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Resumo

Seja $(M^n, g, \nabla f, \lambda)$ um quase soliton de Ricci gradiente, compacto com fronteira. Neste trabalho obtemos teoremas de rigidez para $(M^n, g, \nabla f, \lambda)$ de modo que, sob determinadas hipóteses, podemos mostrar se ele é isométrico a um hemisfério de uma esfera Euclidiana, a uma bola Euclidiana fechada ou a um domínio hiperbólico. Além disso, aplicamos tais teoremas em uma caracterização de quase solitons de Ricci gradientes e compactos sobre o produto warped $M = B \times_h F$, em que B é uma variedade Riemanniana com fronteira.

Palavras-chave: Variedades com fronteira; Campos vetoriais conformes; Quase soliton de Ricci gradiente; Produto warped.

Título: Rigidez de quase-solitons de Ricci gradiente compactos com fronteira.

Abstract

Let $(M^n, g, \nabla f, \lambda)$ be a compact gradient Ricci almost soliton with boundary. In this thesis, we obtain rigidity theorems for $(M^n, g, \nabla f, \lambda)$ so that we can show if it is isometric to a closed hemisphere of an Euclidean sphere, or a closed Euclidean ball, or a domain in $\mathbb{H}^n$. Furthermore, we apply such theorems to characterize gradient Ricci almost solitons on warped product $M = B \times_h F$, where B is a compact Riemannian manifold with boundary.

Keywords: Manifolds with boundary; Conformal vector fields; Gradient Ricci almost solitons; Warped product.

Contents

Introduction

We shall consider $(M, g, \nabla f, \lambda)$ a gradient Ricci almost soliton with boundary, i.e., (M, g) is a Riemannian manifold with boundary, which satisfies the following fundamental equation

$$\mathrm{Ric} = \lambda g + \mathrm{Hess}(f),$$

where λ and f are smooth functions on M, and ∇f and $\mathrm{Hess}(f)$ denote the gradient of f and the Hessian of f, respectively. If the gradient vector field ∇f vanishes, then the gradient Ricci almost soliton is just an Einstein manifold with boundary.

This work is meant to be a first step into the characterization of compact gradient Ricci almost solitons with boundary. We have taken the same perspective by [7], [19], and [10]. Rigidity results for the without boundary case have been studied in [15], [2], and [1]. Our motivation to study the "boundary case" is from [5], and [6], where the authors investigate characterizations of Einstein metrics on warped products.

This work is divided in four chapters.

Chapter 1 we developed the basic theory of manifolds with boundary, we define and prove some properties of Killing vector fields on smooth manifolds with and without boundary, we recall the generalized Bochner formula for manifolds with or without boundary, and at the last section we study some aspects of the topology of manifolds with boundary, namely, the equivalence the topology induced from the distance function and the topology induced from the smooth structure, and we give the statement of the Hopf-Rinow theorem for manifolds with boundary, which was obtained by D. Burago, Y. Burago, S. Ivanov, S. Pigola, and G. Veronelli..

Chapter 2 we start by recalling the definition of Ricci solitons (with or without boundary) and we prove some properties by following the same steps in [7]. In the second section we define warped product, where the base is a Riemannian manifold with boundary and the fiber is a Riemannian manifold without boundary, and we show some identities for the Christoffel symbol and the Hessian of an arbitrary smooth function defined on the warped product. We calculated the Ricci tensor on the warped product. We finish this chapter by proving that a warped product $B \times_h F$, where the fiber F has dimension bigger than

3, which is Einstein with a boundary condition

$$\int_{\partial B} h\frac{\partial h}{\partial \mathcal{N}}d(\partial B) \geq 0,$$

is a Riemannian product.

Chapter 3 is dedicated to obtain rigidity theorems for gradient Ricci almost solitons with boundary. Precisely, the three most important theorems that we proved in this chapter are the following.

Theorem 3.2.9. *Let $(M^n, g, \nabla f, \lambda)$, $n \geq 2$, be a compact connected gradient Ricci almost soliton with connected boundary. Suppose f satisfies*

$$\begin{aligned} f(p) &> c_0 \quad p \in int(M), \\ f(p) &= c_0 \quad\ \ p \in \partial M, \end{aligned}$$

where $c_0 > 0$ is a constant, ∇f does not vanish on ∂M. Suppose the scalar curvature S of M is positive, and ∇f is a conformal vector field. Assume $Hess(f) = \xi\, g$, where $\xi \leq 0$ on ∂M. Then the mean curvature of ∂M is non negative, and there exists a positive constant $\rho \in \mathbb{R}$ such that the Ricci curvature of M satisfies

$$\mathrm{Ric}_p(v, v) \geq (n-1)\rho^2,$$

for all $p \in \partial M$, and $v \in T_p\partial M$, $|v| = 1$. Moreover, the first eigenvalue $\lambda_1(\Delta)$ of the Laplacian on M satisfies the inequality $\lambda_1(\Delta) \geq n\rho^2$. The equality holds if and only if M is isometric to a closed hemisphere of the Euclidean sphere $\mathbb{S}^n(\rho^2)$ of radius $\dfrac{1}{\rho}$.

The section 8 is dedicated to prove the following theorem.

Theorem 3.8.2. *Let $(M^n, g, \nabla f, \lambda)$, $n \geq 2$, be a compact gradient Ricci almost soliton with boundary. Suppose f is constant on ∂M, and the Ricci curvature of M is non negative. If*

$$g_p(\nabla f(p), \eta(p)) < H(p), \text{ and } \lambda(p) < \mathrm{Ric}_{\partial M}(v, v) + K_M(\eta(p), v),$$

for all $p \in \partial M$, where $\eta(p)$ is the inward unit normal vector to the boundary at p, H(p) is the mean curvature of ∂M at p, and $v \in T_p\partial M$, $|v| = 1$, is a principal direction of the shape operator $S_{\eta(p)}$. Here, $Ric_{\partial M}$ and K_M denote the Ricci curvature of ∂M and the sectional curvature of M, respectively. Then, the following assertions are satisfies:

(i) *The first eigenvalue $\lambda_1(\Delta_{\partial M})$ of the Laplacian on ∂M satisfies $\lambda_1(\Delta_{\partial M}) \geq (n-1)\rho^2$, for some positive constant $\rho \in \mathbb{R}$. Moreover, the equality holding if and only if M is isometric to an n-dimensional closed Euclidean ball of radius $\dfrac{1}{\rho}$.*

(ii) *The mean curvature satisfies*

$$\int_{\partial M} \frac{1}{H(p)} d(\partial M) \geq n\mathrm{vol}(M).$$

The equality holds if and only if M is isometric the a closed Euclidean ball.

The section 9 is dedicated to prove the following theorem.

Theorem 3.9.3. *Let $(M^n, g, \nabla f, \lambda)$, $n \geq 3$, be a compact gradient Ricci almost soliton with simply connected boundary. Suppose f satisfies*

$$\begin{aligned} f(p) &> c_0 \quad p \in int(M), \\ f(p) &= c_0 \quad\;\; p \in \partial M, \end{aligned}$$

where c_0 is a constant, ∇f does not vanish on ∂M. If there exists an isometric immersion of ∂M into the $\mathbb{H}^{n+n_0}$, $n_0 \geq 0$, and

$$1 - n - \lambda(p) \leq \mathrm{Hess}(f)(v, v) \leq -|\nabla f(p)||II_p^{\mathbb{H}}(v, v)|_{\mathbb{L}},$$

for all $p \in \partial M$, $v \in T_p\partial M$, $|v|_{\mathbb{L}} = 1$. Here, $II^{\mathbb{H}}$ denotes the vector-valued second fundamental form of ∂M in $\mathbb{H}^{n+n_0}$, then M is isometric to a domain in $\mathbb{H}^n$.

In the Chapter 4 we characterize gradient Ricci almost solitons on warped product with boundary by applying the rigidity theorems that we obtained in the Chapter 3. In the first section we obtain the following theorem.

Corollary 4.1.6. *Let (B^m, g_B) and (F^k, g_F) be Riemannian manifolds, where B is a oriented connected Riemannian manifold with boundary and F is a Riemannian manifold without boundary. Let $(M = B \times_h F, g)$ be an warped product, where h is a positive function which satisfies*

$$\begin{aligned} h(p) &> c_0 \quad p \in int(B), \\ h(p) &= c_0 \quad\;\; p \in \partial B, \end{aligned}$$

where c_0 is a constant. Suppose $(M, g, \nabla f, \lambda)$ is a gradient Ricci almost soliton such that f is non constant on F. Then, the following assertions are satisfies:

(i) *If B is compact, then $\nabla_B h \neq 0$ and $H_{\partial B} > 0$, where $H_{\partial B}$ is the mean curvature of ∂B. In particular, the second fundamental form of ∂B is positive.*

(ii) *If $\nabla_B h \neq 0$ on ∂B, then $|\nabla_B h|$ is constant.*

(iii) *If $\nabla_B h \neq 0$ on ∂B, then ∂B is totally geodesic.*

(iv) *Let $\Lambda : B \longrightarrow \mathbb{R}$ be the funcion which satisfies $f = \Lambda + h\Phi$. Suppose Λ is constant on ∂B, and $m \geq 3$. Then, $g_B(\nabla_B h, \nabla_B \Lambda)$ is constant in B if and only if B is Ricci flat and $\nabla_B \Lambda$ is a Killing vector field.*

The second section is dedicated to demonstrate the following result.

Theorem 4.2.2. *Let (B^m, g_B) and (F^k, g_F) be Riemannian manifolds, where B is a oriented connected compact Riemannian manifold with connected boundary, $m \geq 2$, and F is a Riemannian manifold with no boundary. Let $(M = B \times_h F, g)$ be an warped product, where h is a positive function which satisfies*

$$\begin{aligned} h(p) &> c_0 \quad p \in int(B), \\ h(p) &= c_0 \quad p \in \partial B, \end{aligned}$$

where c_0 is a constant. Suppose $(M, g, \nabla f, \lambda)$ is a gradient Ricci almost soliton such that f is constant on F. Moreover, suppose that $\nabla_B h$ is a conformal vector field. Then, the following assertions are satisfies:

(i) *If f satisfies*

$$\begin{aligned} f(p) &> c_0 \quad p \in int(B), \\ f(p) &= c_0 \quad p \in \partial B, \end{aligned}$$

where c_0 is a constant, $\nabla f \neq 0$ on ∂M, the scalar curvature of B is positive, and $Hess(f) = \xi g$, where $\xi \leq 0$ on ∂B, then there exists a positive constant $\rho \in \mathbb{R}$ such that the first eigenvalue $\lambda_1(\Delta_B)$ of the Laplacian on B satisfies the inequality $\lambda_1(\Delta_B) \geq m\rho^2$. Moreover, the equality holds if and only if B is isometric to an Euclidean sphere $\mathbb{S}^m(\rho^2)$ of radius $\frac{1}{\rho}$.

(ii) *Let $H_{\partial B}$, $Ric_{\partial B}$, and K_B be the mean curvature of ∂B, the Ricci curvature of ∂B and the sectional curvature of B, respectively. Suppose f constant on ∂M, and the Ricci curvature of B is non negative. Assume $H_{\partial B}$, $Ric_{\partial B}$, and K_B are such that the following system*

$$\begin{aligned} k g_B(\nabla_B h(p), \eta(p)) + h(p)|\nabla f(p)| &< h(p) H_{\partial B}(p), \\ \lambda(p) &< Ric_{\partial B}(v, v) + K_B(\eta(p), v), \end{aligned} \tag{1}$$

is satisfied, for all $p \in \partial B$, where $\eta(p) = \frac{\nabla f(p)}{|\nabla f(p)|}$. Then the first eigenvalue $\lambda_1(\Delta_{\partial B})$ of the Laplacian on ∂B satisfies $\lambda_1(\Delta_{\partial B}) \geq (m-1)\rho^2$, for some positive constant $\rho \in \mathbb{R}$. Moreover, the equality holds if and only if B is isometric to an m-dimensional closed Euclidean ball of radius $\frac{1}{\rho}$.

(iii) *Let f, $H_{\partial B}$, $Ric_{\partial B}$, and K_B be as in item (ii). If $H_{\partial B}$, $Ric_{\partial B}$, and K_B are such that (1) is satisfied, then the mean curvature $H_{\partial B}$ satisfies*

$$\int_{\partial B} \frac{1}{H_{\partial B}(p)} d(\partial B) \geq m vol(B).$$

Moreover, the equality holds if and only if B is isometric to a closed Euclidean ball.

(iv) *Suppose that* ∂B *is simply connected,* $m \geq 3$*, and* f *satisfies*

$$\begin{aligned} f(p) > c_0 \quad p \in int(B), \\ f(p) = c_0 \quad\quad p \in \partial B, \end{aligned}$$

where c_0 *is a constant, and* ∇f *does not vanish on* ∂B*. If there exists an isometric immersion of* ∂B *into the* $\mathbb{H}^{m+m_0}$*,* $m_0 \geq 0$*, and the Hessian of* f *is such that*

$$\begin{aligned} \frac{(1-n-\lambda(p))h(p)|\nabla f(p)|}{k\, g_B(\nabla_B h(p), \eta(p)) + h(p)|\nabla f(p)|} \leq Hess(f)(v,v) \leq \\ \leq -|\nabla f(p)|_{\mathbb{L}} |II_p^{\mathbb{H}}(v,v)|_{\mathbb{L}}, \end{aligned}$$

is satisfied, then B *is isometric to a* m*-dimensional hyperbolic domain. Here,* $II^{\mathbb{H}}$ *denotes the vector-valued second fundamental form of* ∂B *in* $\mathbb{H}^{m+m_0}$*.*

Chapter 1

Preliminaries

Since in this thesis we study Ricci almost solitons with boundary, then the two first sections are dedicated to recall some basic definitions and properties of smooth manifolds with boundary and Riemannian metrics. For a treatment much more systematic about manifolds with boundary see [9]. The third section we give some basic definitions and properties about Killing vector fields on a smooth manifold with or without boundary. The fourth section we give the generalized Bochner formula. At the fifth section we study the relationship of the topology induced from the smooth structure on a manifold with boundary and the topology induced from the distance function. We finish the last section by giving the statement of the Hopf-Rinow theorem for manifolds with boundary.

