \phi e_n^{\frac{1}{p}} dx + \int_{\omega_l \cap \Omega_{n,\varepsilon}} \phi e_n^{\frac{1}{p}} dx. \tag{D.0.5}$$

Step 2.1 Estimate over $\omega_l \cap \Omega_{n,\varepsilon}$

Hölder inequality and the boundedness of (e_n) provide

$$\begin{aligned}
\int_{\omega_l \cap \Omega_{n,\varepsilon}} \phi e_n^{\frac{1}{p}} dx &\le \left(\int_{\omega_l \cap \Omega_{n,\varepsilon}} e_n dx\right)^{\frac{1}{p}} \left(\int_{\omega_l \cap \Omega_{n,\varepsilon}} \phi^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \\
&\le c \left(\int_{\omega_l \cap \Omega_{n,\varepsilon} \cap B_{m+1}(0)} \phi^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \\
&\le c \ (\text{meas } \ (\Omega_{n,\varepsilon} \cap B_{m+1}(0)))^{\frac{p-1}{p}}.
\end{aligned}$$

As meas $(B_{m+1}(0)) < \infty$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$, we have that $(u_n|_{B_{m+1}(0)})$ converges to u in measure, then there exists $n_0 \in \mathbb{N}$ such that

$$\text{meas } \left(\left\{x \in \mathbb{R}^N; |u_n(x) - u(x)| \geq \frac{\varepsilon}{2}\right\} \cap B_{m+1}(0)\right) < \frac{\varepsilon}{2} \text{ for all } n \geq n_0.$$

Then

$$(\text{meas } (\Omega_{n,\varepsilon} \cap B_{m+1}(0))) < \varepsilon \text{ for all } n \geq n_0,$$

hence

$$\limsup_{n\to\infty} \int_{\omega_l \cap \Omega_{n,\varepsilon}} e_n^{\frac{1}{p}} dx < c\varepsilon^{\frac{p-1}{p}}. \tag{D.0.6}$$

Step 2.2 Estimate over $\omega_l \cap \omega_{n,\varepsilon}$.
Hölder inequality provides

$$\begin{aligned}
\int_{\omega_l \cap \omega_{n,\varepsilon}} \phi e_n^{\frac{1}{p}} dx &= \int_{\omega_l \cap \omega_{n,\varepsilon} \cap B_{m+1}(0)} \phi e_n^{\frac{1}{p}} dx \\
&\leq \left(\int_{\omega_l \cap \omega_{n,\varepsilon} \cap B_{m+1}(0)} \phi e_n dx\right)^{\frac{1}{p}} \left(\int_{\omega_l \cap \omega_{n,\varepsilon} \cap B_{m+1}(0)} \phi dx\right)^{\frac{p-1}{p}} \\
&\leq c\left(\int_{\omega_l \cap \Omega_{n,\varepsilon} \cap B_{m+1}(0)} \phi e_n dx\right)^{\frac{1}{p}}.
\end{aligned} \tag{D.0.7}$$

By definition of e_n,

$$\begin{aligned}
\int_{\omega_l \cap \omega_{n,\varepsilon} \cap B_{m+1}(0)} \phi e_n dx &= \int_{\omega_l \cap \omega_{n,\varepsilon} \cap B_{m+1}(0)} |x|^{-ap} |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) \phi dx \\
&\quad - \int_{\omega_l \cap \omega_{n,\varepsilon} \cap B_{m+1}(0)} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla (u_n - u) \phi dx \\
&= \int_{\omega_l \cap \omega_{n,\varepsilon} \cap B_{m+1}(0)} |x|^{-ap} |\nabla u_n|^{p-2} \nabla u_n \nabla \tau_\varepsilon (u_n - u) \phi dx \\
&\quad - \int_{\omega_l \cap \omega_{n,\varepsilon} \cap B_{m+1}(0)} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla \tau_\varepsilon (u_n - u) \phi dx \\
&\leq \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u_n|^{p-2} \nabla u_n \nabla \tau_\varepsilon (u_n - u) \phi dx \\
&\quad - \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla \tau_\varepsilon (u_n - u) \phi dx.
\end{aligned}$$

This and (D.0.7) provide

$$\begin{aligned}
c\left(\int_{\omega_l \cap \omega_{n,\varepsilon}} \phi e_n^{\frac{1}{p}} dx\right)^p &\leq \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u_n|^{p-2} \nabla u_n \nabla \tau_\varepsilon (u_n - u) \phi dx \\
&\quad - \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla \tau_\varepsilon (u_n - u) \phi dx.
\end{aligned} \tag{D.0.8}$$

Step 2.2.1 Estimate the second integral in the right-hand side of (D.0.8).
Observe the functional H defined in E_0 by

$$H(w) = \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla w \phi dx$$

is bounded and, as $u_n \rightharpoonup u$, we have

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla \tau_\varepsilon(u_n - u)\phi dx = H(u_n - u) \to 0. \tag{D.0.9}$$

Step 2.2.2 Estimate the first integral in the right-hand side of (D.0.8).
Observe that

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla \tau_\varepsilon(u_n - u)\phi dx &= \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla(\tau_\varepsilon(u_n - u)\phi) dx \\ &\quad - \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla\phi\tau_\varepsilon(u_n - u) dx\end{aligned} \tag{D.0.10}$$

Step 2.2.2.1 Estimate the second integral in the right-hand side of (D.0.10).
Hölder's inequality, the boundedness of (e_n) in E_0 and $\tau_\varepsilon \leq \varepsilon$ ensure that

$$\left|\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla\phi\tau_\varepsilon(u_n - u) dx\right| \leq \varepsilon \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^{p-1}|\nabla\phi| dx \leq \varepsilon\|u_n\|^{p-1}\|\phi\| \leq c\varepsilon. \tag{D.0.11}$$

Step 2.2.2.2 Estimate the first integral in the right-hand side of (D.0.10).
By definition of $\widetilde{I}$, we have

$$\begin{aligned}&\partial_u\widetilde{I}(\theta_n, u_n)(\phi\tau_\varepsilon(u_n - u)) \\ &= \exp((N-p)\theta_n)\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla(\phi\tau_\varepsilon(u_n - u)) dx \\ &+ \exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^{p-2}u_n(\phi\tau_\varepsilon(u_n - u)) dx \\ &- \exp(N\theta_n)\int_{\mathbb{R}^N} |x|^{-bp^*}h(u_n)(\phi\tau_\varepsilon(u_n - u)) dx\end{aligned} \tag{D.0.12}$$

Observe that $u_n \rightharpoonup u$ in E_0, then (u_n) is bounded in E_0 and it is bounded in $L^p_b(\mathbb{R}^N)$, hence $u_n \rightharpoonup u$ in $L^p_b(\mathbb{R}^N)$. This and $\tau_\varepsilon \leq \varepsilon$ ensure that

$$\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^{p-2}u_n(\phi\tau_\varepsilon(u_n - u)) dx \leq \varepsilon \int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^{p-2}u_n\phi dx = \varepsilon \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p-2}u\phi dx + o_n(1). \tag{D.0.13}$$