1.1 Smooth Manifolds with boundary

Let M be a topological space. We say M is an n-**dimensional topological manifold with boundary** if M is a Hausdorff space such that admit a countable basis for its topology and every point in M has a neighborhood homeomorphic to an open subset of

$$\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_n \geq 0\}, \tag{1.1}$$

where $\mathbb{R}^n_+$ is provided by the inherited topology from the $\mathbb{R}^n$. If $n > 0$ we set

$$\text{int}(\mathbb{R}^n_+) = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}, \tag{1.2}$$

$$\partial\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_n = 0\}. \tag{1.3}$$

If $n = 0$, then $\text{int}(\mathbb{R}^0_+) = \{0\}$ and $\partial\mathbb{R}^0_+ = \varnothing$.

Let M be an n-dimensional topological manifold with boundary. An **chart for** M is a pair (U, φ) such that U is an open subset of M and $\varphi : U \longrightarrow \mathbb{R}^n_+$ is an homeomorphism on the open subset $\varphi(U) \subset \mathbb{R}^n_+$. It follows from the definition that if M is an n-dimensional topological manifold with boundary, then there exists a collection $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ such that $(U_\alpha, \varphi_\alpha)$ is a chart for M, for all α, and

$$M = \bigcup_\alpha U_\alpha.$$

A point $p \in M$ is called an **interior point of** M, if $\varphi(p) \in \mathrm{int}(\mathbb{R}^n_+)$ for some chart (U, φ) for M. On the other hand, it is called a **boundary point of** M, if $\varphi(p) \in \partial\mathbb{R}^n_+$, for some chart (U, φ) for M.

A given point cannot be simultaneously an interior point with respect to one chart and a boundary point with respect to another. For convenience, we state the theorem here.

Theorem 1.1.1 (**Topological Invariance of the Boundary.**). *If M is a topological manifold with boundary, then each point of M is either a boundary point or an interior point, but not both.*

If M is a topological manifold with boundary, then the subset of M whose points are boundary points we denoted by ∂M, and the subset of M whose points are interior points we denoted by $\mathrm{int}(M)$. From the Theorem 1.1.1 we have $M = \partial M \cup \mathrm{int}(M)$, $\partial M \cap \mathrm{int}(M) = \varnothing$.

Proposition 1.1.2. *Let M be a topological n-manifold with boundary. Then ∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.*

One of the reasons for introducing smooth structures was to enable us to define smooth functions on manifolds and smooth maps between manifolds with boundary.

Definition 1.1.3. *Let U be an open subset of $\mathbb{R}^n_+$. A map $f : U \longrightarrow \mathbb{R}^k$ is called* ***smooth*** *if for each point $p \in U$ there exist an open subset $V \subset \mathbb{R}^n$ and a smooth map $F : V \longrightarrow \mathbb{R}^k$ such that $p \in V$ and $F|_{U\cap V} = f|_{U\cap V}$.*

Example 1.1.4. *Let $B^2 \subset \mathbb{R}^2$ be the set $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Let $U = B^2 \cap \mathbb{R}^2_+$, and define $f : U \longrightarrow \mathbb{R}$ by $f(x, y) = \sqrt{1 - x^2 - y^2}$. Since the function $F : B^2 \longrightarrow \mathbb{R}$ given by $F(x, y) = \sqrt{1 - x^2 - y^2}$ is smooth and $F|U = f$, then f is smooth in the sense that we just defined.*

Example 1.1.5. *Let U be like in Example 1.1.4, define $h : U \longrightarrow \mathbb{R}$ by $h(x, y) = \sqrt{y}$. Observe that h is continuous in U and smooth in $U \cap$ int$(\mathbb{R}^2_+)$, but it has no smooth extension to any neighborhood of the origin in $\mathbb{R}^2$, because*

$$\frac{\partial h}{\partial y}(x, y) \longrightarrow \infty, \text{ as } (x, y) \longrightarrow (0, 0).$$

Therefore, h is not smooth in U.

Let M be an n-dimensional topological manifold with boundary. If there exists a collection $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ of charts for M such that

(i) $M = \bigcup_\alpha U_\alpha$,

(ii) $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth, $\forall \alpha, \beta$, $U_\alpha \cap U_\beta \neq \varnothing$,

M is called a **smooth manifold with boundary**. If M is a smooth manifold with boundary every $(U_\alpha, \varphi_\alpha)$ is called **smooth chart**.

Some trivial examples of smooth manifolds with boundary are $\mathbb{R}^n_+$, with the standard structure, and $\mathbb{S}^n_+ = \{x = (x_1, ..., x_{n+1}) \in \mathbb{S}^{n+1} : x_{n+1} \geq 0\}$, with the inherit structure from the $\mathbb{S}^n$.

Suppose M is a smooth n-manifold with boundary, k is a nonnegative integer, and $f : M \longrightarrow \mathbb{R}^k$ is any function. We say that f is a **smooth function** if for every $p \in M$, there exists a smooth chart (U, φ) for M whose domain contain p and such that the composite function $f \circ \varphi^{-1} : \varphi(U) \longrightarrow \mathbb{R}^k$ is smooth in the sense that we just defined in Definition 1.1.3.

Let M, N be smooth manifolds with boundary, and let $F : M \longrightarrow N$ be any map. We say that F is a **smooth map** if for every point $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and the composite map

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \longrightarrow \psi(V)$$

is smooth in the sense that we defined in Definition 1.1.3.

If M and N are smooth manifolds with or without boundary, a **diffeomorphism from** M **to** N is a smooth bijective map $F : M \longrightarrow N$ that has a smooth inverse. We say that M **and** N **are diffeomorphic** if there exists a diffeomorphism between them.

Theorem 1.1.6 (**Diffeomorphism Invariance of the Boundary.**)**.** *Suppose M and N are smooth manifolds with boundary and $F : M \longrightarrow N$ is a diffeomorphism. Then $F(\partial M) = \partial N$, and F restricts to a diffeomorphism from int(M) to int(N).*

For the proof, see Theorem 2.18 in [9].

Let $C^\infty(\mathbb{R}^n)$ be the set of all smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$. For every $p, v \in \mathbb{R}^n$ define a map $v_p : C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$ by

$$v_p(f) = df_p(v) = \frac{d}{dt}\bigg|_{t=0} f(p + tv). \tag{1.4}$$

The set of all maps defined by (1.4) is called **tangent space to** $\mathbb{R}^n$ **at** p, and it is denoted by $T_p\mathbb{R}^n$. Let $\{e_i\}_{i=1}^n$ be the standard basis of $\mathbb{R}^n$, then, for each i and $p \in \mathbb{R}^n$, we have the following map

$$(e_i)_p(f) = \left.\frac{d}{dt}\right|_{t=0} f(p + te_i) = \frac{\partial f}{\partial x_i}(p),$$

for all $f \in C^\infty(\mathbb{R}^n)$. Since $v = \sum_{i=1}^n v_i e_i$ for all $v \in \mathbb{R}^n$, then by using the chain rule it follows

$$v_p(f) = \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}(p).$$

If v_p is such that $v_p(f) = 0$, for all $f \in C^\infty(\mathbb{R}^n)$, then if $\pi_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the map given by $\pi_i(x_1, ..., x_n) = x_i$ which implies $(e_j)_p(\pi_i) = \delta_{ji}$. Therefore

$$0 = v_p(\pi_i) = v_i, \forall i.$$

Then, $\{(e_i)_p\}_i$ is a basis for $T_p\mathbb{R}^n$.

So, from now on, we will follow the notation

$$(e_i)_p = \left.\frac{\partial}{\partial x_i}\right|_p .$$

With the sum and scalar product standard $T_p\mathbb{R}^n$ is a linear space. Since

$$\left\{ \left.\frac{\partial}{\partial x_1}\right|_p, ..., \left.\frac{\partial}{\partial x_n}\right|_p \right\}$$

is a basis for $T_p\mathbb{R}^n$, then $\dim(T_p\mathbb{R}^n) = n$.

Let M be a smooth manifold with or without boundary, $C^\infty(M)$ the set of all smooth functions from M to $\mathbb{R}$, and let p be a point of M. A linear map $v_p : C^\infty(M) \longrightarrow \mathbb{R}$ is called a **tangent vector on M at p** if it satisfies

$$v_p(f_1 f_2) = f_1(p) v_p(f_2) + f_2(p) v_p(f_1), \forall f_1, f_2 \in C^\infty(M). \tag{1.5}$$

The set of all tangent vectors on M at p is denoted by T_pM, and it is called the **tangent space to M at p**.

Proposition 1.1.7. *Let M be a smooth manifold with or without boundary. Then for each $p \in M$, T_pM is a linear space.*

Now it makes sense to define the differential of a smooth map between smooth manifolds with or without boundary.

Definition 1.1.8. *Let M and N be smooth manifolds with or without boundary. Let $F : M \longrightarrow N$ be a smooth map, for each $p \in M$ the map $dF_p : T_pM \longrightarrow T_{F(p)}N$ given by*

$$dF_p(v)(f) = v(f \circ F), \forall f \in C^\infty(N), \tag{1.6}$$

is called the ***differential of*** F ***at*** p. *Indeed, a straight computation show us that* $dF_p(v) \in T_{F(p)}N$, *for all* $v \in T_pM$, $p \in M$.

Theorem 1.1.9 (**Dimension of Tangent Spaces on a Manifold with Boundary**). *Let* M *be an* n*-dimensional smooth manifold with boundary. For each* $p \in M$, *the tangent space* T_pM *is an* n*-dimensional linear space.*

Proof. See [9]. ■

Let $M_1, ..., M_k$ be smooth manifolds without boundary, with dimension n_1,..., n_k, respectively. If the topology on $M_1 \times ... \times M_k$ is the product topology, then $M_1 \times ... \times M_k$ is an $n_1 + ... + n_k$-dimensional topological manifold. The collection

$$\{(U_1 \times ... \times U_k, \varphi_1 \times ... \times \varphi_k) : (U_i, \varphi_i) \text{ is a smooth chart on } M_i, \text{ for all } i = 1, ..., k\}$$

define a smooth structure on $M_1 \times ... \times M_k$. Therefore, $M_1 \times ... \times M_k$ is a smooth manifold with dimension $n_1 + ... + n_k$.

For smooth manifolds with boundary we have the following proposition.

Proposition 1.1.10. *Let* M_1,..., M_k *be smooth manifolds without boundary and let* N *be a smooth manifold with boundary. Then* $N \times M_1 \times ... \times M_k$ *is a smooth manifold with boundary, and*

$$\partial(N \times M_1 \times ... \times M_k) = \partial N \times M_1 \times ... \times M_k. \tag{1.7}$$

Proof. See [9]. ■

In many cases, it is useful to consider the set of all tangent vectors at all points of a manifold. Let M be an n-dimensional smooth manifold with or without boundary, we denoted by TM the set

$$\{(p, v) : p \in M \text{ and } v \in T_pM\}. \tag{1.8}$$

The set TM is called the **tangent bundle of** M. For each $p \in M$, we will often identify T_pM with its image under the map $v \mapsto (p, v)$. The tangent bundle is provided by the **disjoint union topology** (see [9]). In order to show that TM is a smooth manifold let p be any point in M, let (x_i) be some coordinate system around p locally defined in an open set $U \subset M$, let $\pi : TM \longrightarrow M$ the projection map, i.e., $\pi(p, v) = p$, and define a map $\varphi : \pi^{-1}(U) \longrightarrow \mathbb{R}^{2n}$ be by

$$\varphi\left(p, \sum_{i=1}^{n} v_i \left.\frac{\partial}{\partial x_i}\right|_p\right) = (x_1(p), ..., x_n(p), v_1, ..., v_n).$$

A straight computation (see [9]) shows us the collection of maps defined as above define a smooth structure on TM. In particular, we have that TM has dimension $2n$.

Now, remember that, if E and F are topological spaces, a map $f : E \longrightarrow F$ is said to be **proper** if for every compact set $K \subset F$, the preimage $f^{-1}(K) \subset E$ is compact.

Suppose M is a smooth manifold with or without boundary. An **embedded submanifold of** M is a subset $S \subset M$ that is a manifold (without boundary) endowed with the induced topology from M, and endowed with a smooth structure with respect to which the inclusion map $S \hookrightarrow M$ is a smooth embedded, i.e., the inclusion map is a smooth homeomorphism on its own range, and in the range we consider the induced topology from the M. Furthermore, an embedded $S \subset M$ is said to be **properly embedded** if the inclusion $S \hookrightarrow M$ is a proper map.

Theorem 1.1.11. *If M is a smooth n-manifold with boundary, then ∂M endowed with the induced topology from the M has a smooth structure such that it is a properly embedded submanifold of M. And its dimension is $n-1$.*

Proof. See [9]. ■

Next, we define vector fields on an abstract smooth manifold.

Let M be a smooth manifold with or without boundary. A **vector field on** M is a continuous map $\mathcal{X} : M \longrightarrow TM$ which satisfies

$$\pi \circ \mathcal{X} = \mathrm{id}_M, \tag{1.9}$$

where $\pi : TM \longrightarrow M$ is the given by $\pi(p, v) = p$. It follows from (1.9) that $\mathcal{X}_p = \mathcal{X}(p) \in T_pM$, for each $p \in M$.

Let $(U, (x_i))$ be a chart for M such that $p \in U$ with coordinates functions (x_i). Then

$$\mathcal{X}_p = \sum_{i=1}^{n} a_i(p) \left.\frac{\partial}{\partial x_i}\right|_p. \tag{1.10}$$

If p is implicit, we just write

$$\mathcal{X} = \sum_{i=1}^{n} a_i \mathcal{X}_i, \text{ in } U,$$

where $\mathcal{X}_i = \frac{\partial}{\partial x_i}$. The functions $\mathcal{X}_p : M \longrightarrow \mathbb{R}$ are called **component functions of** $\mathcal{X}$. Then $\mathcal{X} : M \longrightarrow TM$ is a **smooth vector field** if, only if, for each chart (U, φ), with coordinates (x_i) the component functions $\mathcal{X}_i$ are smooth, for every i. We denote by $\mathscr{X}(M)$ the set of all smooth vector fields on M.

Suppose M a smooth manifold with or without boundary. A **vector field locally defined on** M is a continuous map $\mathcal{X} : U \longrightarrow TM$, where $U \subset M$ is an open subset, which satisfies (1.9).

Let M^n be a smooth manifold with or without boundary. A **local frame for** M is a set $\{\mathcal{E}_1, ..., \mathcal{E}_n\}$ of vector fields locally defined on an open subset $U \subset M$ such that $\{\mathcal{E}_1(p), ..., \mathcal{E}_n(p)\}$ is a basis for T_pM, for every $p \in M$.