Since $(\tau_\varepsilon(u_n - u))$ is bounded, we see that $\partial_u\widetilde{I}(\theta_n, u_n)(\phi\tau_\varepsilon(u_n - u)) \to 0$. From (D.0.12), (D.0.13), Hölder's inequality, $\tau_\varepsilon \leq \varepsilon$, the (1.3.2) and $\theta_n \to 0$, we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla(\phi\tau_\varepsilon(u_n-u))dx\right| \\
&\le \exp(-(N-p)\theta_n)|\partial_u\widetilde{I}(\theta_n,u_n)(\phi\tau_\varepsilon(u_n-u))| \\
&+\exp(-(N-p)\theta_n)\left|\exp(N\theta_n)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p-2}u_n(\phi\tau_\varepsilon(u_n-u))dx\right| \\
&+\exp(-(N-p)\theta_n)\exp(p\theta_n)\left|\int_{\mathbb{R}^N}|x|^{-bp^*}h(u_n)(\phi\tau_\varepsilon(u_n-u))dx\right| \\
&\le o_n(1)+\varepsilon c\exp(p\theta_n)\left(\int_{B_{m+1}(0)}|x|^{-bp^*}|u|^p dx\right) \\
&+\varepsilon\exp(p\theta_n)\left(\int_{B_{m+1}(0)}|x|^{-bp^*}h(u_n)dx\right) \\
&\le o_n(1)+\varepsilon c\left(\int_{B_{m+1}(0)}|x|^{-bp^*}|u|^p dx\right) \\
&+\varepsilon\exp(p\theta_n)\left(\int_{B_{m+1}(0)}|x|^{-bp^*}(\varepsilon|u_n|^{p-1}+C_\varepsilon|u_n|^{q-1})\phi dx\right)
\end{aligned} \tag{D.0.14}
$$

The boundedness of (u_n) in E_0, Hölder's inequality and $E_0 \hookrightarrow L_b^s(\mathbb{R}^N)$ for $s\in[p,p^*]$ ensure that

$$
\begin{aligned}
&\varepsilon\exp(p\theta_n)\left(\int_{B_{m+1}(0)}|x|^{-bp^*}(\varepsilon|u_n|^{p-1}+C_\varepsilon|u_n|^{q-1})\phi dx\right) \\
&\le \varepsilon\exp(p\theta_n)\left(\varepsilon\left(\int_{B_{m+1}(0)}|x|^{-bp^*}|u_n|^p dx\right)^{\frac{p-1}{p}}\left(\int_{B_{m+1}(0)}|x|^{-bp^*}|\phi|^p dx\right)^{\frac{1}{p}}\right. \\
&\left.+C_\varepsilon\left(\int_{B_{m+1}(0)}|x|^{-bp^*}|u_n|^q dx\right)^{\frac{q-1}{q}}\left(\int_{B_{m+1}(0)}|x|^{-bp^*}|\phi|^q dx\right)^{\frac{1}{q}}\right) \\
&\le \varepsilon c(1+o_n(1)).
\end{aligned} \tag{D.0.15}
$$

From (D.0.14) and (D.0.15),

$$
\left|\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla(\phi\tau_\varepsilon(u_n-u))dx\right| \le \varepsilon c(1+o_n(1)). \tag{D.0.16}
$$

Step 3. Combine the estimates to conclude.
Take the limit in (D.0.10) and use (D.0.11) and (D.0.16), then

$$
\begin{aligned}
&\limsup_{n\to\infty}\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla\tau_\varepsilon(u_n-u)\phi dx \\
&=\limsup_{n\to\infty}\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla(\tau_\varepsilon(u_n-u)\phi)dx \\
&-\limsup_{n\to\infty}\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla\phi\tau_\varepsilon(u_n-u)dx \\
&\le \limsup_{n\to\infty}(\varepsilon c(1+o_n(1)))+\limsup_{n\to\infty}(c\varepsilon)\le c\varepsilon,
\end{aligned} \tag{D.0.17}
$$

Take the limit in (D.0.8) and use (D.0.9) and (D.0.17), then

$$\begin{aligned}\limsup_{n\to\infty} c\left(\int_{\omega_l\cap\omega_{n,\varepsilon}} \phi e_n^{\frac{1}{p}}dx\right)^p &\le \limsup_{n\to\infty}\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla\tau_\varepsilon(u_n-u)\phi dx\\ &-\limsup_{n\to\infty}\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla\tau_\varepsilon(u_n-u)\phi dx\\ &\le \limsup_{n\to\infty}(c\varepsilon)\\ &\le c\varepsilon,\end{aligned}$$

hence

$$\limsup_{n\to\infty}\int_{\omega_l\cap\omega_{n,\varepsilon}}\phi e_n^{\frac{1}{p}}dx\le c\varepsilon. \tag{D.0.18}$$

Take the limit in (D.0.5) and use (D.0.6) and (D.0.18), then

$$\begin{aligned}\limsup_{n\to\infty}\int_{\omega_l}\phi e_n^{\frac{1}{p}}dx &= \limsup_{n\to\infty}\int_{\omega_l\cap\omega_{n,\varepsilon}}\phi e_n^{\frac{1}{p}}dx+\limsup_{n\to\infty}\int_{\omega_l\cap\Omega_{n,\varepsilon}}\phi e_n^{\frac{1}{p}}dx\\ &\le \limsup_{n\to\infty}(c\varepsilon^{\frac{p-1}{p}})+\limsup_{n\to\infty}(c\varepsilon)\\ &\le o(\varepsilon).\end{aligned} \tag{D.0.19}$$

Use (D.0.4) and (D.0.19) in (D.0.3). This shows that

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\phi e_n^{\frac{1}{p}}dx &= \limsup_{n\to\infty}\int_{\Omega_l}\phi e_n^{\frac{1}{p}}dx+\limsup_{n\to\infty}\int_{\omega_l}\phi e_n^{\frac{1}{p}}dx\\ &\le \limsup_{n\to\infty}\left(\frac{c}{l^{\frac{p-1}{p}}}\right)+\limsup_{n\to\infty} o(\varepsilon)\\ &\le \frac{c}{l^{\frac{p-1}{p}}}+o(\varepsilon).\end{aligned}$$

Thus, $\varepsilon\to 0$ and $l\to\infty$ ensure that

$$\limsup_{n\to\infty}\int_{B_m(0)} e_n^{\frac{1}{p}}dx\le\limsup_{n\to\infty}\int_{\mathbb{R}^N}\phi e_n^{\frac{1}{p}}dx=0,$$

hence

$$e_n^{\frac{1}{p}}\to 0 \text{ in } L^1(B_m(0)).$$

If $p>2$, then it follows by (C.0.1) that

$$\int_{B_m(0)}|\nabla u_n-\nabla u|dx\le\int_{B_m(0)} e_n^{\frac{1}{p}}dx\to 0.$$

Then

$$\nabla u_n\to\nabla u \text{ a.e. in } B_m(0).$$

If $1 < p < 2$, let $s = \frac{p^2-p+2}{p^2} > 1$ and $t = \frac{1}{sp} > 0$. Use Hölder's inequality and (C.0.1) to get

$$\begin{aligned}\int_{B_m(0)} |\nabla u_n - \nabla u|^{2t} dx &\leq \int_{B_m(0)} e_n^t (|\nabla u_n| + |\nabla u|)^{(2-p)t} dx \\ &\leq \left(\int_{B_m(0)} e_n^{st} dx\right)^{\frac{1}{s}} \left(\int_{B_m(0)} (|\nabla u_n| + |\nabla u|)^{\frac{(2-p)ts}{s-1}} dx\right)^{\frac{s-1}{s}} \\ &\leq \left(\int_{B_m(0)} e_n^{\frac{1}{p}} dx\right)^{\frac{1}{s}} \left(\int_{B_m(0)} (|\nabla u_n| + |\nabla u|)^{\frac{(2-p)ts}{s-1}} dx\right)^{p} \to 0.\end{aligned}$$

Thus,

$$|\nabla u_n - \nabla u| \to 0 \text{ in } L^1(B_m(0)),$$

hence

$$\nabla u_n \to \nabla u \text{ a.e. in } B_m(0),$$

up to a subsequence. A diagonal argument shows that

$$\nabla u_n \to \nabla u \text{ a.e. in } \mathbb{R}^N.$$

□

An adaptation on Step 2.2.2.2 allow to prove the next three lemmas.