One of the most important properties of vector fields on a smooth manifold M with or without boundary is that they define operators on $C^\infty(M)$. Indeed, if $f \in C^\infty(M)$ and $\mathcal{X} \in \mathscr{X}(M)$, then the function $\mathcal{X}f : M \longrightarrow \mathbb{R}$ defined by

$$(\mathcal{X}f)(p) = \mathcal{X}_p f$$

is smooth. Conversely, if $\mathcal{X}$ is a vector field such that $\mathcal{X}f \in C^\infty(M)$, for every, $f \in C^\infty(M)$, then $\mathcal{X} : M \longrightarrow TM$ is smooth.

Define the operator $[\mathcal{X}, \mathcal{Y}] : C^\infty(M) \longrightarrow C^\infty(M)$, where $\mathcal{X}, \mathcal{Y} \in \mathscr{X}(M)$, by

$$[\mathcal{X}, \mathcal{Y}]f = \mathcal{X}\mathcal{Y}f - \mathcal{Y}\mathcal{X}f. \tag{1.11}$$

This operator is called the **Lie bracket of $\mathcal{X}$ and $\mathcal{Y}$**.

Let M be a smooth manifold with or without boundary. Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth vector fields on M. If (x_i) is a smooth local coordinate for M, then there exists smooth functions a_i's and b_i's locally defined on an open subset of M such that $\mathcal{X}$ and $\mathcal{Y}$ are locally given by

$$\mathcal{X} = \sum_i a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad \mathcal{Y} = \sum_i b_i \frac{\partial}{\partial x_i}.$$

By using the definition we can show that

$$[\mathcal{X}, \mathcal{Y}] = \sum_{i,j} \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}, \tag{1.12}$$

which implies

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] = 0. \tag{1.13}$$

Proposition 1.1.12 (Properties of the Lie Bracket.). *Let M be a smooth manifold with or without boundary. The Lie bracket satisfies the following identities for all $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathscr{X}(M)$:*

(i) Bilinearity: For every $\alpha, \beta \in \mathbb{R}$,

$$[\alpha\mathcal{X} + \beta\mathcal{Y}, \mathcal{Z}] = \alpha[\mathcal{X}, \mathcal{Z}] + \beta[\mathcal{Y}, \mathcal{Z}],$$

$$[\mathcal{X}, \alpha\mathcal{Y} + \beta\mathcal{Z}] = \alpha[\mathcal{X}, \mathcal{Y}] + \beta[\mathcal{X}, \mathcal{Z}].$$

(ii) Antisymmetry: $[\mathcal{X}, \mathcal{Y}] = -[\mathcal{Y}, \mathcal{X}]$.

(iii) Jacobi's identity:

$$[\mathcal{X}, [\mathcal{Y}, \mathcal{Z}]] + [\mathcal{Y}, [\mathcal{Z}, \mathcal{X}]] + [\mathcal{Z}, [\mathcal{X}, \mathcal{Y}]] = 0.$$

(iv) For $f, h \in C^\infty(M)$,

$$[f\mathcal{X}, h\mathcal{Y}] = gh[\mathcal{X}, \mathcal{Y}] + f\mathcal{X}(h)\mathcal{Y} - h\mathcal{Y}(f)h.$$

For while, let M be a topological space. A **vector bundle over** M **of rank** m is a topological space E, called the **total space of the bundle**, together with a surjective continuous map $\pi : E \longrightarrow M$, called the **projection**, satisfying the following properties:

(i) For each $p \in M$ the preimage $\pi^{-1}(p)$ is endowed with the structure of a linear space of dimension m. We denote the preimage of p under the projection by E_p, and this space is called the **fiber of** p.

(ii) For each $p \in M$, there exists a neighborhood U of p in M and a homeomorphism $\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^m$, satisfying the following conditions:

- $\mathrm{Proj}_U \circ \Phi = \pi$ (where $\mathrm{Proj}_U : U \times \mathbb{R}^k \longrightarrow U$ is the projection);
- for each $q \in U$, the restriction of Φ to E_q is an isomorphism from E_q to $\{q\} \times \mathbb{R}^m$.

If M and E are smooth manifolds with or without boundary, π is a smooth map and Φ is a diffeomosphism, for each $p \in M$, then E is called a **smooth vector bundle**.

Example 1.1.13 (The Möbius Band.). *Define an equivalence relation on* $\mathbb{R}^2$ *by declaring that* $(x_1, y_1) \sim (x_2, y_2)$ *if and only if*

$$(x_2, y_2) = (x_1 + n, (-1)^n y_1),$$

for some $n \in \mathbb{Z}$. *Let* $E = \mathbb{R}/\sim$ *denote the quotient space, and let* $\xi : \mathbb{R}^2 \longrightarrow E$ *be the quotient map. For any* $r > 0$, *the image under* ξ *of the rectangle* $[0, 1] \times [-r, r]$ *is a smooth compact manifold with boundary called* ***Möbius band****.*

Moreover, we have the following proposition.

Proposition 1.1.14. *Let M^n be a smooth manifold with or without boundary. Then TM is a smooth vector bundle over M of rank n.*

Proof. See [9]. ■

Definition 1.1.15. *Let (E, M^n, π) be a vector bundle. A* ***section of*** *E is a section of the map π, that is, a continuous map $\sigma : M \longrightarrow E$ satisfying $\pi \circ \sigma = id_M$. This means that $\sigma(p)$ is an element of the fiber E_p for each $p \in M$. More generally, a* ***local section of*** *E is a continuous map $\sigma : U \longrightarrow E$ defined on some open subset $U \subset M$ and satisfying $\pi \circ \sigma = id_U$. When a section is defined on all of M is called a* ***global section****. If M is a smooth manifold with or without boundary and E is a smooth vector bundle, a* ***smooth (local or global) section of*** *E is one that is a smooth map from its domain to E.*

Now let M^n be a smooth manifold with or without boundary. For each $p \in M$, we define the **cotangent space at** p, denoted by T_p^*M, to be the dual space to T_pM, that is,

$$T_p^*M = (T_pM)^*.$$

Elements of T_p^*M are called **tangent covectors at** p, or just **covectors at** p.

Given smooth local coordinates (x_i) on an open subset $U \subset M$, for each $p \in U$ the coordinate basis $\left\{ \left.\frac{\partial}{\partial x_i}\right|_p \right\}$ gives rise to a dual basis for T_p^*M, which we denote by $(dx_i|_p)$. Any covector $\omega(p) \in T_p^*M$ can thus be written uniquely as

$$\omega(p) = \sum_{i=1}^{n} a_i(p) dx_i|_p, \text{ where } a_i(p) = \omega(p) \left(\left.\frac{\partial}{\partial x_i}\right|_p \right).$$

The disjoint union

$$T^*M = \coprod_{p \in M} T_p^*M = \{(p, v^*) : p \in M \text{ and } v^* \in T_p^*M\},$$

is called the **cotangent bundle of** M. It has a natural projection map $\pi : T^*M \longrightarrow M$ sending $\omega(p) \in T_p^*M$ to $p \in M$.

Since $(dx_i|_p)$ is a dual basis for T_p^*M, for each $p \in M$, then this fact defines n maps $dx_1, ..., dx_n : U \longrightarrow T^*M$, called **coordinate covector fields**. The next proposition tell us that the cotangent bundle is a vector bundle.

Proposition 1.1.16 (The Cotangent Bundle as a Vector Bundle.). *Let M^n be a smooth manifold with or without boundary. With its standard projection map and the natural vector space structure on each fiber, the cotangent bundle T^*M has a unique topology and smooth structure making it into a smooth n-rank vector bundle over M for which all coordinate covector fields are smooth local sections.*

Definition 1.1.17. *Let M and N be smooth manifolds with or without boundary. Let $F : M \longrightarrow N$ be a smooth map, let $p \in M$ be arbitrary. For every covector field ω on N, we define a* ***covector field*** $F^*\omega$ ***on*** *M by*

$$(F^*\omega)_p(v) = \omega_{F(p)}(dF_p(v)), \tag{1.14}$$

for all $v \in T_pM$.

So, by using local coordinates and the Definition 1.5.1, we can show the following proposition.

Proposition 1.1.18. *Let M and N be smooth manifolds with or without boundary. Let $F : M \longrightarrow N$ be a smooth map and let ω be a covector field on N. If ω is smooth, then $F^*\omega$ is smooth.*

Proof. See [9]. ■

Let M be a smooth manifold with or without boundary. We define the **bundle of covariant k-tensors on** M by

$$T^kT^*M = \coprod_{p\in M} T^k(T_p^*M).$$

Proposition 1.1.19. *Let M^n be a smooth manifold with or without boundary. Then T^kT^*M has a natural structure as smooth vector bundle over M, and its rank is n^k.*

A section of a bundle of covariant k-tensors on M is called a **covariant k-tensor field on** M. A **smooth covariant k-tensor field** is a section that is smooth in the usual sense of smooth section of vector bundles.

The space of smooth covariant k-tensor fields on M,

$$\Gamma(T^kT^*M),$$

is an infinite-dimensional vector space over $\mathbb{R}$, and modules over $C^\infty(M)$. In any smooth local coordinates (x_i), covariant k-tensor fields $F \in \Gamma(T^kT^*M)$ can be written (using Einstein's convention) as

$$F = F_{i_1\cdots i_k} dx_{i_1} \otimes \cdots \otimes dx_{i_k}.$$

The functions $F_{i_1\cdots i_k}$ are called the **component functios of** F in the chosen coordinates. We denote the space of all smooth covariant k-tensor fields by

$$\mathscr{T}^k(M) = \Gamma(T^kT^*M).$$

Denote by $\mathscr{X}(M)$ the set of all smooth vector fields on M.

Proposition 1.1.20 (Smoothness Criteria for Tensor Fields.). *Let M be a smooth manifold with or without boundary, and let $F : M \longrightarrow T^kT^*M$ be a rough section, The following statements are equivalent.*

(i) F is smooth.

(ii) In every smooth coordinate chart, the component functions of F are smooth.

(iii) Each point of M is contained in some coordinate chart in which F has smooth component functions.

(iv) If $\mathcal{X}_1, ..., \mathcal{X}_k \in \mathscr{X}(M)$, then the function $F(\mathcal{X}_1, ..., \mathcal{X}_k) : M \longrightarrow \mathbb{R}$, defined by

$$F(\mathcal{X}_1, ..., \mathcal{X}_k)(p) = F_p(\mathcal{X}_1|_p, ..., \mathcal{X}_k|_p),$$

is smooth.

(v) Whenever $\mathcal{X}_1, ..., \mathcal{X}_k$ are smooth vector fields defined on some open subset $U \subset M$, the function $F(\mathcal{X}_1, ..., \mathcal{X}_k)$ is smooth on U.

Theorem 1.1.21 (Tensor Characterization.). *Let M be a smooth manifold with or without boundary. A map*

$$F : \underbrace{\mathscr{X}(M) \times ... \times \mathscr{X}(M)}_{k} \longrightarrow C^\infty(M),$$

is induced by a smooth covariant k-tensor field as above if and only if it is multilinear over $C^\infty(M)$.

For more details about this section we indicated [9] and [8].

1.2 Riemannian metrics, Connections and Curvature

Let M be a smooth manifold with or without boundary. A **Riemannian metric on** M is a smooth symmetric covariant 2-tensor field on M that is positive definite at each point. A **Riemannian manifold** is a pair (M, g), where M is a smooth manifold and g is a Riemannian metric on M. When the Riemannian metric on M is understood we simply call M by Riemannian manifold.

In some (x_i) local coordinates of a Riemannian manifold M with or without boundary, a Riemannian metric can be written

$$g = g_{ij} dx_i \otimes dx_j.$$

Proposition 1.2.1 (**Existence of Riemannian Metrics.**). *Every smooth manifold with or without boundary admits a Riemannian metric.*

Definition 1.2.2. *Let (M^n, g) be a Riemannian manifold with or without boundary. We say that a* ***local frame*** *$\{\mathcal{E}_1, ..., \mathcal{E}_n\}$ for M on an open subset $U \subset M$ is an orthonormal frame if the vectors $\{(\mathcal{E}_1)_p, ..., (\mathcal{E}_n)_p\}$ form an orthonormal basis for T_pM at each point $p \in U$.*

Proposition 1.2.3 (**Existence of Local Orthonormal Frame.**). *Let (M, g) be a Riemannian manifold with or without boundary. For each $p \in M$, there is a smooth orthonormal frame on a neighborhood of p.*

A **linear connection** on a smooth manifold M with or without boundary is a map $\nabla : \mathscr{X}(M) \times \mathscr{X}(M) \longrightarrow \mathscr{X}(M)$ which satisfies the following properties:

(i) $\nabla_{f\mathcal{X}+h\mathcal{Y}}\mathcal{Z} = f\nabla_{\mathcal{X}}\mathcal{Z} + h\nabla_{\mathcal{Y}}\mathcal{Z}$, for every $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathscr{X}(M)$ and $f, g \in C^\infty(M)$.

(ii) $\nabla_{\mathcal{X}}(\alpha\mathcal{Y} + \beta\mathcal{Z}) = \alpha\nabla_{\mathcal{X}}\mathcal{Y} + \beta\nabla_{\mathcal{X}}\mathcal{Z}$, for every $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathscr{X}(M)$ and $\alpha, \beta \in \mathbb{R}$.

(iii) $\nabla_{\mathcal{X}}(f\mathcal{Y}) = f\nabla_{\mathcal{X}}\mathcal{Y} + \mathcal{X}(f)\mathcal{Y}$, for every $\mathcal{X}, \mathcal{Y} \in \mathscr{X}(M)$ and $f \in C^\infty(M)$.

In particular, we can show (see Petersen) that $(\nabla_{\mathcal{X}}\mathcal{Y})(p)$, $p \in M$, only depend of the value of $\mathcal{X}$ at p and the values that $\mathcal{Y}$ assumes in any curve that has $\mathcal{X}(p)$ as a tangent vector.