Lemma D.0.2 (convergence a.e. of the gradient)**.** *Let $\widetilde{I}_0$ the functional defined in* (1.4.3) *and (u_n) a bounded sequence in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $\mathcal{D}_a^{1,p}(\mathbb{R}^N)$ and $\widetilde{I}_0'(u_n) \to 0$, then $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$ up to a subsequence.*

Lemma D.0.3 (convergence a.e. of the gradient)**.** *Let $I_{\mu,0}$ the functional defined in* (2.2.1) *when $\varrho = 0$ and (u_n) a bounded sequence in E such that $u_n \rightharpoonup u$ in E and $I_{\mu,0}'(u_n) \to 0$, then $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$ up to a subsequence.*

Lemma D.0.4 (convergence a.e. of the gradient)**.** *Let $I_{\mu,1}$ the functional defined in* (2.2.1) *when $\varrho = 1$ and (u_n) a bounded sequence in E such that $u_n \rightharpoonup u$ in E and $I_{\mu,1}'(u_n) \to 0$, then $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$ up to a subsequence.*

Lemma D.0.5. *Let $u_n \rightharpoonup u$ in $E_{0,rad}$. Then*

$$\int_{\mathbb{R}^N} |x|^{-bp^*} |u_n|^{p-2} u_n \varphi dx \to \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p-2} u \varphi dx, \forall \varphi \in C_{0,rad}^{\infty}(\mathbb{R}^N). \tag{D.0.20}$$

Proof. From $u_n \rightharpoonup u$ in $E_{0,rad}$,

$$u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N.$$

Observe that $\left(|x|^{-b\left(\frac{p-1}{p}\right)p^*} |u_n|^{p-2} u_n\right) \subset L^{\frac{p}{p-1}}(\mathbb{R}^N)$ and $|x|^{-b\left(\frac{p-1}{p}\right)p^*} |u|^{p-2} u \in L^{\frac{p}{p-1}}(\mathbb{R}^N)$ because

$$\int_{\mathbb{R}^N} \left(|x|^{-b\left(\frac{p-1}{p}\right)p^*} |u_n|^{p-1}\right)^{\frac{p}{p-1}} dx = \int_{\mathbb{R}^N} |x|^{-bp^*} |u_n|^p dx \leq \sup_{n\in\mathbb{N}} \|u_n\|_0^p < \infty$$

and

$$\int_{\mathbb{R}^N} \left(|x|^{-b\left(\frac{p-1}{p}\right)p^*} |u|^{p-1} \right)^{\frac{p}{p-1}} dx = \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^p dx \leq \|u\|_0^p < \infty.$$

Also, note that

$$\int_{\mathbb{R}^N} \left| |x|^{-b\left(\frac{1}{p}\right)p^*} \varphi \right|^p dx = \int_{\mathbb{R}^N} |x|^{-bp^*} |\varphi|^p dx \leq \|\varphi\|_0^p < \infty,$$

i.e., $|x|^{-b\left(\frac{1}{p}\right)p^*} \varphi \in L^p(\mathbb{R}^N)$.

As $|x|^{-b\left(\frac{p-1}{p}\right)p^*} |u_n(x)|^{p-2} u_n(x) \to |x|^{-b\left(\frac{p-1}{p}\right)p^*} |u(x)|^{p-2} u(x)$ a.e. in $\mathbb{R}^N$, $\left(|x|^{-b\left(\frac{p-1}{p}\right)p^*} |u_n|^{p-2} u_n \right)$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{R}^N)$ and $|x|^{-b\left(\frac{p-1}{p}\right)p^*} |u|^{p-2} u \in L^{\frac{p}{p-1}}(\mathbb{R}^N)$, Brezis-Lieb's Theorem implies that

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^{-bp^*} |u_n|^{p-2} u_n \varphi dx &= \int_{\mathbb{R}^N} |x|^{-b\left(\frac{p-1}{p}\right)p^*} |u_n|^{p-2} u_n |x|^{-b\left(\frac{1}{p}\right)p^*} \varphi dx \\ &\to \int_{\mathbb{R}^N} |x|^{-b\left(\frac{p-1}{p}\right)p^*} |u|^{p-2} u |x|^{-b\left(\frac{1}{p}\right)p^*} \varphi dx \\ &= \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p-2} u \varphi dx, \forall \varphi \in C_{0,rad}^{\infty}(\mathbb{R}^N), \end{aligned}$$

which proves (D.0.20). □

We proceed to show that u is a critical point of I restricted to $E_{0,rad}$.

Theorem D.0.6. *Let $u_n \rightharpoonup u$ in $E_{0,rad}$. Then u is a critical point of I restricted to $E_{0,rad}$.*

Proof. Let $\varphi \in C_{0,rad}^{\infty}(\mathbb{R}^N)$ and fix it.

We want to show that

$$\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi dx \to \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla \varphi dx, \tag{D.0.21}$$

$$\int_{\mathbb{R}^N} |x|^{-bp^*} |u_n|^{p-2} u_n \varphi dx \to \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p-2} u \varphi dx \tag{D.0.22}$$

and

$$\int_{\mathbb{R}^N} |x|^{-bp^*} h(u_n) \varphi dx \to \int_{\mathbb{R}^N} |x|^{-bp^*} h(u) \varphi dx. \tag{D.0.23}$$

Observe that $\left(|x|^{-a\left(\frac{p-1}{p}\right)p} |\nabla u_n|^{p-2} \nabla u_n \right) \subset L^{\frac{p}{p-1}}(\mathbb{R}^N)$ and $|x|^{-a\left(\frac{p-1}{p}\right)p} |\nabla u|^{p-2} \nabla u \in L^{\frac{p}{p-1}}(\mathbb{R}^N)$ because

$$\int_{\mathbb{R}^N} \left(|x|^{-a\left(\frac{p-1}{p}\right)p} |\nabla u_n|^{p-1} \right)^{\frac{p}{p-1}} dx = \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u_n|^p dx \leq \sup_{n \in \mathbb{N}} \|u_n\|_0^p < \infty$$

and

$$\int_{\mathbb{R}^N} \left(|x|^{-a\left(\frac{p-1}{p}\right)p}|\nabla u|^{p-1}\right)^{\frac{p}{p-1}} dx = \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx \leq \|u\|_0^p < \infty.$$

Also, note that

$$\int_{\mathbb{R}^N} \left| |x|^{-a\left(\frac{1}{p}\right)p} \nabla\varphi \right|^p dx = \int_{\mathbb{R}^N} |x|^{-ap}|\nabla\varphi|^p dx \leq \|\varphi\|_0^p < \infty,$$

i.e., $|x|^{-a\left(\frac{1}{p}\right)p}\nabla\varphi \in L^p(\mathbb{R}^N)$.

As $|x|^{-a\left(\frac{p-1}{p}\right)p}|\nabla u_n(x)|^{p-2}\nabla u_n(x) \to |x|^{-a\left(\frac{p-1}{p}\right)p}|\nabla u(x)|^{p-2}\nabla u(x)$ a.e. in $\mathbb{R}^N$ from Lemma D.0.1, $\left(|x|^{-a\left(\frac{p-1}{p}\right)p}|\nabla u_n|^{p-2}\nabla u_n\right)$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{R}^N)$ and $|x|^{-a\left(\frac{p-1}{p}\right)p}|\nabla u|^{p-2}\nabla u \in L^{\frac{p}{p-1}}(\mathbb{R}^N)$, Brezis-Lieb's Theorem implies that

$$\begin{aligned}
\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx &= \int_{\mathbb{R}^N} |x|^{-a\left(\frac{p-1}{p}\right)p}|\nabla u_n|^{p-2}\nabla u_n |x|^{-a\left(\frac{1}{p}\right)p}\nabla\varphi dx \\
&\to \int_{\mathbb{R}^N} |x|^{-a\left(\frac{p-1}{p}\right)p}|\nabla u|^{p-2}\nabla u |x|^{-a\left(\frac{1}{p}\right)p}\nabla\varphi dx \\
&= \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla\varphi dx, \forall \varphi \in C_0^\infty(\mathbb{R}^N),
\end{aligned}$$

which proves (D.0.21).

(D.0.22) is true by the Lemma D.0.5.

$u_n \rightharpoonup u$ in $E_{0,rad}$ implies that $u_n \to u$ in $L_b^s(\text{supp}\varphi)$ for $1 \leq s < p^*$ from Theorem C.0.7. The Vainberg's theorem implies that there exists $g_s \in L_b^s(\text{supp}\varphi)$ with $1 \leq s < p^*$ such that

$$\begin{cases} u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N; \\ |u_n(x)| \leq |g_s(x)| \text{ a.e. in } \mathbb{R}^N \end{cases}$$

up to a subsequence.

By continuity of h,

$$|x|^{-bp^*}h(u_n(x))\varphi(x) \to |x|^{-bp^*}h(u(x))\varphi(x) \text{ a.e. in } \mathbb{R}^N.$$

By (1.3.2),

$$\begin{aligned}
|x|^{-bp^*}h(u_n(x))\varphi(x) &\leq \epsilon|x|^{-bp^*}|u_n(x)|^{p-1}|\varphi(x)| + C_\epsilon|x|^{-bp^*}|u_n(x)|^{q-1}|\varphi(x)| \\
&\leq \epsilon|x|^{-bp^*}|g_p(x)|^{p-1}|\varphi(x)| + C_\epsilon|x|^{-bp^*}|g_q(x)|^{q-1}|\varphi(x)|, \qquad \text{(D.0.24)}
\end{aligned}$$

therefore Holder's inequality and the boundedness of (u_n) in $L_b^p(\mathbb{R}^N)$ imply that

$$
\begin{aligned}
\int_{\mathbb{R}^N} |x|^{-bp^*} h(u_n)\varphi dx &\le \epsilon \int_{\mathbb{R}^N} |x|^{-bp^*} |g_p|^{p-1}|\varphi| dx + C_\epsilon \int_{\mathbb{R}^N} |x|^{-bp^*} |g_q(x)|^{q-1}|\varphi| dx \\
&= \epsilon \int_{\mathbb{R}^N} |x|^{-b\left(\frac{p-1}{p}\right)p^*} |g_p|^{p-1} |x|^{-b\left(\frac{1}{p}\right)p^*} |\varphi| dx \\
&+ C_\epsilon \int_{\mathbb{R}^N} |x|^{-b\left(\frac{q-1}{q}\right)p^*} |g_q|^{q-1} |x|^{-b\left(\frac{1}{q}\right)p^*} |\varphi| dx \\
&\le \epsilon \left(\int_{\mathbb{R}^N} |x|^{-bp^*} |g_p|^p dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |x|^{-bp^*} |\varphi|^p dx \right)^{\frac{1}{p}} \\
&+ C_\epsilon \left(\int_{\mathbb{R}^N} |x|^{-bp^*} |g_q|^q dx \right)^{\frac{q-1}{q}} \left(\int_{\mathbb{R}^N} |x|^{-bp^*} |\varphi|^q dx \right)^{\frac{1}{q}} \\
&< \infty.
\end{aligned}
$$

This and (D.0.24) imply that $|x|^{-bp^*} h(u_n(x))\varphi(x)$ is dominated by $\epsilon |x|^{-bp^*} |g_p(x)|^{p-1}|\varphi(x)| + C_\epsilon |x|^{-bp^*} |g_q(x)|^{q-1}|\varphi(x)| \in L^1(\mathbb{R}^N)$. Thus, the Dominated Convergence Theorem provides the convergence (D.0.20). Since $\theta_n \to 0$ from Lemma 1.3.9, $\lim_{n\to+\infty} \exp((N-p)\theta_n) = \lim_{n\to+\infty} \exp(N\theta_n) = 1$. These, (D.0.21), (D.0.22), (D.0.23) and (1.3.13) ensure that

$$
I'(u)\varphi = 0.
$$

Since $\varphi \in C^\infty_{0,rad}(\mathbb{R}^N)$ is arbitrary,

$$
I'(u)\varphi = 0 \text{ for all } \varphi \in C^\infty_{0,rad}(\mathbb{R}^N).
$$

By density, u is a critical point of I restricted to $E_{0,rad}$. □

We proceed to show that u is a critical point of I_0 restricted to $\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$.

Theorem D.0.7. *Let $u_n \rightharpoonup u$ in $\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$. Then u is a critical point of I_0 restricted to $\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$.*

Proof. Let $\varphi \in C^\infty_{0,rad}(\mathbb{R}^N)$ and fix it.

Observe that the proof that

$$
\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi dx \to \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla \varphi dx \tag{D.0.25}
$$

is similar to the previous theorem once that the Lemma D.0.2 holds.

We show that

$$
\int_{\mathbb{R}^N} |x|^{-bp^*} f(u_n)\varphi dx \to \int_{\mathbb{R}^N} |x|^{-bp^*} f(u)\varphi dx. \tag{D.0.26}
$$

From f_1), given $\varepsilon > 0$, there exist $\delta > 0$ and $A > 1$ such that

$$
f(t) \le \frac{1}{2}|t|^{p^*-1}, \text{ for all } t \in (0, \delta),
$$

$$
f(t) \le \varepsilon |t|^{p^*-1}, \text{ for all } t \in (A, \infty)
$$

and the continuity of f over the compact interval $[\delta, A]$ ensures that there exists $C > 0$ such that

$$f(t) \leq C|t|^{p^*-1}, \text{ for all } t \in [\delta, A].$$

The last three inequalities ensure that

$$f(t) \leq \varepsilon|t|^{p^*-1} + C_\varepsilon|t|^{p^*-1}, \quad \text{for all} \quad t \in \mathbb{R}.$$

Then $|x|^{-b\left(\frac{p^*-1}{p^*}\right)p^*} f(u_n) \in L^{\frac{p^*}{p^*-1}}(\mathbb{R}^N)$ and it is bounded since (u_n) is bounded in $L_b^{p^*}(\mathbb{R}^N)$.

From $u_n \rightharpoonup u$ in $\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$,

$$u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N.$$

By continuity of f,

$$|x|^{-b\left(\frac{p^*-1}{p^*}\right)p^*} f(u_n(x)) \to |x|^{-b\left(\frac{p^*-1}{p^*}\right)p^*} f(u(x)) \text{ a.e. in } \mathbb{R}^N.$$

Thus, Brezis-Lieb's Theorem provides (D.0.26).

Since $\theta_n \to 0$ from Lemma 1.4.7, $\lim\limits_{n\to+\infty} \exp((N-p)\theta_n) = \lim\limits_{n\to+\infty} \exp(N\theta_n) = 1$. These, (D.0.25), (D.0.26) and (1.4.8) ensure that

$$I_0'(u)\varphi = 0.$$

Since $\varphi \in C^\infty_{0,rad}(\mathbb{R}^N)$ is arbitrary,

$$I_0'(u)\varphi = 0 \text{ for all } \varphi \in C^\infty_{0,rad}(\mathbb{R}^N).$$

By density, u is a critical point of I_0 restricted to $\mathcal{D}^{1,p}_{a,rad}(\mathbb{R}^N)$. □

We proceed to show that u_μ is a critical point of $I_{\mu,0}$.