Let M be a smooth manifold with or without boundary. Let (x_i) be a local coordinates for M. Set $\mathcal{X}_i = \frac{\partial}{\partial x_i}$, for each i. Then

$$\nabla_{\mathcal{X}_i}\mathcal{X}_j = \sum_k \Gamma_{ij}^k \mathcal{X}_k. \tag{1.15}$$

Each term Γ_{ij}^k is called **Christoffel symbol** (**of** ∇) with respect to the frame $\{\mathcal{X}_i\}_i$. For any $\mathcal{X}, \mathcal{Y} \in \mathscr{X}(M)$ a straight computation show us that

$$\nabla_{\mathcal{X}}\mathcal{Y} = \sum_{i,j,k} (\mathcal{X}(b_k) + a_i b_j \Gamma_{ij}^k)\mathcal{X}_k, \tag{1.16}$$

where $\mathcal{X} = \sum_i a_i\mathcal{X}_i$ and $\mathcal{Y} = \sum_j b_j\mathcal{X}_j$.

Definition 1.2.4. *Let (M, g) be a Riemannian manifold with or without boundary. We say that a linear connection ∇ on M is* ***compatible with the metric*** *g when*

$$\mathcal{X}g(\mathcal{Y}, \mathcal{Z}) = g(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{Z}) + g(\mathcal{Y}, \nabla_{\mathcal{X}}\mathcal{Z}), \tag{1.17}$$

for all $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathscr{X}(M)$.

Definition 1.2.5. *Let M be a smooth manifold with or without boundary. A linear connection ∇ on M is called* ***symmetric*** *when*

$$\nabla_{\mathcal{X}}\mathcal{Y} - \nabla_{\mathcal{Y}}\mathcal{X} = [\mathcal{X}, \mathcal{Y}], \tag{1.18}$$

for all $\mathcal{X}, \mathcal{Y} \in \mathscr{X}(M)$.

For the proof of the following theorem see [12] or [13].

Theorem 1.2.6 (Levi-Civita.). *Let (M, g) be a Riemannian manifold with or without boundary. There exists only one linear connection ∇ on M which is symmetric and compatible with the metric g. Such connection is called the Levi-Civita connection of M (with respect the metric g).*

From now on, every Riemannian manifold with or without boundary shall be provided with its Levi-Civita connection.

Let (M, g) be a Riemannian manifold with or without boundary. Let (x_i) be local coordinates for M. Set $\mathcal{X}_i = \frac{\partial}{\partial x_i}$, for each i. Denote by (g^{km}) the inverse matrix of (g_{ij}), where $g_{ij} = g(\mathcal{X}_i, \mathcal{X}_j)$. We can show that (see [12])

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \left\{ \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right\} g^{km}. \tag{1.19}$$

Definition 1.2.7 (Hessian of a smooth function.). *Let (M, g) be a Riemannian manifold with or without boundary. For each $f \in C^\infty(M)$ we define the* ***Hessian of the function*** *f as the tensor $Hess(f) \in \mathscr{T}^2(M)$ given by*

$$Hess(f)(\mathcal{X}, \mathcal{Y}) = g(\nabla_{\mathcal{X}} \nabla f, \mathcal{Y}). \tag{1.20}$$

Moreover, a straight computation show us that $Hess(f)$ is symmetric, for all $f \in C^\infty(M)$.

Next, we define the **covariant derivative** of a tensor with respect to a vector field, and the **covariant differential** of a tensor.

Definition 1.2.8. *Let M be a smooth manifold with or without boundary. For each $\mathcal{T} \in \mathscr{T}^k(M)$ the* ***covariant derivative of $\mathcal{T}$ with respect to $\mathcal{X} \in \mathscr{X}(M)$*** *is given by*

$$\begin{aligned} (\nabla_{\mathcal{X}}\mathcal{T})(\mathcal{Y}_1, ..., \mathcal{Y}_k) &= \mathcal{X}(\mathcal{T}(\mathcal{Y}_1, ..., \mathcal{Y}_k)) - \\ &\quad - \sum_{i=1}^{k} \mathcal{T}(\mathcal{Y}_1, ..., \mathcal{Y}_{i-1}, \nabla_{\mathcal{X}}\mathcal{Y}_i, \mathcal{Y}_{i+1} ..., \mathcal{Y}_k). \end{aligned} \tag{1.21}$$

Definition 1.2.9. *Let M be a smooth manifold with or without boundary. The* ***covariant differential*** *of a tensor $\mathcal{T} \in \mathscr{T}^k(M)$ is a tensor in $\mathscr{T}^{k+1}(M)$, denoted by $\nabla\mathcal{T}$, is given* *by*

$$\nabla\mathcal{T}(\mathcal{Y}_1, ..., \mathcal{Y}_k, \mathcal{X}) = (\nabla_{\mathcal{X}}\mathcal{T})(\mathcal{Y}_1, ..., \mathcal{Y}_k). \tag{1.22}$$

Proposition 1.2.10. *Let (M, g) be a Riemannian manifold with or without boundary. For any $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathscr{X}(M)$, we have from the Definition 1.2.8*

$$(\nabla_{\mathcal{X}}g)(\mathcal{Y}, \mathcal{Z}) = \mathcal{X}(g(\mathcal{Y}, \mathcal{Z})) - g(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{Z}) - g(\mathcal{Y}, \nabla_{\mathcal{X}}\mathcal{Z}). \tag{1.23}$$

Since ∇ is the Levi-Civita connection for M, then from the identity (1.17) *we concluded that $(\nabla_{\mathcal{X}}g)(\mathcal{Y}, \mathcal{Z}) = 0$.*

In what follows we give some very useful definitions.

Definition 1.2.11. *Given a vector field $\mathcal{X} \in \mathscr{X}(M)$, we define the map $\mathcal{L}_{\mathcal{X}} : C^\infty(M) \longrightarrow C^\infty(M)$ by*

$$(\mathcal{L}_{\mathcal{X}}f)(p) = \mathcal{X}_p f. \tag{1.24}$$

This map is called the ***Lie derivative (on functions) with respect to $\mathcal{X}$.***

Definition 1.2.12. *Given a vector field $\mathcal{X} \in \mathscr{X}(M)$, we define a map $\mathcal{L}_{\mathcal{X}} : \mathscr{X}(M) \longrightarrow \mathscr{X}(M)$ by*

$$\mathcal{L}_{\mathcal{X}}(\mathcal{Y}) = [\mathcal{X}, \mathcal{Y}]. \tag{1.25}$$

This map is called the ***Lie derivative (with respect to $\mathcal{X}$).***

Definition 1.2.13. *Given $\mathcal{X} \in \mathscr{X}(M)$, we define the* ***Lie derivative of a tensor with respect to $\mathcal{X}$*** *by the map $\mathcal{L}_{\mathcal{X}} : \mathscr{T}^k(M) \longrightarrow \mathscr{T}^k(M)$ given by*

$$\begin{aligned}(\mathcal{L}_{\mathcal{X}}(T))(\mathcal{Y}_1, \cdots, \mathcal{Y}_k) &= \mathcal{X}(T(\mathcal{Y}_1, \cdots, \mathcal{Y}_k)) - \\ &- \sum_{j=1}^{k} T(\mathcal{Y}_1, \cdots, \mathcal{Y}_{j-1}, \mathcal{L}_{\mathcal{X}}(\mathcal{Y}_j), \mathcal{Y}_{j+1}, \cdots, \mathcal{Y}_k).\end{aligned} \tag{1.26}$$

Proposition 1.2.14. *Let (M, g) be a Riemannian manifold with or without boundary. If $f \in C^\infty(M)$, then*

$$(\mathcal{L}_{\nabla f}g)(\mathcal{X}, \mathcal{Y}) = 2Hess(f)(\mathcal{X}, \mathcal{Y}), \tag{1.27}$$

for all $\mathcal{X}, \mathcal{Y} \in \mathscr{X}(M)$.

Proof. Just apply the definitions. ■

In what follows we define the curvature tensor.

Definition 1.2.15. *Let (M, g) be a Riemannian manifold with or without boundary. The* ***curvature tensor*** *is the tensor*

$$R : \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \longrightarrow C^\infty(M)$$

given by

$$R(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}) = g(R(\mathcal{X}, \mathcal{Y})\mathcal{Z}, \mathcal{W}), \tag{1.28}$$

where

$$R(\mathcal{X}, \mathcal{Y})\mathcal{Z} = \nabla_{\mathcal{X}}\nabla_{\mathcal{Y}}\mathcal{Z} - \nabla_{\mathcal{Y}}\nabla_{\mathcal{X}}\mathcal{Z} - \nabla_{[\mathcal{X},\mathcal{Y}]}\mathcal{Z}, \tag{1.29}$$

for all $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W} \in \mathscr{X}(M)$.

Furthermore, we can show that the value of $R(\mathcal{X}, \mathcal{Y})\mathcal{Z}$ at p only depends of the values of $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ at p (see [12]).

The curvature tensor satisfies the properties in the following proposition.

Proposition 1.2.16. *Let (M, g) be a Riemannian manifold with or without boundary. The tensor curvature*

$$R : \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \longrightarrow C^\infty(M)$$

satisfies the following properties, for any $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W} \in \mathscr{X}(M)$:

(i) $R(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}) = -R(\mathcal{Y}, \mathcal{X}, \mathcal{Z}, \mathcal{W}) = R(\mathcal{Y}, \mathcal{X}, \mathcal{W}, \mathcal{Z})$.

(ii) $R(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}) = R(\mathcal{Z}, \mathcal{W}, \mathcal{X}, \mathcal{Y})$.

(iii) Bianchi's first identity

$$R(\mathcal{X}, \mathcal{Y})\mathcal{Z} + R(\mathcal{Z}, \mathcal{X})\mathcal{Y} + R(\mathcal{Y}, \mathcal{Z})\mathcal{X} = 0. \tag{1.30}$$

(iv) Bianchi's second identity

$$(\nabla_{\mathcal{Z}}R)(\mathcal{X}, \mathcal{Y})W + (\nabla_{\mathcal{X}}R)(\mathcal{Y}, \mathcal{Z})W + (\nabla_{\mathcal{Y}}R)(\mathcal{Z}, \mathcal{X})W = 0. \tag{1.31}$$

Proof. See [13]. ∎

Proposition 1.2.17. *Let $(M^n.g)$ be a Riemannian manifold with or without boundary, where $n \geq 2$. Let P be a 2-dimensional linear subspace of T_pM, for some $p \in M$. Let*

$\{u, v\}$ be a basis for P. Then, the expression

$$K_p(u,v) = \frac{g_p(R(u,v)u,v)}{g_p(u,u)g_p(v,v) - g_p(u,v)^2} \tag{1.32}$$

does not depend of the basis $\{u, v\}$.

Proof. Just apply elementary operations. ∎

Definition 1.2.18. *Let M be a Riemannian manifold with or without boundary. For any $p \in M$ and a 2-dimensional linear subspace P of T_pM the real numnber $K_p(u,v) = K_p(P)$, where $\{u, v\}$ is a basis for P, is called the **sectional curvature of** P **at** p.*

Definition 1.2.19. *Let (M^n, g) be a Riemannian manifold with or without boundary. For each $p \in M$, let $\{e_1, ..., e_n\}$ be an orthonormal basis for T_pM. So, the **Ricci tensor on** M **at** p is given by*

$$Ric_p(u,v) = \sum_{i=1}^{n} g_p(R(e_i,v)u,e_i), \tag{1.33}$$

*where $u, v \in T_pM$. For each $p \in M$, and an unit vector $v \in T_pM$, the **Ricci curvature of** M **at** p is given by*

$$Ric_p(v) = Ric_p(v,v) = \sum_{i=1}^{n-1} g_p(R(e_i,v)v,e_i),$$

where $\{e_1, ..., e_{n-1}, v\}$ is an orthonormal basis for T_pM.

Definition 1.2.20. *Let (M, g) be a Riemannian manifold with or without boundary. We say that M is an **Einstein manifold** (with or without boundary) if for all $\mathcal{X}, \mathcal{Y} \in \mathscr{X}(M)$ we have*

$$Ric(\mathcal{X},\mathcal{Y}) = \lambda g(\mathcal{X},\mathcal{Y}), \tag{1.34}$$

where $\lambda \in C^\infty(M)$.

The following proposition gives an interesting property of Einstein manifolds.

Proposition 1.2.21. *Let (M^n, g) be an Einstein manifold with or without boundary. Let $\lambda \in C^\infty(M)$ be a function which satisfies (1.34). If M is connected and $n \geq 3$, then λ is constant. Furthermore, if $n = 3$, then M has constant sectional curvature.*

Proof. Use Bianchi's second identity. ∎

The last curvature that we define in this section is the *scalar curvature.*

Definition 1.2.22. *Let (M^n, g) be a Riemannian manifold with or without boundary. For each $p \in M$, let $\{e_1, ..., e_n\}$ be an orthonormal basis for T_pM. So, the* ***scalar curvature of*** *M* ***at*** *p is given by*

$$S_p(u, v) = \sum_{j,i=1}^{n} g_p(R(e_i, e_j)e_j, e_i). \tag{1.35}$$

For more details about this section we indicated [12], [13], and [8].

1.3 Killing Vector Fields

Let M be a smooth manifold with or without. Let $\mathcal{X}$ be a smooth vector field on M. If $p \in \mathrm{int}(M)$, then there exists an open subset $U \subset M$ which contains p, $\delta > 0$ and a smooth map $\varphi : (-\delta, \delta) \times U \longrightarrow M$ such that

$$t \mapsto \varphi(t, q), t \in (-\delta, \delta), q \in U$$

is the unique curve which satisfies

$$\frac{\partial \varphi}{\partial t} = \mathcal{X}(\varphi(t, q)), \text{ and } \varphi(0, q) = q. \tag{1.36}$$

An smooth vector field $\mathcal{X}$ on M is called **inward** at each boundary point $p \in \partial M$ if there exists an open subset $U \subset M$ which contains p, $\delta > 0$ and a smooth map $\varphi : [0, \delta) \times U \longrightarrow M$ such that

$$t \mapsto \varphi(t, q), t \in [0, \delta), q \in U$$

is the unique curve which satisfies (1.36). The map φ is called **the local flow generated by** $\mathcal{X}$. and we say that $\mathcal{X}$ is **outward** if there exists such curve whose domain is $(-\delta, 0]$. The proof of these statements are in [9].