Theorem D.0.8. *Let $u_n \rightharpoonup u_\mu$ in E. Then u_μ is a critical point of $I_{\mu,0}$.*

Proof. Let Ω as in (V_2), $\varphi \in C_0^\infty(\Omega)$ and fix it.

From $u_n \rightharpoonup u_\mu$ in E,

$$u_n(x) \to u_\mu(x) \text{ a.e. in } \mathbb{R}^N$$

and

$$u_n\phi \rightharpoonup u_\mu\phi \text{ in } E.$$

From Theorem C.0.7, it follows that $E(\text{supp } \phi) \overset{c}{\hookrightarrow} L_b^s(\text{supp } \phi)$ with $s \in [1, p^*)$ and

$$u_n\phi \to u_\mu\phi \text{ in } L_b^s(\text{supp } \phi).$$

By the Vainberg's Theorem, there exists $h \in L_b^s(\text{supp } \phi)$ such that

$$u_n(x)\phi(x) \to u(x)\phi(x) \text{ a.e. in supp } \phi$$

and

$$|u_n(x)\phi(x)| \leq h(x) \text{ a.e. in supp } \phi$$

up to a subsequence.
By continuity of f,

$$f(u_n(x))\phi(x) \to f(u_\mu(x))\phi(x) \text{ a.e. in supp } \phi.$$

Observe that

$$\begin{aligned}|x|^{-bp^*}|f(u_n(x))\phi(x)| &\leq |x|^{-bp^*}(\xi|u_n(x)|^{p-1} + C_\xi|u_n(x)|^{r-1})|\phi(x)| \\ &= \xi|x|^{-bp^*}|u_n(x)|^{p-1}|\phi(x)| + C_\xi|x|^{-bp^*}|u_n(x)|^{r-1}|\phi(x)| \\ &\leq \xi|x|^{-bp^*}|h(x)|^{p-1}|\phi(x)| + C_\xi|x|^{-bp^*}|h(x)|^{r-1}|\phi(x)| \text{ a.e. in supp } \phi.\end{aligned}$$

Holder's inequality ensures that

$$\int_{\mathbb{R}^N} |x|^{-bp^*}|h(x)|^{p-1}|\phi(x)|dx < \infty$$

and

$$\int_{\mathbb{R}^N} |x|^{-bp^*}|h(x)|^{r-1}|\phi(x)|dx < \infty,$$

then $\xi|x|^{-bp^*}|h(x)|^{p-1}|\phi(x)| + C_\xi|x|^{-bp^*}|h(x)|^{r-1}|\phi(x)| \in L^1(\text{supp } \phi)$.
From Dominated Convergence Theorem,

$$\int_{\mathbb{R}^\mathbb{N}} |x|^{-bp^*}f(u_n)\phi dx \to \int_{\mathbb{R}^N} |x|^{-bp^*}f(u_\mu)\phi dx.$$

From Brezis-Lieb's theorem and Lemma D.0.3,

$$\int_{\mathbb{R}^\mathbb{N}} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla\phi dx \to \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_\mu|^{p-2}\nabla u_\mu \nabla\phi dx$$

and

$$\int_{\mathbb{R}^\mathbb{N}} |x|^{-bp^*}|u_n|^{p-2}u_n\phi dx \to \int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{p-2}u_\mu\phi dx.$$

Recall that $I'_{\mu,0}(u_n)\phi = o_n(1)$.
Combine these convergences and take the limit when $n \to +\infty$ to get

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_\mu|^{p-2}\nabla u_\mu \nabla\phi dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{p-2}u_\mu\phi dx - \int_{\mathbb{R}^N} |x|^{-bp^*}f(u_\mu)\phi dx = 0,$$

i.e.,

$$I'_{\mu,0}(u_\mu)\phi = 0.$$

Since $\phi \in C_0^\infty(\Omega)$ is arbitrary, we have

$$I'_{\mu,0}(u_\mu)\phi = 0, \forall\phi \in C_0^\infty(\Omega).$$

By density, u_μ is a critical point of $I_{\mu,0}$. □

The next lemma is an adaptation of [[46], p.711 and 712].

Lemma D.0.9. *Let (u_n) a $(PS)_{c_{\mu,1}}$ sequence bounded in E such that $u_n \rightharpoonup u$ in E and $c_{\mu,1} < \left(\frac{1}{p} - \frac{1}{p^*}\right) S_{a,b}^{N/pd}$, for all $\mu \geq 0$ and for all $\lambda > \lambda^*$. Then $\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^{p^*}dx \to \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*}dx$.*

Proof. Let $\phi \in C_0^\infty(\Omega)$ and fix it. Then there exists $R > 0$ with supp $\phi \subset B_R(0)$.

Since (u_n) a bounded $(PS)_{c_{\mu,1}}$ sequence in E, (u_n) is a bounded sequence in $E(\Omega)$. Then

$$\begin{cases} (u_n) \text{ is bounded in } \mathcal{D}_a^{1,p}(\Omega), \\ (|u_n|^{p^*-2}u_n) \text{ is bounded in } L_b^{(p^*)'}(\Omega), \\ (|u_n|^{p-2}u_n) \text{ is bounded in } L_b^{p'}(\Omega) \end{cases}$$

and

$$\begin{cases} |u_n(x)|^{p^*-2}u_n(x) \to |u(x)|^{p^*-2}u(x) \text{ a.e. in } \Omega, \\ |u_n(x)|^{p-2}u_n(x) \to |u(x)|^{p-2}u(x) \text{ a.e. in } \Omega. \end{cases}$$

Brezis-Lieb's Theorem and Lemma D.0.4 ensure that

$$\begin{cases} |\nabla u_n|^{p-2}\nabla u_n \rightharpoonup |\nabla u|^{p-2}\nabla u \text{ in } (L^{p'}(\Omega, |x|^{-ap}))^N, \\ |u_n|^{p^*-2}u_n \rightharpoonup |u|^{p^*-2}u \text{ in } L_b^{(p^*)'}(\Omega), \\ |u_n|^{p-2}u_n \rightharpoonup |u|^{p-2}u \text{ in } L_b^{p'}(\Omega). \end{cases} \tag{D.0.27}$$

On the other hand,

$$E(\Omega) \overset{c}{\hookrightarrow} L_b^s(\Omega), 1 \leq s < p^*.$$

By the Vainberg's theorem, there exist $h_s \in L_b^s(\Omega)$ such that

$$\begin{cases} u_n(x) \to u(x) \text{ a.e. in } \Omega, \\ |u_n(x)| \leq h_s(x) \text{ a.e. in } \Omega, \end{cases}$$

then

$$\begin{cases} f(u_n(x)) \to f(u(x)) \text{ a.e. in } \Omega, \\ |x|^{-bp^*}|f(u_n(x))\phi(x)| \leq \xi|x|^{-bp^*}|h_p(x)|^{p-1}|\phi(x)| + C_\xi|x|^{-bp^*}|h_r(x)|^{r-1}|\phi(x)| \text{ a.e. in } \Omega, \end{cases}$$

where $|x|^{-bp^*}|h_p|^{p-1}|\phi| + |x|^{-bp^*}|h_r|^{r-1}|\phi| \in L^1(\Omega)$.

The Dominated Convergence Theorem implies that

$$\int_{\mathbb{R}^N} |x|^{-bp^*}f(u_n)\phi dx \to \int_{\mathbb{R}^N} |x|^{-bp^*}f(u)\phi dx. \tag{D.0.28}$$

As (u_n) is a $(PS)_{c_{\mu,1}}$ sequence, holds that

$$I'_{\mu,1}(u_n)\phi = o_n(1) \tag{D.0.29}$$

Do $n \to +\infty$ in (D.0.29), use (D.0.27) and (D.0.28) to have

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla\phi dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p-2}u\phi dx \tag{D.0.30}$$
$$= \int_{\mathbb{R}^N} |x|^{-bp^*}f(u)\phi dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*-2}u\phi dx.$$

Let $\Phi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \Phi(x) \leq 1$ for all $x \in \mathbb{R}^N$ and

$$\Phi(x) = \begin{cases} 1, & \text{if } x \in B_{\frac{1}{2}}(0), \\ 0, & \text{if } x \in B_1^c(0). \end{cases}$$

For each $\varepsilon > 0$, define

$$\psi_\varepsilon(x) = \Phi\left(\frac{x - x_j}{\varepsilon}\right),$$

where $\{x_j\}_{j\in J}$ is a set of points of Ω will be fixed later.
Observe that

$$\psi_\varepsilon(x) = \begin{cases} 1, & \text{if } x \in B_{\frac{\varepsilon}{2}}(x_j), \\ 0, & \text{if } x \in B_\varepsilon^c(x_j). \end{cases}$$

Let $1 > \varepsilon > 0$ such that $B_\varepsilon(x_j) \subset \Omega$. We show that $(\psi_\varepsilon u_n)$ is bounded in E. Since $0 \leq \psi_\varepsilon(x) \leq 1$ for all $x \in \mathbb{R}^N$,

$$\begin{aligned} \int_{\mathbb{R}^N} (|x|^{-ap}|\nabla(\psi_\varepsilon u_n)|^p + |x|^{-bp^*}|\psi_\varepsilon u_n|^p)dx &\leq \int_{\mathbb{R}^N} (|x|^{-ap}|\nabla\psi_\varepsilon u_n|^p + |x|^{-bp^*}|u_n|^p)dx \\ &= \int_{\mathbb{R}^N} |x|^{-ap}|\nabla\psi_\varepsilon|^p|u_n|^p dx + \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^p|\psi_\varepsilon|^p dx \\ &+ \int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^p dx \\ &\leq \int_{\mathbb{R}^N} |x|^{-ap}|\nabla\psi_\varepsilon|^p|u_n|^p dx + \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^p dx \\ &+ \int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^p dx. \end{aligned}$$

It is sufficient to show that $\int_{\mathbb{R}^N} |x|^{-ap}|\nabla\psi_\varepsilon|^p|u_n|^p dx$ is bounded independent of ε.
Let $y = \frac{x-x_j}{\varepsilon}$, then

$$\nabla\psi_\varepsilon(x) = \frac{1}{\varepsilon}\nabla\Phi(y).$$

Since $\Phi \in C_0^\infty(\mathbb{R}^N)$, we have

$$|\nabla\psi_\varepsilon(x)| \leq \frac{\|\nabla\Phi\|_\infty}{\varepsilon}. \tag{D.0.31}$$

From Hölder's inequality,

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^{-ap}|\nabla\psi_\varepsilon|^p|u_n|^p dx &= \int_{\mathbb{R}^N} |x|^{-bp}|u_n|^p|x|^{-p(a-b)}|\nabla\psi_\varepsilon|^p dx \\ &\leq \left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^{p^*} dx\right)^{\frac{p}{p^*}} \\ &\left(\int_{\mathbb{R}^N} |x|^{-\frac{pp^*(a-b)}{p^*-p}}|\nabla\psi_\varepsilon|^p dx\right)^{\frac{p^*-p}{p^*}}. \end{aligned}$$

From (1.2.1), the boundedness of the sequence (u_n) in E and (D.0.31)

$$
\begin{aligned}
&\leq \left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^{p^*}dx\right)^{\frac{p}{p^*}} \\
&\left(\int_{B_\varepsilon(x_j)} |x|^{-\frac{pp^*(a-b)}{p^*-p}}|\nabla\psi_\varepsilon|^p dx\right)^{\frac{p^*-p}{p^*}} \\
&\leq \frac{1}{S_{a,b}}\left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^p dx\right) \\
&\left(\int_{B_\varepsilon(x_j)} |x|^{-\frac{pp^*(a-b)}{p^*-p}}|\nabla\psi_\varepsilon|^p dx\right)^{\frac{p^*-p}{p^*}} \\
&\leq \frac{1}{S_{a,b}}\sup_{n\in\mathbb{N}}\|u_n\|_\mu^p\left(\int_{B_\varepsilon(x_j)} |x|^{-\frac{pp^*(a-b)}{p^*-p}}\left|\frac{\|\nabla\Phi\|_\infty}{\varepsilon}\right|^p dx\right)^{\frac{p^*-p}{p^*}} \\
&= \frac{1}{S_{a,b}}\sup_{n\in\mathbb{N}}\|u_n\|_\mu^p\left|\frac{\|\nabla\Phi\|_\infty}{\varepsilon}\right|^p\left(\int_{B_\varepsilon(x_j)} |x|^{-\frac{pp^*(a-b)}{p^*-p}} dx\right)^{\frac{p^*-p}{p^*}}.
\end{aligned}
$$

From Proposition C.0.5,

$$
\begin{aligned}
\left(\int_{B_\varepsilon(x_j)} |x|^{-\frac{pp^*(a-b)}{p^*-p}} dx\right)^{\frac{p^*-p}{p^*}} &= \omega_{N-1}\left(\int_0^\varepsilon r^{-\frac{pp^*(a-b)}{p^*-p}}r^{N-1}dr\right)^{\frac{p^*-p}{p^*}} \\
&= \omega_{N-1}\left(\int_0^\varepsilon r^{\frac{pp^*(b-a)}{p^*-p}}r^{N-1}dr\right)^{\frac{p^*-p}{p^*}} \\
&= \omega_{N-1}\left(\int_0^\varepsilon r^{\frac{Np(b-a)}{dp}}r^{N-1}dr\right)^{\frac{dp}{N}} \\
&= \omega_{N-1}\left(\int_0^\varepsilon r^{\frac{Np(b-a)+Ndp}{dp}-1}dr\right)^{\frac{dp}{N}} \\
&= \omega_{N-1}\left(\int_0^\varepsilon r^{\frac{N}{d}-1}dr\right)^{\frac{dp}{N}} \\
&= \omega_{N-1}\left(\frac{d}{N}\right)^{\frac{dp}{N}}\varepsilon^p.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_{\mathbb{R}^N} |x|^{-ap}|\nabla\psi_\varepsilon|^p|u_n|^p dx &\leq \frac{1}{S_{a,b}}\sup_{n\in\mathbb{N}}\|u_n\|_\mu^p\left|\frac{\|\nabla\Phi\|_\infty}{\varepsilon}\right|^p\left(\int_{B_\varepsilon(x_j)} |x|^{-\frac{pp^*(a-b)}{p^*-p}} dx\right)^{\frac{p^*-p}{p^*}} \\
&= \frac{1}{S_{a,b}}\sup_{n\in\mathbb{N}}\|u_n\|_\mu^p\left|\frac{\|\nabla\Phi\|_\infty}{\varepsilon}\right|^p\omega_{N-1}\left(\frac{d}{N}\right)^{\frac{dp}{N}}\varepsilon^p \\
&= \frac{1}{S_{a,b}}\sup_{n\in\mathbb{N}}\|u_n\|_\mu^p\|\nabla\Phi\|_\infty^p\omega_{N-1}\left(\frac{d}{N}\right)^{\frac{dp}{N}},
\end{aligned}
$$

hence the boundedness of $(\psi_\varepsilon u_n)$ in E follows.