Definition 1.3.1. *Let M be a smooth manifold with or without boundary and $\mathcal{X} \in \mathscr{X}(M)$ (in the case $\partial M \neq \varnothing$, $\mathcal{X}$ is an inward smooth vector field in each point of the boundary). Let p be any point in int(M) (resp. ∂M), U a neighborhood of p in M, and a local flow $\varphi : (-\epsilon, \epsilon) \times U \longrightarrow M$ (resp. $\varphi : [0, \epsilon) \times U \longrightarrow M$) generated by $\mathcal{X}$. The vector field $\mathcal{X}$ is called* ***Killing vector field*** *if $\varphi_t = \varphi(t, \cdot) : U \longrightarrow M$ is an isometry, for each $t \in (-\epsilon, \epsilon)$ (resp.$t \in [0, \epsilon)$).*

In what follows we show a relation between Killing vector fields and the Lie derivative.

Lemma 1.3.2. *Let M be a smooth manifold with or without boundary. Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth vector fields on M, where $\mathcal{X}$ is inward in each point of the ∂M, $p \in M$ and*

$\varphi_t : U \longrightarrow M$ *a local flow generated by* $\mathcal{X}$, $p \in U$. *Then*

$$[\mathcal{X}, \mathcal{Y}](p) = \lim_{t\to 0} \frac{1}{t}[Y - d\varphi_t Y](\varphi_t(p)).$$

In particular

$$[\mathcal{X}, \mathcal{Y}](p) = -\left.\frac{d}{dt} d\varphi_t Y(\varphi_t(p))\right|_{t=0}.$$

Proof. See [12]. ∎

Proposition 1.3.3. *Let* (M^n, g) *be a Riemannian manifold with or without boundary. A smooth vector field* $\mathcal{X}$ *on* M *is a Killing vector field if only if* $\mathcal{L}_{\mathcal{X}} g = 0$.

Proof. From the definition

$$(\mathcal{L}_{\mathcal{X}} g)(\mathcal{Y}, \mathcal{Z}) = \mathcal{X}(g(\mathcal{Y}, \mathcal{Z})) - g([\mathcal{X}, \mathcal{Y}], \mathcal{Z}) - g(\mathcal{Y}, [\mathcal{X}, \mathcal{Z}]).$$

On the other hand,

$$\begin{aligned}\frac{g_{\varphi_t(p)}(d\varphi_t\mathcal{Y}, d\varphi_t\mathcal{Z}) - g_p(\mathcal{Y}, \mathcal{Z})}{t} &= \frac{g_{\varphi_t(p)}(d\varphi_t\mathcal{Y}, d\varphi_t\mathcal{Z}) - g_{\varphi_t(p)}(\mathcal{Y}, d\varphi_t\mathcal{Z})}{t} + \\ &\quad + \frac{g_{\varphi_t(p)}(\mathcal{Y}, d\varphi_t\mathcal{Z}) - g_{\varphi_t(p)}(\mathcal{Y}, \mathcal{Z})}{t} + \\ &\quad + \frac{g_{\varphi_t(p)}(\mathcal{Y}, \mathcal{Z}) - g_p(\mathcal{Y}, \mathcal{Z})}{t}.\end{aligned}$$

Then

$$\begin{aligned}\lim_{t\to 0} \frac{g_{\varphi_t(p)}(d\varphi_t\mathcal{Y}, d\varphi_t\mathcal{Z}) - g_p(\mathcal{Y}, \mathcal{Z})}{t} &= \lim_{t\to 0} g_{\varphi_t(p)}\left(\frac{d\varphi_t\mathcal{Y} - \mathcal{Y}}{t}, d\varphi_t\mathcal{Z}\right) + \\ &\quad + \lim_{t\to 0} g_{\varphi_t(p)}\left(\mathcal{Y}, \frac{d\varphi_t\ddagger - \mathcal{Z}}{t}\right) + \\ &\quad + \lim_{t\to 0} \frac{g_{\varphi_t(p)}(\mathcal{Y}, \mathcal{Z}) - g_p(\mathcal{Y}, \mathcal{Z})}{t}.\end{aligned}$$

Since all the limits above exist, then

$$\left.\frac{d}{dt} g_{\varphi_t(p)}(d\varphi_t\mathcal{Y}, d\varphi_t\mathcal{Z})\right|_{t=0} = -g_p([\mathcal{X}, \mathcal{Y}], \mathcal{Z}) - g_p(\mathcal{Y}, [\mathcal{X}, \mathcal{Z}]) + \left.\frac{d}{dt} g_{\varphi_t(p)}(\mathcal{Y}, \mathcal{Z})\right|_{t=0}.$$

Since

$$\begin{aligned}\left.\frac{d}{dt} g_{\varphi_t(p)}(\mathcal{Y}, \mathcal{Z})\right|_{t=0} &= g_p\left(\left.\frac{D}{dt}\mathcal{Y}\right|_{t=0}, \mathcal{Z}\right) + g_p\left(\mathcal{Y}, \left.\frac{D}{dt}\mathcal{Z}\right|_{t=0}\right) \\ &= g_p(\nabla_{\frac{\partial\varphi}{\partial t}(0,p)}\mathcal{Y}, \mathcal{Z}) + g_p(\mathcal{Y}, \nabla_{\frac{\partial\varphi}{\partial t}(0,p)}\mathcal{Z}) \\ &= g(\nabla_{\mathcal{X}}\mathcal{Y}, \mathcal{Z}) + g(\mathcal{Y}, \nabla_{\mathcal{X}}\mathcal{Z}) \\ &= \mathcal{X}(g(\mathcal{Y}, \mathcal{Z})).\end{aligned}$$

Therefore

$$(\mathcal{L}_{\mathcal{X}} g)(\mathcal{Y}, \mathcal{Z}) = \frac{d}{dt} g_{\varphi_t(p)}(d\varphi_t \mathcal{Y}, d\varphi_t \mathcal{Z}) \Big|_{t=0}.$$

On the other hand

$$\begin{aligned} \frac{d}{dt} g_{\varphi_t(p)}(d\varphi_t \mathcal{Y}, d\varphi_t \mathcal{Z}) \Big|_{t=t_0} &= \frac{d}{dt} g_{\varphi_t(p)}(d\varphi_{t-t_0} d\varphi_{t_0} \mathcal{Y}, d\varphi_{t-t_0} d\varphi_{t_0} \mathcal{Z}) \Big|_{t=t_0} \\ &= \frac{d}{ds} g_{\varphi_t(p)}(d\varphi_s d\varphi_{t_0} \mathcal{Y}, d\varphi_s d\varphi_{t_0} \mathcal{Z}) \Big|_{s=0} \\ &= (\mathcal{L}_{\mathcal{X}} g)(d\varphi_{t_0} \mathcal{Y}, d\varphi_{t_0} \mathcal{Z}). \end{aligned}$$

This shows that $\mathcal{L}_{\mathcal{X}} g = 0$ if only if $t \mapsto g_{\varphi_t(p)}(d\varphi_t \mathcal{Y}, d\varphi_t \mathcal{Z})$ is constant. ■

1.4 Generalized Bochner formula

In this section, we recall the generalized Bochner formula for Riemannian manifold with or without boundary. The proof is in [14].

First, we recall some basic definitions and properties.

Definition 1.4.1. *Let (M^n, g) be a Riemannian manifold with or without boundary. For any $f \in C^\infty(M)$, the* ***gradient of the function*** *f is the map $\nabla f : M \longrightarrow TM$ which satisfies*

$$df(\mathcal{X}) = g(\nabla f, \mathcal{X}). \tag{1.37}$$

Moreover, if $\{\mathcal{E}_1, ..., \mathcal{E}_n\}$ is an orthonormal frame for M locally defined, then we can show that

$$\nabla f = \sum_{i=1}^{n} \mathcal{E}_i(f)\mathcal{E}_i. \tag{1.38}$$

Definition 1.4.2. *Let (M^n, g) be a Riemannian manifold with or without boundary. The* ***divergence of a vector field*** *$\mathcal{X} \in \mathscr{X}(M)$ is defined by*

$$div(\mathcal{X}) = \sum_{i=1}^{n} g(\nabla_{\mathcal{E}_i} \mathcal{X}, \mathcal{E}_i), \tag{1.39}$$

where $\{\mathcal{E}_1, ..., \mathcal{E}_n\}$ is an orthonormal frame locally defined in some open subset of M.

Definition 1.4.3. *Let (M^n, g) be a Riemannian manifold with or without boundary. For any $f \in C^\infty(M)$, we define the* ***Laplacian of*** *f by*

$$\Delta f = -div(\nabla f). \tag{1.40}$$

It follows from (1.40) *that* $\Delta f = -tr\ Hess(f)$.

Definition 1.4.4. *Let* (M^n, g) *be a Riemannian manifold with or without boundary. The* ***divergence of a tensor*** $\mathcal{T} \in \mathscr{T}^k(M)$ ***at*** $p \in M$*, where* k *is an integer bigger than zero, is given by*

$$(div(\mathcal{T}))(\mathcal{X}_1, ..., \mathcal{X}_{k-1}) = \sum_{i=1}^{n} (\nabla_{\mathcal{E}_i} \mathcal{T})(\mathcal{E}_i, \mathcal{X}_1, ..., \mathcal{X}_{k-1}), \tag{1.41}$$

where $\{\mathcal{E}_1, ..., \mathcal{E}_n\}$ *is an orthonormal frame locally defined in a neighborhood of* p*. It follows from* (1.41) *that* $div(\mathcal{T}) \in \mathscr{T}^{k-1}(M)$*. In particular, the divergence of a smooth function is zero.*

Proposition 1.4.5. *Let* (M^n, g) *be a Riemannian manifold with or without boundary. If* $\mathcal{X} \in \mathscr{X}(M)$ *and* $\lambda \in C^\infty(M)$*, then*

$$div(\lambda g)(\mathcal{X}) = \mathcal{X}(\lambda),$$

where $\{\mathcal{E}_1, ..., \mathcal{E}_n\}$ *is an orthonormal frame for* M *locally defined in an open subset of* M*.*

Proposition 1.4.6. *Let* (M^n, g) *be a Riemannian manifold with or without boundary. Then*

$$div\ Ric = \frac{1}{2} dS, \tag{1.42}$$

where S *is the scalar curvature of* M*.*

Proof. Use Definition 1.4.4. and (1.31). ∎

The reader can find the following lemma in [14].

Lemma 1.4.7 (**Generalized Bochner formula.**). *Let* (M^n, g) *be a Riemannian manifold with or without boundary. For any* $\mathcal{X} \in \mathscr{X}(M)$ *we have*

$$div(\mathcal{L}_{\mathcal{X}} g)(\mathcal{X}) = -\frac{1}{2}\Delta(|\mathcal{X}|^2) - |\nabla \mathcal{X}|^2 + Ric(\mathcal{X}, \mathcal{X}) + \mathcal{X}(div(\mathcal{X})). \tag{1.43}$$

In particular, if $\mathcal{X} = \nabla f$*, for some* $f \in C^\infty(M)$*, then*

$$div(\mathcal{L}_{\nabla f} g)(\mathcal{Y}) = 2Ric(\mathcal{Y}, \nabla f) + 2\mathcal{Y}(div(\nabla f)). \tag{1.44}$$

for every $\mathcal{Y} \in \mathscr{X}(M)$*. Therefore,*

$$div(Hess(f))(\mathcal{Y}) = Ric(\mathcal{Y}, \nabla f) - \mathcal{Y}(\Delta f), \tag{1.45}$$

for every $\mathcal{Y} \in \mathscr{X}(M)$*.*

Proof. See [14]. ■

1.5 Geodesic completeness of manifolds with boundary

Now our goal is define a *metric* on a Riemannian manifold with boundary M which induces the same topology on M as the topology on M induced from the smooth structure. The way to do this is similar as we do for Riemannian manifolds without boundary, i.e., we define the *distance* function by the infimum of the length of a set of curves. However, to prove the topology induced from such function is the same as the topology induced from the smooth structure of the manifold is not similar as we do for Riemannian manifolds without boundary. The proof of this topological equivalence is the main result of this section.

Definition 1.5.1. *Let E be a non empty set. Two metrics d_1 and d_2 on E are called **equivalent** if there exist positive constants α and β such that*

$$\beta d_1(u,v) \leq d_2(u,v) \leq \alpha d_1(u,v),$$

for all $u, v \in E$.

Let (M^n, g), $n \geq 2$, be a connected Riemannian manifold with boundary. Let $C([0,1], M)$ be the set of all **piecewise smooth curves** $\gamma : [0,1] \longrightarrow M$, i.e., there exists a finite subdivision $a = t_0 < t_1 < ... < t_k = b$ such that $\gamma|_{[t_{i-1},t_i]}$ for every $i = 1, 2, ..., k$. For any $p, q \in M$ define

$$\mathscr{C}_{p,q}(M) = \{\gamma \in C([0,1], M) : \gamma(0) = p, \text{ and } \gamma(1) = q\}.$$

For all $p, q \in M$, and $\gamma \in \mathscr{C}_{p,q}(M)$, the **length of** γ is

$$L(\gamma) = \int_0^1 |\gamma'(t)| dt. \tag{1.46}$$

Let $d_M : M \times M \longrightarrow \mathbb{R}$ be the function given by

$$d_M(p,q) = \inf_{\gamma \in \mathscr{C}_{p,q}(M)} L(\gamma). \tag{1.47}$$

From the definition of infimum we obtain

$$d_M(p,q) \leq d_M(p,\overline{p}) + d_M(\overline{p},q), \tag{1.48}$$
$$d_M(p,q) = d_M(q,p), \tag{1.49}$$

for all $p, \overline{p}, q \in M$. The next theorem show that $d_M(p,q) = 0$ if and only if $p = q$, i.e., d_M is a metric on M. Furthermore, the topology on M induced from d_M is the same as the topology on M induced from the smooth structure of M.

For brevity, denote by $\tau(g)$ the topology on M induced from the smooth structure.

Theorem 1.5.2. *Let (M^n, g) be a Riemannian manifold with boundary. Then the function d_M defined in (1.47) is a metric on M, and the topology induced from d_M is the same as the topology $\tau(g)$.*

Proof. Let $p \in M$ be arbitrary. Let U_p be a precompact coordinate neighborhood in M of p. Thus we have two cases.