Choose $\phi = \psi_\varepsilon u_n$ in (D.0.29), then

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla(\psi_\varepsilon u_n)dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^{p-2}u_n(\psi_\varepsilon u_n)dx$$
$$= \int_{\mathbb{R}^N} |x|^{-bp^*} f(u_n)(\psi_\varepsilon u_n)dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^{p^*-2}u_n(\psi_\varepsilon u_n)dx,$$

i.e.,

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla(\psi_\varepsilon)u_n dx + \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_n|^p\psi_\varepsilon dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^p\psi_\varepsilon dx$$
$$= \int_{\mathbb{R}^N} |x|^{-bp^*} f(u_n)(\psi_\varepsilon u_n)dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u_n|^{p^*}\psi_\varepsilon dx, \qquad \text{(D.0.32)}$$

Observe that

$$E(\Omega) \overset{c}{\hookrightarrow} L^s_b(\Omega), 1 \le s < p^*.$$

By the Vainberg's theorem, there exist $h_s \in L^s_b(\Omega)$ such that

$$\begin{cases} u_n(x) \to u(x) \text{ a.e. in } \Omega, \\ |u_n(x)| \le h_s(x) \text{ a.e. in } \Omega, \end{cases}$$

then

$$\begin{cases} f(u_n(x)) \to f(u(x)) \text{ a.e. in } \Omega, \\ |x|^{-bp^*}|f(u_n(x))\psi(x)u_n(x)| \le \xi|x|^{-bp^*}|h_p(x)|^p|\psi(x)| + C_\xi|x|^{-bp^*}|h_r(x)|^r|\psi(x)| \text{ a.e. in } \Omega, \end{cases}$$

where $|x|^{-bp^*}|h_p|^p|\psi| + C_\xi|x|^{-bp^*}|h_r|^r|\psi| \in L^1(\Omega)$ once that $\psi \in C_0^\infty(\mathbb{R}^N)$.

The Dominated Convergence Theorem implies that

$$\int_{\mathbb{R}^N} |x|^{-bp^*} f(u_n)(\psi u_n)dx \to \int_{\mathbb{R}^N} |x|^{-bp^*} f(u)(\psi u)dx. \qquad \text{(D.0.33)}$$

Do $n \to +\infty$ in (D.0.32), use (D.0.27), (D.0.33) and Theorem C.0.8 to have

$$\int_{\mathbb{R}^N} |x|^{-ap}u|\nabla u|^{p-2}\nabla u\nabla\psi_\varepsilon dx + \int_{\mathbb{R}^N} \mu\psi_\varepsilon dx = \int_{\mathbb{R}^N} |x|^{-bp^*} f(u)\psi_\varepsilon u dx + \int_{\mathbb{R}^N} \nu\psi_\varepsilon dx,$$

i.e.,

$$\int_{\mathbb{R}^N} |x|^{-ap}u|\nabla u|^{p-2}\nabla u\nabla\psi_\varepsilon dx + \int_{\mathbb{R}^N} \psi_\varepsilon d\mu = \int_{\mathbb{R}^N} |x|^{-bp^*} f(u)\psi_\varepsilon u dx + \int_{\mathbb{R}^N} \psi_\varepsilon d\nu. \quad \text{(D.0.34)}$$

Do $\phi = \psi_\varepsilon u$ in (D.0.30) to have

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla(\psi_\varepsilon u)dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p-2}u(\psi_\varepsilon u)dx$$
$$= \int_{\mathbb{R}^N} |x|^{-bp^*} f(u)(\psi_\varepsilon u)dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*-2}u(\psi_\varepsilon u)dx,$$

i.e.,

$$\int_{\mathbb{R}^N} |x|^{-ap}u|\nabla u|^{p-2}\nabla u\nabla\psi_\varepsilon dx + \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p\psi_\varepsilon dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^p\psi_\varepsilon dx$$
$$= \int_{\mathbb{R}^N} |x|^{-bp^*} f(u)(\psi_\varepsilon u)dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*}\psi_\varepsilon dx. \qquad \text{(D.0.35)}$$

From (D.0.34) and (D.0.35),

$$\begin{aligned}\int_{\mathbb{R}^N} \psi_\varepsilon d\mu - \int_{\mathbb{R}^N} \psi_\varepsilon d\nu &= -\int_{\mathbb{R}^N} |x|^{-ap} u |\nabla u|^{p-2} \nabla u \nabla \psi_\varepsilon dx + \int_{\mathbb{R}^N} |x|^{-bp^*} f(u)(\psi_\varepsilon u) dx \\ &= \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p \psi_\varepsilon dx + \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^p \psi_\varepsilon dx - \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^{p^*} \psi_\varepsilon dx.\end{aligned}$$

From this and Theorem C.0.8,

$$\int_{\mathbb{R}^N} \psi_\varepsilon d\mu = \sum_{j \in J} \nu_j \psi_\varepsilon(x_j) + \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p \psi_\varepsilon dx + \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^p \psi_\varepsilon dx, \tag{D.0.36}$$

We proceed to show that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p \psi_\varepsilon dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^p \psi_\varepsilon dx = 0,$$

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \psi_\varepsilon d\mu = \int_{\{x_j\}} d\mu$$

and

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \psi_\varepsilon d\nu = \int_{\{x_j\}} d\nu.$$

For each $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p \psi_\varepsilon dx = \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p \psi_\varepsilon \chi_{B_\varepsilon(x_j)} dx$$

and

$$||x|^{-ap} |\nabla u(x)|^p \psi_\varepsilon(x) \chi_{B_\varepsilon(x_j)}(x)| \leq |x|^{-ap} |\nabla u(x)|^p,$$

where $|x|^{-ap} |\nabla u|^p \in L^1(\mathbb{R}^N)$. If $\varepsilon \to 0$,

$$|x|^{-ap} |\nabla u(x)|^p \psi_\varepsilon(x) \chi_{B_\varepsilon(x_j)}(x) \to 0 \text{ a.e. in } \mathbb{R}^N.$$

Dominated Convergence Theorem ensures that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p \psi_\varepsilon dx = 0. \tag{D.0.37}$$

Similar argument shows that

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} |x|^{-bp^*} |u|^p \psi_\varepsilon dx = 0. \tag{D.0.38}$$

For each $\varepsilon > 0$,

$$|\psi_\varepsilon(x) \chi_{B_\varepsilon(x_j)}(x)| \leq 1.$$

If $\varepsilon \to 0$,

$$\psi_\varepsilon(x) \chi_{B_\varepsilon(x_j)}(x) \to \chi_{\{x_j\}}(x).$$

Since Radon measures are finite, we have that 1 is integrable with respect to ν. Thus, Dominated Convergence Theorem ensures that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\psi_\varepsilon d\nu = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\psi_\varepsilon \chi_{B_\varepsilon(x_j)} d\nu = \int_{\mathbb{R}^N}\chi_{\{x_j\}} d\nu = \int_{\{x_j\}} d\nu. \tag{D.0.39}$$

Similar argument shows that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\psi_\varepsilon d\mu = \int_{\{x_j\}} d\mu. \tag{D.0.40}$$

From (D.0.36), (D.0.37), (D.0.38), (D.0.39) and (D.0.40),

$$\nu_j = \nu(x_j) = \mu(x_j) = \mu_j,$$

then

$$S_{a,b}(\nu_j)^{\frac{p}{p^*}} \leq \mu_j = \nu_j,$$

i.e.,

$$\nu_j \geq S_{a,b}^{\frac{N}{dp}} \text{ for } \nu_j \neq 0. \tag{D.0.41}$$

As (u_n) is a $(PS)_{c_{\mu,1}}$ sequence, holds that

$$\begin{aligned} c_{\mu,1} + o_n(1) &= I_{\mu,1}(u_n) - \frac{1}{p} I'_{\mu,1}(u_n)u_n \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p^*}dx \\ &+ \int_{\mathbb{R}^N}|x|^{-bp^*}\left(\frac{1}{p}f(u_n)u_n - F(u_n)\right)dx. \end{aligned}$$

The hypothesis (f_4) implies that

$$t \longmapsto \frac{1}{p}f(t)t - F(t), \text{ is increasing for } t \in (0, +\infty).$$

Hence

$$\int_{\mathbb{R}^N}|x|^{-bp^*}\left(\frac{1}{p}f(u_n)u_n - F(u_n)\right)dx \geq 0,$$

then

$$c_{\mu,1} + o_n(1) \geq \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p^*}dx.$$

$0 \leq \psi_\varepsilon(x) \leq 1$ for all $x \in \mathbb{R}^N$ provides that

$$\left(\frac{1}{p} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p^*}dx \geq \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p^*}\psi_\varepsilon dx.$$

If $n \to \infty$, Theorem C.0.8 ensures that

$$\lim_{n\to+\infty}\left(\frac{1}{p} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p^*}\psi_\varepsilon dx = \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N}\left(|x|^{-bp^*}|u|^{p^*} + \sum_{j\in J}\nu_j\delta_{x_j}\right)\psi_\varepsilon dx,$$

then

$$\begin{aligned} c_{\mu,1} &\geq \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} \left(|x|^{-bp^*}|u|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j} \right) \psi_\varepsilon dx \\ &\geq \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} \left(\sum_{j\in J} \nu_j \delta_{x_j} \right) \psi_\varepsilon dx \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} \psi_\varepsilon \nu dx \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_{\mathbb{R}^N} \psi_\varepsilon d\nu. \end{aligned}$$

If $\varepsilon \to 0$, (D.0.40) ensures that

$$\begin{aligned} c_{\mu,1} &\geq \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_{\{x_j\}} d\nu \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right) \nu_j \end{aligned}$$

From (D.0.41),

$$c_{\mu,1} \geq \left(\frac{1}{p} - \frac{1}{p^*}\right) \nu_j \geq S_{a,b}^{\frac{N}{dp}} \text{ for } \nu_j \neq 0,$$

which is a contradiction with the Lemma 2.4.1. Thus, $\nu_j = 0$ for all $j \in J$ and Theorem C.0.8 ensures the convergence.

□

We proceed to show that u_μ is a critical point of $I_{\mu,1}$.

Theorem D.0.10. *Let $u_n \rightharpoonup u_\mu$ in E. Then u_μ is a critical point of $I_{\mu,1}$.*

Proof. Let Ω as in (V_2), $\phi \in C_0^\infty(\Omega)$ and fix it.

From $u_n \rightharpoonup u_\mu$ in E,

$$u_n(x) \to u_\mu(x) \text{ a.e. in } \mathbb{R}^N$$

and

$$u_n\phi \rightharpoonup u_\mu\phi \text{ in } E.$$

From Theorem C.0.7 it follows that $E(\text{supp } \phi) \overset{c}{\hookrightarrow} L_b^s(\text{supp } \phi)$ with $s \in [1, p^*)$ and

$$u_n\phi \to u_\mu\phi \text{ in } L_b^s(\text{supp } \phi).$$

By the Vainberg's theorem, there exists $h \in L_b^s(\text{supp } \phi)$ such that

$$u_n(x)\phi(x) \to u(x)\phi(x) \text{ a.e. in supp } \phi$$

and

$$|u_n(x)\phi(x)| \leq h(x) \text{ a.e. in supp } \phi$$

up to a subsequence.

By continuity of f,

$$f(u_n(x))\phi(x) \to f(u_\mu(x))\phi(x) \text{ a.e. in supp } \phi.$$

Observe that

$$\begin{aligned}|x|^{-bp^*}|f(u_n(x))\phi(x)| &\le |x|^{-bp^*}(\xi|u_n(x)|^{p-1} + C_\xi|u_n(x)|^{r-1})|\phi(x)| \\ &= \xi|x|^{-bp^*}|u_n(x)|^{p-1}|\phi(x)| + C_\xi|x|^{-bp^*}|u_n(x)|^{r-1}|\phi(x)| \\ &\le \xi|x|^{-bp^*}|h(x)|^{p-1}|\phi(x)| + C_\xi|x|^{-bp^*}|h(x)|^{r-1}|\phi(x)| \text{ a.e. in supp } \phi.\end{aligned}$$

Holder's inequality ensures that

$$\int_{\mathbb{R}^N} |x|^{-bp^*}|h(x)|^{p-1}|\phi(x)|dx < \infty$$

and

$$\int_{\mathbb{R}^N} |x|^{-bp^*}|h(x)|^{r-1}|\phi(x)|dx < \infty,$$

then $\xi|x|^{-bp^*}|h(x)|^{p-1}|\phi(x)| + C_\xi|x|^{-bp^*}|h(x)|^{r-1}|\phi(x)| \in L^1(\text{supp } \phi)$.
From Dominated Convergence Theorem,

$$\int_{\mathbb{R}^{\mathbb{N}}} |x|^{-bp^*}f(u_n)\phi dx \to \int_{\mathbb{R}^N} |x|^{-bp^*}f(u_\mu)\phi dx.$$

From the Lemma D.0.9,

$$u_n\phi \to u_\mu\phi \text{ in } L_b^{p^*}(\text{supp } \phi).$$

By the Vainberg's theorem, there exists $h \in L_b^{p^*}(\text{supp } \phi)$ such that

$$|u_n(x)|^{p^*-2}u_n(x)\phi(x) \to u(x)|u_\mu(x)|^{p^*-2}u_\mu(x)\phi(x) \text{ a.e. in supp } \phi$$

and

$$|u_n(x)\phi(x)| \le h(x) \text{ a.e. in supp } \phi$$

up to a subsequence.
Observe that

$$|x|^{-bp^*}||u_n(x)|^{p^*-2}u_n(x)\phi(x)| \le |x|^{-bp^*}|h(x)|^{p^*-1}|\phi(x)| \text{ a.e. in supp } \phi.$$

Holder's inequality ensures that

$$\int_{\mathbb{R}^N} |x|^{-bp^*}|h(x)|^{p^*-1}|\phi(x)|dx < \infty,$$

then $|x|^{-bp^*}|h(x)|^{p^*-1}|\phi(x)| \in L^1(\text{supp } \phi)$.
From Dominated Convergence Theorem,

$$\int_{\mathbb{R}^{\mathbb{N}}} |x|^{-bp^*}|u_n|^{p^*-2}u_n\phi dx \to \int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{p^*-2}u_\mu\phi dx.$$

From Brezis-Lieb's theorem and Lemma D.0.4,

$$\int_{\mathbb{R}^{\mathbb{N}}} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla\phi dx \to \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_\mu|^{p-2}\nabla u_\mu\nabla\phi dx$$

and

$$\int_{\mathbb{R}^{\mathbb{N}}} |x|^{-bp^*}|u_n|^{p-2}u_n\phi dx \to \int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{p-2}u_\mu\phi dx.$$

Recall that $I'_{\mu,1}(u_n)\phi = o_n(1)$.
Combine these convergences and take the limit when $n \to +\infty$ to get

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u_\mu|^{p-2}\nabla u_\mu \nabla\phi dx + \int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{p-2}u_\mu\phi dx - \int_{\mathbb{R}^N} |x|^{-bp^*}f(u_\mu)\phi dx$$
$$- \int_{\mathbb{R}^N} |x|^{-bp^*}|u_\mu|^{p^*-2}u_\mu\phi dx = 0,$$

i.e.,

$$I'_{\mu,1}(u_\mu)\phi = 0.$$

Since $\phi \in C_0^\infty(\Omega)$ is arbitrary, we have

$$I'_{\mu,1}(u_\mu)\phi = 0, \forall\phi \in C_0^\infty(\Omega).$$

By density, u_μ is a critical point of $I_{\mu,1}$. □

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