CASE 1: $p \in \text{int}(M)$. Since U_p is a coordinate neighborhood, then $U_p \subset \text{int}(M)$. For any $q \in M$, $q \neq p$, there exists a geodesic ball $B_\epsilon(p) \subset U_p$ centered at p of radius $\epsilon > 0$ such that $q \notin B_\epsilon(p)$. Therefore, every $\gamma \in \mathscr{C}_{p,q}(M)$ has length bigger than $\epsilon > 0$, which implies that

$$d_M(p,q) \geq \epsilon > 0.$$

Thus $d_M|_{\text{int}(M)\times\text{int}(M)}$ is a metric on $\text{int}(M) \times \text{int}(M)$. Set

$$B_M(p,r) = \{q \in M : d_M(p,q) < r\}.$$

So, we have that for all $p \in \text{int}(M)$, there exists $\epsilon > 0$ such that $B_\epsilon(p) = B_M(p,\epsilon)$. Therefore, $d_M|_{\text{int}(M)\times\text{int}(M)}$ induces the same topology as the given $\tau(g)$.

CASE 2: $p \in \partial M$. For $q \in U_p$ arbitrary, we have two situations. First, if $q \in \text{int}(M)$, then from CASE 1 we have that $d_M(p,q) > 0$. Second, $q \in \partial M$, $q \neq p$.

Since $(U_p, \tau(g)|_{U_p})$ is a metrizable topological space, then there exists a metric d_p on U_p which induces the topology $\tau(g)|_{U_p}$. We shall prove there exist positive constants $\alpha_p, \beta_p \in \mathbb{R}$ such that

$$\beta_p d_p(p,q) \leq d_M(p,q) \leq \alpha_p d_p(p,q), \tag{1.50}$$

for all $q \in U_p$.

Let $\gamma \in C([0,1], U_p)$ be a piecewise smooth curve such that $\gamma(1) = p$, and $\gamma([0,1)) \subset U_p \cap \text{int}(M)$. Set

$$R = \sup_{0 \leq t \leq 1} |\gamma'(t)|.$$

For any $0 < t < 1$, define $\gamma_t : [0,1] \longrightarrow U_p$ by

$$\gamma_t(s) = \gamma(t + (1-t)s),$$

which implies $\gamma_t(0) = \gamma(t)$, $\gamma_t(1) = p$, and

$$\gamma_t'(s) = (1-t)\gamma'(t + (1-t)s), \forall s \in [0,1].$$

Thus

$$L(\gamma_t) = (1-t)\int_0^1 |\gamma'(t+(1-t)s)|ds \leq (1-t)R.$$

We obtain $d_M(\gamma(t),p) \leq (1-t)R$, for all $t \in (0,1)$. Then

$$\lim_{t\to 1} d_M(\gamma(t),p) = 0. \tag{1.51}$$

Let q be an interior point of U_p, i.e., $q \in U_p \cap \operatorname{int}(M)$. Let γ be a curve in $C([0,1],U_p)$ such that $\gamma(0) = q$, $\gamma(1) = p$, and $\gamma([0,1)) \subset \operatorname{int}(M)$. Let $(t_k) \subset [0,1)$ be an arbitrary sequence such that $\lim_{k\to\infty} t_k = 1$. We have

$$d_M(\gamma(t_k),q) \leq d_M(\gamma(t_k),\gamma(t_m)) + d_M(\gamma(t_m),q),$$

which implies

$$d_M(\gamma(t_k),q) - d_M(\gamma(t_m),q) \leq d_M(\gamma(t_k),\gamma(t_m)).$$

Analogously, we obtain

$$d_M(\gamma(t_m),q) - d_M(\gamma(t_k),q) \leq d_M(\gamma(t_k),\gamma(t_m)).$$

Therefore

$$|d_M(\gamma(t_k),q) - d_M(\gamma(t_m),q)| \leq d_M(\gamma(t_k),\gamma(t_m)), \tag{1.52}$$

for all $k,m \in \mathbb{N}$. We have that

$$d_M(\gamma(t_k),\gamma(t_m)) \leq d_M(\gamma(t_k),p) + d_M(p,\gamma(t_m)),$$

for all $k,m \in \mathbb{N}$. From (1.51) it follows that

$$\lim_{k,m\to\infty} d_M(\gamma(t_k),\gamma(t_m)) = 0.$$

Thus the sequence $d_M(\gamma(t_k),q)$ is a Cauchy sequence of real numbers, then there exists the following limit

$$\lim_{t\to 1} d_M(\gamma(t),q).$$

We have

$$d_M(p,q) \leq d_M(p,\gamma(t)) + d_M(\gamma(t),q),$$

and

$$d_M(\gamma(t),q) \leq d_M(\gamma(t),p) + d_M(p,q),$$

for all $t \in [0,1]$. From (1.51) we obtain

$$d_M(p,q) = \lim_{t\to 1} d_M(\gamma(t), q).$$

From the CASE 1 the topology induced from d_M on $U_p \cap \mathrm{int}(M)$ is the same as the topology $\tau(g)|_{U_p \cap \mathrm{int}(M)}$. Then, d_M and d_p are equivalent on $U_p \cap \mathrm{int}(M)$, i.e., there exist positive constants $\alpha_p, \beta_p \in \mathbb{R}$ such that

$$\beta_p d_p(\overline{p}, \overline{q}) \leq d_M(\overline{p}, \overline{q}) \leq \alpha_p d_p(\overline{p}, \overline{q}),$$

for all $\overline{p}, \overline{q} \in U_p \cap \mathrm{int}(M)$. Since $\gamma(t), q \in U_p \cap \mathrm{int}(M)$, for all $t \in [0,1)$, then

$$\beta_p d_p(\gamma(t), q) \leq d_M(\gamma(t), q) \leq \alpha_p d_p(\gamma(t), q),$$

for all $t \in [0,1)$. So, by taking $t \longrightarrow 1$ we obtain

$$\beta_p d_p(p,q) \leq d_M(p,q) \leq \alpha_p d_p(p,q),$$

for all $q \in U_p \cap \mathrm{int}(M)$.

Now, let q be a boundary point of U_p, i.e., $q \in U_p \cap \partial M$, $q \neq p$. Let $\gamma, \overline{\gamma}$ be curves in $C([0,1], U_p)$ such that $\gamma(0) = q$, $\gamma(1) = p$, $\gamma((0,1)) \subset \mathrm{int}(M)$, and $\overline{\gamma}(s) = \gamma(1-s)$. Since

$$d_M(p,q) \leq d_M(p, \gamma(t)) + d_M(\gamma(t), q),$$

for all $t \in [0,1]$. Then by taking $t \to 1$ we obtain

$$d_M(p,q) \leq \liminf_{t\to 1} d_M(\gamma(t), q).$$

On the other hand,

$$d_M(\gamma(t), q) \leq d_M(\gamma(t), p) + d_M(p,q),$$

for all $t \in [0,1]$. Which implies

$$\limsup_{t\to 1} d_M(\gamma(t), q) \leq d_M(p,q).$$

Therefore

$$\lim_{t\to 1} d_M(\gamma(t), q) = d_M(p,q).$$

For all $s, t \in (0,1)$, we have $\overline{\gamma}(s), \gamma(t) \in U_p \cap \mathrm{int}(M)$. Then

$$\beta_p d_p(\overline{\gamma}(s), \gamma(t)) \leq d_M(\overline{\gamma}(s), \gamma(t)) \leq \alpha_p d_p(\overline{\gamma}(s), \gamma(t)), \forall s, t \in (0,1).$$

By taking $s \longrightarrow 1$ we obtain

$$\beta_p d_p(q, \gamma(t)) \leq d_M(q, \gamma(t)) \leq \alpha_p d_p(q, \gamma(t)), \forall t \in (0, 1).$$

By taking $t \longrightarrow 1$ we obtain

$$\beta_p d_p(p, q) \leq d_M(p, q) \leq \alpha_p d_p(p, q),$$

for all $q \in U_p \cap \partial M$. Therefore,

$$\beta_p d_p(p, q) \leq d_M(p, q) \leq \alpha_p d_p(p, q),$$

for all $q \in U_p$. Then the function d_M is a metric on U_p such that the topology induced from d_M is the same the $\tau(g)|_{U_p}$. Since p is arbitrary, then d_M define a metric on M, and its induced topology is the same as the one we use by $\tau(g)$. ■

From now on we say that a Riemannian manifold with boundary M^n, $n \geq 2$, is **metrically complete** if (M, d_M) is complete as a metric space, where d_M is defined by (1.47).

In what follows we prove the equivalence of metrically complete Riemannian manifold with boundary that we just defined and the geodesically complete Riemannian manifold with boundary defined by Pigola in [16].

Let M be a smooth manifold with or without boundary. Denote by $C^0([0, 1], M)$ the set of all continuous paths from $[0, 1]$ to M.

Definition 1.5.3. *Let* (M, d_M) *a Riemannian manifold with or without boundary, where* d_M *is defined in* (1.47)*. A path* $\gamma \in C^0([0, 1], M)$ *is called* ***rectifiable*** *if*

$$R(\gamma, d_M) = \sup_{\mathcal{P}} \sum_{i=1}^{k} d_M(\gamma(t_{i-1}), \gamma(t_i)) < \infty, \tag{1.53}$$

where $\mathcal{P}$ *is the set of all partitions* $P = \{0 = t_0 < t_1 < ... < t_k = 1\}$ *of the interval* $[0, 1]$*. The number* $R(\gamma)$ *is called the* ***metric-length of*** γ*.*

Definition 1.5.4. *Let* $\mathcal{R}_{[0,1]}(M, d_M)$ *be the set of all rectifiable paths in* $C^0([0, 1], M)$*, where the Riemannian manifold (with or without boundary)* M *is provided with the metric* d_M *defined in* (1.47)*. For any* $p, q \in M$ *set*

$$\mathscr{R}_{p,q}(M) = \{\gamma \in \mathcal{R}_{[0,1]}(M, d_M) : \gamma(0) = p, \text{ and } \gamma(1) = q\}. \tag{1.54}$$

So define on (M, d_M) the ***length-distance*** *by*

$$d_R(p,q) = \inf_{\mathscr{R}_{p,q}(M)} R(\gamma, d_M), \tag{1.55}$$

for all $p, q \in M$.

The next proposition show that $d_M = d_R$.

Proposition 1.5.5. *Let (M, d_M) be a connected Riemannian manifold with or without boundary, where d_M is defiend in (1.47). If d_R is the length-distance on M, then $d_M = d_R$.*

Proof. For any path $\gamma \in \mathscr{R}_{p,q}(M)$, from the triangle inequality we have

$$d_M(p,q) \leq \sum_{i=1}^{k} d_M(\gamma(t_{i-1}), \gamma(t_i)),$$

for all partition $P = \{0 = t_0 < t_1 < ... < t_k = 1\} \in \mathcal{P}$. Then

$$d_M(p,q) \leq R(\gamma, d_M),$$

for all $\gamma \in \mathscr{R}_{p,q}(M)$. We obtain

$$d_M(p,q) \leq d_R(p,q),$$

for all $p, q \in M$.

For all $\gamma \in \mathscr{C}_{p,q}(M)$ we have

$$\sum_{i=1}^{k} d_M(\gamma(t_{i-1}), \gamma(t_i)) \leq L(\gamma),$$

for all partition $P\{0 = t_0 < t_1 < ... < t_k = 1\}$ of the interval $[0,1]$, where $L(\gamma)$ is defined in (1.46). Then

$$R(\gamma, d_M) \leq L(\gamma),$$

for all $\gamma \in \mathscr{C}_{p,q}(M)$. Since $\mathscr{C}_{p,q}(M) \subset \mathscr{R}_{p,q}(M)$, then

$$d_R(p,q) \leq d_M(p,q).$$

Therefore, $d_M = d_R$. ■

Next we have the definition of a geodesically complete Riemannian manifold with boundary. Such definition is given by Pigola in [16].

Definition 1.5.6. *Let (M, g) be a Riemannian manifold with boundary. We say that*

*M is **geodesically complete** if every geodesic $\gamma : [0, a) \longrightarrow M$ can be extended to a continuous path $\overline{\gamma} : [0, a] \longrightarrow M$.*

The following definition is for Riemannian manifold with or without boundary, but in [16] the reader can find a more general version.

Definition 1.5.7. *Let (M, g) be a Riemannian manifold with or without boundary. A piecewise smooth curve $\gamma \in C([0, 1), M)$ is called a **divergent path** if for all compact subset $K \subset M$ there exists $0 < t_K < 1$ such that $\gamma(t) \notin K$ for any $t_K \leq t < 1$.*

Next, we give a "Hopf-Rinow" version for Riemannian manifolds with boundary.

Theorem 1.5.8. *Let (M, d_M) be a Riemannian manifold with boundary, where d_M is defined by (1.47). Then the following assertions are equivalent.*

(i) *(M, d_M) is metrically complete.*

(ii) *Every bounded and closed subset of M is compact.*

(iii) *(M, d_M) is geodesically complete.*

(iv) *Every Lipschitz path $\gamma : [0, a) \longrightarrow M$ can be extended to a continuous path $\overline{\gamma} : [0, a] \longrightarrow M$.*

(v) *Every divergent path $\gamma : [0, 1) \longrightarrow M$ has infinite length, i.e., $R(\gamma, d_M) = \infty$.*

Proof. In [16], the Theorem 3.7.6 is proved for locally compact length spaces, i.e., a space metric (X, d) which is locally compact and $d = d_R$, where d_R is defined as in (1.53). So, (M, d_M) is a locally compact and we have that $d_M = d_R$ from Proposition 3.7.3. For the prove of *(i)* $\Leftrightarrow$ *(ii)* $\Leftrightarrow$ *(iii)* see [3], and for *(iii)* $\Leftrightarrow$ *(iv)* $\Leftrightarrow$ *(v)* see [16]. ■

The following results follow immediately from the Theorem 3.7.6.

Corollary 1.5.9. *Let (M, g) be a compact Riemannian manifold with or without boundary. Then M is geodesically complete.*

Corollary 1.5.10. *Let (M, g) be a Riemannian manifold with boundary. If M is metrically complete, then ∂M is geodesically complete.*

Chapter 2

Gradient Ricci Solitons with Boundary

In this chapter we shall take the perspective in [7]. We extend some results first obtained in [7] to manifolds with boundary. For every $(M, g, \nabla f, \lambda)$ such that the potential function f satisfies an inequality and λ is a non positive constant, we obtain that f is constant. At the last section we define warped product, where the basis is a Riemannian manifold with boundary, and then we prove some identities to the Christoffel symbol, the Hessian of a smooth function defined on warped product, and some applications to the Ricci curvature. We finish by proving that a warped product Einstein which the warping function satisfies an inequality has to be a Riemannian product.

2.1 Ricci Solitons with Boundary

A **Ricci soliton with or without boundary** $(M, g, \mathcal{X}, \lambda)$ is a Riemannian manifold (M, g) with or without boundary with a vector field $\mathcal{X} \in \mathscr{X}(M)$, and a constant λ satisfying the following equation

$$\text{Ric} = \lambda g + \frac{1}{2}\mathcal{L}_{\mathcal{X}} g. \tag{2.1}$$

If the vector field $\mathcal{X}$ vanishes, then the Ricci soliton is just an Einstein manifold (with or without boundary). If the vector field $\mathcal{X}$ is the gradient of some smooth function $f : M \longrightarrow \mathbb{R}$, then the soliton is called **gradient Ricci soliton**. In this case, it is denoted by $(M, g, \nabla f, \lambda)$. By replacing the expression in Proposition 1.2.14 in 2.1 we obtain

$$\text{Ric} = \lambda g + \text{Hess}(f). \tag{2.2}$$

The following theorem is an extension of the Theorem 1 in [7].

Theorem 2.1.1. *Let $(M^k, g, \nabla f, \lambda)$ be a connected compact gradient Ricci soliton with boundary. Suppose that λ is a non-positive constant, the maximum point of f is an interior point, and f satisfies*

$$\int_{\partial M} \frac{\partial f}{\partial \mathcal{N}} d(\partial M) \geq 0, \tag{2.3}$$

where $\mathcal{N}$ is the outward unit normal vector field along ∂M. Then f is constant.

Proof. By taking the trace in (2.2) we obtain

$$S = k\lambda - \Delta f. \tag{2.4}$$

From (2.4) we obtain

$$dS = -d(\Delta f).$$

By taking the divergence in (2.2) we obtain the following identity

$$\text{div}(\text{Ric}) = \text{div}(\text{Hess})(f).$$

From (1.45) and (1.42) we have, respectively,

$$\begin{aligned} \text{div}(\text{Hess}(f))(\mathcal{X}) &= \text{Ric}(\nabla f, \mathcal{X}) - d(\Delta f)(\mathcal{X}),, \\ \mathcal{X}(S) &= 2\text{div}(\text{Ric})(\mathcal{X}), \end{aligned}$$

for all $\mathcal{X} \in \mathscr{X}(X)$. Then

$$-\mathcal{X}(\Delta f) = 2\text{Ric}(\nabla f, \mathcal{X}) - 2\mathcal{X}(\Delta f),$$

which implies

$$2\lambda\mathcal{X}(f) + 2\text{Hess}(f)(\nabla f, \mathcal{X}) - \mathcal{X}(\Delta f) = 0. \tag{2.5}$$

From the definition of Hessian, we have

$$\text{Hess}(f)(\nabla f, \mathcal{X}) = \frac{1}{2}\mathcal{X}(|\nabla f|^2). \tag{2.6}$$

So, by replacing (2.6) in to (2.5) we obtain

$$2\lambda\mathcal{X}(f) + \mathcal{X}(|\nabla f|^2) - \mathcal{X}(\Delta f) = 0,$$

for all $\mathcal{X} \in \mathscr{X}(M)$, which implies

$$\mathcal{X}(2\lambda f + |\nabla f|^2 - \Delta f) = 0,$$

for all $\mathcal{X} \in \mathscr{X}(M)$. Therefore, there exists a constant μ such that

$$\mu = 2\lambda f + |\nabla f|^2 - \Delta f. \tag{2.7}$$

By integrating (2.7) we obtain

$$\mu = \frac{2\lambda}{\text{vol}(M)} \int_M f + \frac{1}{\text{vol}(M)} \int_M |\nabla f|^2 - \frac{1}{\text{vol}(M)} \int_M \Delta f. \tag{2.8}$$

We have

$$\frac{\partial f}{\partial \mathcal{N}} = g(\nabla f, \mathcal{N}) \quad \text{and} \quad \Delta f = -\text{div}(\nabla f).$$

By applying the Divergence theorem on third integral in (2.8) we obtain

$$\mu = \frac{2\lambda}{\text{vol}(M)} \int_M f + \frac{1}{\text{vol}(M)} \int_M |\nabla f|^2 + \frac{1}{\text{vol}(M)} \int_{\partial M} \frac{\partial f}{\partial \mathcal{N}}.$$

Let $p_{max} \in M$ be the maximum point of f. Then, $\nabla f(p_{max}) = 0$, and since $\Delta f = -\text{tr Hess}(f)$, then $\Delta f(p_{max}) \geq 0$. So, by using (2.7), (2.8), the hypothesis that $\lambda \leq 0$ and (2.3), respectively, we obtain the following inequalities

$$\begin{aligned}
0 &\leq \Delta f(p_{max}) \\
&= 2\lambda f(p_{max}) - \mu \\
&= \frac{2\lambda}{\text{vol}(M)} \int_M f(p_{max}) - \frac{2\lambda}{\text{vol}(M)} \int_M f - \frac{1}{\text{vol}(M)} \int_M |\nabla f|^2 - \frac{1}{\text{vol}(M)} \int_{\partial M} \frac{\partial f}{\partial \mathcal{N}} \\
&= \frac{2\lambda}{\text{vol}(M)} \int_M (f(p_{max}) - f) - \frac{1}{\text{vol}(M)} \int_M |\nabla f|^2 - \frac{1}{\text{vol}(M)} \int_{\partial M} \frac{\partial f}{\partial \mathcal{N}} \\
&\leq 0.
\end{aligned}$$

Then $|\nabla f| = 0$ on M. Therefore, f is constant. ■

2.2 Einstein Warped Products

Let (B^n, g_B) and (F^m, g_F) be Riemannian manifolds, where B is a smooth manifold with boundary, and F is a smooth manifold without boundary. Let $h : B \longrightarrow \mathbb{R}$ be a positive smooth function. The product $M = B \times F$ provided with the metric

$$g = \pi^* g_B + h^2 \sigma^* g_F,$$

where π and σ denote the projections of $B \times F$ onto B and F, respectively, and the $*$ denotes the pullback, is called the **warped product of** B **and** F, where B is called **base of the warped product**, F is called the **fiber of the warped product**, and the function h is called **warping function**. We denote the product $B \times F$ with the metric g by $M = B \times_h F$. For simplicity we shall write the metric g on $M = B \times_h F$ by

$$g = g_B + h^2 g_F.$$

Let (x_i) and (x_α) be coordinate systems on B and F, respectively. We set

$$X_i = \frac{\partial}{\partial x_i}, \quad \text{and} \quad X_\alpha = \frac{\partial}{\partial x_\alpha},$$

for all i, and α. Then $g_{ij} = g(X_i, X_j) = g_B(X_i, X_j) = (g_B)_{ij}$, $g_{i\alpha} = g(X_i, X_\alpha) = 0$, and $g_{\alpha\beta} = g(X_\alpha, X_\beta) = h^2 g_F(X_\alpha, X_\beta) = h^2(g_F)_{\alpha\beta}$. Here we are identifying $X_i = (X_i, 0)$ and $X_\alpha = (0, X_\alpha)$. The vector fields in TB are called **horizontal vector fields** and the vector fields in TF are called **vertical vector fields**.

Let $M = B \times_h F$ be an warped product on Riemannian manifolds, where B is a smooth manifold with boundary, and F is a smooth manifold without boundary. We denote the Levi-Civita connection and Christoffel symbol on B and F by ∇^B, Γ^B, ∇^F, and Γ^F, respectively.

Lemma 2.2.1. *Let (M, g) be a Riemannian manifold with or without boundary. Let (x_i) be any local coordinate on M. Then*

$$\sum_i \mathcal{X}_k(g^{ij})g_{mi} = -\Gamma^j_{km} - \sum_{i,j} \Gamma^l_{ki} g^{ij} g_{lm}, \tag{2.9}$$

where $\mathcal{X}_k = \frac{\partial}{\partial x_k}$, Γ is the Christoffel symbol of M, and g^{ij} is the ij-th entrance of the inverse matrix of $[g_{ij}]$.

Proof. Since

$$\sum_i g^{ij} g_{mi} = \delta_{mj},$$

then

$$\sum_i [\mathcal{X}_k(g^{ij})g_{mi} + g^{ij}\mathcal{X}_k(g_{mi})] = 0,$$

which implies

$$\sum_i \mathcal{X}_k(g^{ij})g_{mi} = -\sum_i g^{ij}\mathcal{X}_k(g_{mi}).$$

By the properties of the ∇ we have $\mathcal{X}_k(g_{mi}) = g(\nabla_{\mathcal{X}_k}\mathcal{X}_m, \mathcal{X}_i) + g(\nabla_{\mathcal{X}_k}\mathcal{X}_i, \mathcal{X}_m)$, then

$$\mathcal{X}_k(g_{mi}) = \sum_l \Gamma^l_{km} g_{li} + \sum_l \Gamma^l_{ki} g_{lm}.$$

Therefore

$$\begin{aligned}\sum_i \mathcal{X}_k(g^{ij})g_{mi} &= -\sum_{i,l}\Gamma^l_{km}g^{ij}g_{li} - \sum_{i,l}\Gamma^l_{ki}g^{ij}g_{lm} \\ &= -\sum_l \Gamma^l_{km}\delta_{lj} - \sum_{i,l}\Gamma^l_{ki}g^{ij}g_{lm} \\ &= -\Gamma^j_{km} - \sum_{i,l}\Gamma^l_{ki}g^{ij}g_{lm}.\end{aligned}$$

■

We shall prove below some identities for Levi-Civita connection and the Christoffel symbol.

Lemma 2.2.2. *Let (B^n, g_B) and (F^m, g_F) be Rimannian manifolds, where B is a manifold with boundary and F is without boundary. Let $(M = B \times_h F, g)$ be the warped product. Let (x_i) and (x_α) be local coordinates for B and F, respectively. If ∇ and Γ are the Levi-Civita connection and Christoffel symbol of M, then*

$$\Gamma^s_{ij} = (\Gamma^B)^s_{ij}, \Gamma^s_{i\alpha} = 0, \Gamma^\beta_{i\alpha} = \frac{1}{h}\frac{\partial h}{\partial x_i}\delta_{\alpha\beta}, \Gamma^\alpha_{ij} = 0, \Gamma^\lambda_{\alpha\beta} = (\Gamma^F)^\lambda_{\alpha\beta}, \tag{2.10}$$

$$\Gamma^s_{\alpha\beta} = -\sum_i h\frac{\partial h}{\partial x_i}(g_F)_{\alpha\beta}g^{is}_B, \tag{2.11}$$

$$\nabla_{X_i}X_j = \nabla^B_{X_i}X_j, \nabla_{X_\alpha}X_i = \nabla_{X_i}X_\alpha = \frac{1}{h}\frac{\partial h}{\partial x_i}X_\alpha, \tag{2.12}$$

$$\nabla_{X_\alpha}X_\beta = -h(g_F)_{\alpha\beta}\nabla_B h + \sum_\lambda \Gamma^\lambda_{\alpha\beta}X_\lambda, \tag{2.13}$$

for all $i, j, s, \alpha, \beta, \lambda$. The term $(g_B)^{is}$ denote the is-th entrance of the inverse matrix of $[(g_B)_{ij}]$. Here, $\nabla_B h$ is the gradient of h, which is given by

$$\nabla_B h = \sum_{i,j}(g_B)^{ij}\frac{\partial h}{\partial x_j}X_i. \tag{2.14}$$

Proof. For any i, j and s, we have

$$\begin{aligned}\Gamma^s_{ij} &= \frac{1}{2}\sum_k\left\{\frac{\partial}{\partial x_i}g_{jk} + \frac{\partial}{\partial x_j}g_{ki} - \frac{\partial}{\partial x_k}g_{ij}\right\}g^{ks} \\ &\quad + \frac{1}{2}\sum_\alpha\left\{\frac{\partial}{\partial x_i}g_{j\alpha} + \frac{\partial}{\partial x_j}g_{\alpha i} - \frac{\partial}{\partial x_\alpha}g_{ij}\right\}g^{\alpha s}.\end{aligned}$$

Since $g^{\alpha s} = 0$, for all α, then $\Gamma^s_{ij} = (\Gamma^B)^s_{ij}$.

For every i, s and β, we have

$$\begin{aligned}\Gamma^s_{i\beta} &= \frac{1}{2}\sum_k \left\{\frac{\partial}{\partial x_i} g_{\beta k} + \frac{\partial}{\partial x_\beta} g_{ki} - \frac{\partial}{\partial x_k} g_{i\beta}\right\} g^{ks} \\ &+ \frac{1}{2}\sum_\alpha \left\{\frac{\partial}{\partial x_i} g_{\beta\alpha} + \frac{\partial}{\partial x_\beta} g_{\alpha i} - \frac{\partial}{\partial x_\alpha} g_{i\beta}\right\} g^{\alpha s}.\end{aligned}$$

Since $g_{\beta k} = g_{i\beta} = g^{\alpha s} = 0$ and $\frac{\partial}{\partial x_\beta} g_{ki} = 0$, then $\Gamma^s_{i\beta} = 0$.

For every i, λ and β, we have

$$\begin{aligned}\Gamma^\lambda_{i\beta} &= \frac{1}{2}\sum_k \left\{\frac{\partial}{\partial x_i} g_{\beta k} + \frac{\partial}{\partial x_\beta} g_{ki} - \frac{\partial}{\partial x_k} g_{i\beta}\right\} g^{k\lambda} \\ &+ \frac{1}{2}\sum_\alpha \left\{\frac{\partial}{\partial x_i} g_{\beta\alpha} + \frac{\partial}{\partial x_\beta} g_{\alpha i} - \frac{\partial}{\partial x_\alpha} g_{i\beta}\right\} g^{\alpha\lambda}.\end{aligned}$$

Since $g^{k\lambda} = g_{\alpha i} = g_{i\beta} = 0$, then

$$\Gamma^\lambda_{i\beta} = \frac{1}{2}\sum_\alpha \left\{\frac{\partial}{\partial x_i} g_{\beta\alpha}\right\} g^{\alpha\lambda} = \sum_\alpha h \frac{\partial h}{\partial x_i} (g_F)_{\beta\alpha} g^{\alpha\lambda}.$$

It follows that

$$\Gamma^\lambda_{i\beta} = \frac{1}{h}\frac{\partial h}{\partial x_i}\delta_{\beta\lambda}.$$

For every i, j, and β, we have

$$\begin{aligned}\Gamma^\beta_{ij} &= \frac{1}{2}\sum_k \left\{\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij}\right\} g^{k\beta} \\ &+ \frac{1}{2}\sum_\alpha \left\{\frac{\partial}{\partial x_i} g_{j\alpha} + \frac{\partial}{\partial x_j} g_{\alpha i} - \frac{\partial}{\partial x_\alpha} g_{ij}\right\} g^{\alpha\beta}.\end{aligned}$$

Since $g^{k\beta} = g_{j\alpha} = g_{\alpha i} = 0$ and $\frac{\partial}{\partial x_\alpha} g_{ij} = 0$, then $\Gamma^\beta_{ij} = 0$.

For every α, β, and δ, we have

$$\begin{aligned}\Gamma^\delta_{\alpha\beta} &= \frac{1}{2}\sum_k \left\{\frac{\partial}{\partial x_\alpha} g_{\beta k} + \frac{\partial}{\partial x_\beta} g_{k\alpha} - \frac{\partial}{\partial x_k} g_{\alpha\beta}\right\} g^{k\delta} \\ &+ \frac{1}{2}\sum_\gamma \left\{\frac{\partial}{\partial x_\alpha} g_{\beta\gamma} + \frac{\partial}{\partial x_\beta} g_{\gamma\alpha} - \frac{\partial}{\partial x_\gamma} g_{\alpha\beta}\right\} g^{\gamma\delta}.\end{aligned}$$

Since $g^{k\delta} = 0$, then $\Gamma^\delta_{\alpha\beta} = (\Gamma^F)^\delta_{\alpha\beta}$.

For every k, α, and β, we have

$$\begin{aligned}\Gamma^k_{\alpha\beta} &= \frac{1}{2}\sum_i \left\{\frac{\partial}{\partial x_\alpha} g_{\beta i} + \frac{\partial}{\partial x_\beta} g_{i\alpha} - \frac{\partial}{\partial x_i} g_{\alpha\beta}\right\} g^{ik} \\ &= +\frac{1}{2}\sum_\gamma \left\{\frac{\partial}{\partial x_\alpha} g_{\beta\gamma} + \frac{\partial}{\partial x_\beta} g_{\gamma\alpha} - \frac{\partial}{\partial x_\gamma} g_{\alpha\beta}\right\} g^{\gamma k}.\end{aligned}$$

Since $g^{\gamma k} = 0$ e $g_{\beta i} = g_{i\alpha} = 0$, then

$$\Gamma^k_{\alpha\beta} = \frac{1}{2}\sum_i \left\{-\frac{\partial}{\partial x_i} g_{\alpha\beta}\right\} g^{ik}.$$

It follows that

$$\Gamma^k_{\alpha\beta} = \frac{1}{2}\sum_i \left\{-2h\frac{\partial h}{\partial x_i}(g_F)_{\alpha\beta}\right\} g^{ik}.$$

Therefore, one sees that

$$\Gamma^k_{\alpha\beta} = -\sum_i h\frac{\partial h}{\partial x_i}(g_F)_{\alpha\beta} g^{ik},$$

and then

$$\nabla_{X_j} X_i = \sum_k \Gamma^k_{ji} X_k + \sum_\alpha \Gamma^\alpha_{ji} X_\alpha.$$

Since $\Gamma^\alpha_{ji} = 0$, then $\nabla_{X_j} X_i = \sum_k \Gamma^k_{ji} X_k$. Consequently,

$$\nabla_{X_j} X_i = \nabla^B_{X_j} X_i.$$

Then, one obtains that

$$\nabla_{X_\alpha} X_i = \sum_k \Gamma^k_{\alpha i} X_k + \sum_\beta \Gamma^\beta_{\alpha i} X_\beta.$$

Since $\Gamma^k_{\alpha i} = 0$ and $\Gamma^\beta_{\alpha i} = \dfrac{1}{h}\dfrac{\partial h}{\partial x_i}\delta_{\alpha\beta}$, then

$$\nabla_{X_\alpha} X_i = \sum_\beta \frac{1}{h}\frac{\partial h}{\partial x_i}\delta_{\alpha\beta} X_\beta = \frac{1}{h}\frac{\partial h}{\partial x_i} X_\alpha.$$

A similar computation give us $\nabla_{X_\alpha} X_i = \nabla_{X_i} X_\alpha$.

Since

$$\nabla_{X_\alpha} X_\beta = \sum_k \Gamma^k_{\alpha\beta} X_k + \sum_\gamma \Gamma^\gamma_{\alpha\beta} X_\gamma,$$

and $\Gamma^k_{\alpha\beta} = -\sum_i h\dfrac{\partial h}{\partial x_i}(g_F)_{\alpha\beta} g^{ik}$, then

$$\nabla_{X_\alpha} X_\beta = -h(g_F)_{\alpha\beta}\nabla_B h + \sum_\gamma \Gamma^\gamma_{\alpha\beta} X_\gamma.$$

∎

Lemma 2.2.3. *Let (B^n, g_B) and (F^m, g_F) be Rimannian manifolds, where B is a manifold with boundary and F is without boundary. Let $(M = B \times_h F, g)$ be the warped product.*

Let $f : M \longrightarrow \mathbb{R}$ be a smooth function. Let (x_i) be a local coordinate system for B and $X_i = \frac{\partial}{\partial x_i}$ for each i. Consider

$$\nabla_B f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_j} X_i, \tag{2.15}$$

where g^{ij} denote the ij-th entrance of the inverse matrix of $[g_{ij}]$. Then

$$Hess(f)(X,Y) = g_B(\nabla^B_X \nabla_B f, Y) \tag{2.16}$$

Proof. First of all, we can write the gradient of f in the following way

$$\nabla f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_j} X_i + \sum_{\alpha,\beta} \frac{1}{h^2} g_F^{\alpha\beta} \frac{\partial f}{\partial x_\beta} X_\alpha.$$

So, one has

$$\begin{aligned} \nabla_{X_k} \nabla f &= \sum_{i,j} \left[X_k \left(g^{ij} \frac{\partial f}{\partial x_j} \right) X_i + g^{ij} \frac{\partial f}{\partial x_j} \nabla_{X_k} X_i \right] + \\ &\quad + \sum_{\alpha,\beta} \left[X_k \left(\frac{1}{h^2} g_F^{\alpha\beta} \frac{\partial f}{\partial x_\beta} \right) X_\alpha + \frac{1}{h^2} g_F^{\alpha\beta} \frac{\partial f}{\partial x_\beta} \nabla_{X_k} X_\alpha \right]. \end{aligned}$$

Since $\nabla_{X_k} X_\alpha = \dfrac{1}{h} \dfrac{\partial h}{\partial x_i} X_\alpha$ and $g(X_\alpha, X_k) = 0$, then

$$g(\nabla_{X_k} \nabla f, X_l) = g\left(\sum_{i,j} \left[X_k \left(g^{ij} \frac{\partial f}{\partial x_j} \right) X_i + g^{ij} \frac{\partial f}{\partial x_j} \nabla_{X_k} X_i \right], X_l \right).$$

Therefore, $g(\nabla_{X_k} \nabla f, X_l) = g_B(\nabla^B_{X_k} \nabla_B f, X_l)$. ■

We shall use the identities that we just proved in Lemma 2.2.1 and Lemma 2.2.2 to show the following proposition.

Proposition 2.2.4. *Let (B^n, g_B) and (F^m, g_F) be Rimannian manifolds, where B is a manifold with boundary and F is without boundary. Let $(M = B \times_h F, g)$ be the warped product. Let $f : M \longrightarrow \mathbb{R}$ be a smooth function. The following identities for the Hessian* *of f are satisfies*

$$Hess(f)(X_i, X_j) = -\sum_{k=1}^{n} \Gamma_{ij}^{k} \frac{\partial f}{\partial x_k} + \frac{\partial^2 f}{\partial x_i \partial x_j}, \tag{2.17}$$

$$Hess(f)(X_i, X_\alpha) = -\frac{1}{h} \frac{\partial h}{\partial x_i} \frac{\partial f}{\partial x_\alpha} + \frac{\partial^2 f}{\partial x_i \partial x_\alpha}, \tag{2.18}$$

$$Hess(f)(X_\alpha, X_\beta) = h(g_F)_{\alpha\beta} g_B(\nabla_B f, \nabla_B h) - \sum_{\gamma=1}^{m} \Gamma_{\alpha\beta}^{\gamma} \frac{\partial f}{\partial x_\gamma} + \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta}, \tag{2.19}$$

for any horizontal coordinate vector fields X_i, X_j and any vertical coordinate vector fields X_α, X_β. Here, Γ denotes the Christoffel symbol of M.

Proof. From the definition we have that

$$\nabla f = g_B^{ij} \frac{\partial f}{\partial x_j} X_i + \frac{1}{h^2} g_F^{\alpha\beta} \frac{\partial f}{\partial x_\beta} X_\alpha. \tag{2.20}$$

By applying any horizontal vector field,

$$\begin{aligned} \nabla_{X_k} \nabla f &= \nabla_{X_k} \left(g_B^{ij} \frac{\partial f}{\partial x_j} X_i + \frac{1}{h^2} g_F^{\alpha\beta} \frac{\partial f}{\partial x_\beta} X_\alpha \right) \\ &= \nabla_{X_k} \left(g_B^{ij} \frac{\partial f}{\partial x_j} X_i \right) + \nabla_{X_k} \left(\frac{1}{h^2} g_F^{\alpha\beta} \frac{\partial f}{\partial x_\beta} X_\alpha \right) \\ &= g_B^{ij} \frac{\partial f}{\partial x_j} \nabla_{X_k} X_i + X_k(g_B^{ij}) \frac{\partial f}{\partial x_j} X_i + g_B^{ij} \frac{\partial^2 f}{\partial x_k \partial x_j} X_i + \frac{1}{h^2} g_F^{\alpha\beta} \frac{\partial f}{\partial x_\beta} \nabla_{X_k} X_\alpha \\ &\quad - \frac{2}{h^3} \frac{\partial h}{\partial x_k} g_F^{\alpha\beta} \frac{\partial f}{\partial x_\beta} X_\alpha + \frac{1}{h^2} g_F^{\alpha\beta} \frac{\partial^2 f}{\partial x_k \partial x_\beta} X_\alpha. \end{aligned} \tag{2.21}$$

By replacing the third identity in (2.12) in (2.21) we obtain

$$\begin{aligned} \nabla_{X_k} \nabla f &= g_B^{ij} \frac{\partial f}{\partial x_j} \nabla_{X_k} X_i + X_k(g_B^{ij}) \frac{\partial f}{\partial x_j} X_i + g_B^{ij} \frac{\partial^2 f}{\partial x_k \partial x_j} X_i + \frac{1}{h^3} g_F^{\alpha\beta} \frac{\partial f}{\partial x_\beta} \frac{\partial h}{\partial x_k} X_\alpha \\ &\quad - \frac{2}{h^3} \frac{\partial h}{\partial x_k} g_F^{\alpha\beta} \frac{\partial f}{\partial x_\beta} X_\alpha + \frac{1}{h^2} g_F^{\alpha\beta} \frac{\partial^2 f}{\partial x_k \partial x_\beta} X_\alpha, \end{aligned}$$

which implies

$$\begin{aligned} \nabla_{X_k} \nabla f &= g_B^{ij} \frac{\partial f}{\partial x_j} \nabla_{X_k} X_i + X_k(g_B^{ij}) \frac{\partial f}{\partial x_j} X_i + g_B^{ij} \frac{\partial^2 f}{\partial x_k \partial x_j} X_i - \frac{1}{h^3} g_F^{\alpha\beta} \frac{\partial f}{\partial x_\beta} \frac{\partial h}{\partial x_k} X_\alpha \\ &\quad + \frac{1}{h^2} g_F^{\alpha\beta} \frac{\partial^2 f}{\partial x_k \partial x_\beta} X_\alpha. \end{aligned} \tag{2.22}$$

From (2.12) we have $g(\nabla_{X_k} X_\alpha, X_o) = 0$. Since $g(X_i, X_\alpha) = 0$ for every $i = 1, ..., n$ and

$\alpha = 1, ..., m$, by using (2.9) and the first identity in (2.12), we get

$$
\begin{aligned}
g(\nabla_{X_k}\nabla f, X_o) &= g_B^{ij}\frac{\partial f}{\partial x_j}g_B(\nabla^B_{X_k}X_i, X_o) + X_k(g_B^{ij})(g_B)_{io}\frac{\partial f}{\partial x_j} + g_B^{ij}(g_B)_{io}\frac{\partial^2 f}{\partial x_x \partial x_j} \\
&= \Gamma^l_{ki}g_B^{ij}(g_B)_{lo}\frac{\partial f}{\partial x_j} - \Gamma^j_{ko}\frac{\partial f}{\partial x_j} - \Gamma^l_{ki}g_B^{ij}(g_B)_{lo}\frac{\partial f}{\partial x_j} + \delta_{oj}\frac{\partial^2 f}{\partial x_k \partial x_j} \\
&= -\Gamma^j_{ko}\frac{\partial f}{\partial x_j} + \frac{\partial^2 f}{\partial x_k \partial x_o}.
\end{aligned}
$$

Since $\mathrm{Hess}(f)(X_i, X_j) = g(\nabla_{X_i}\nabla f, X_j)$, then

$$\mathrm{Hess}(f)(X_i, X_j) = -\Gamma^k_{ij}\frac{\partial f}{\partial x_k} + \frac{\partial^2 f}{\partial x_i \partial x_j},$$

which implies (2.17)

From (2.22) we have

$$\nabla_{X_k}\nabla f = H - \frac{1}{h^3}g_F^{\alpha\beta}\frac{\partial h}{\partial x_k}\frac{\partial f}{\partial x_\beta}X_\alpha + \frac{1}{h^2}g_F^{\alpha\beta}\frac{\partial^2 f}{\partial x_k \partial x_\beta}X_\alpha,$$

where H is a horizontal term. Therefore, for each $\gamma = 1, ..., m$, we deduce

$$
\begin{aligned}
g(\nabla_{X_k}\nabla f, X_\gamma) &= -\frac{1}{h}g_F^{\alpha\beta}(g_F)_{\alpha\gamma}\frac{\partial h}{\partial x_k}\frac{\partial f}{\partial x_\beta} + g_F^{\alpha\beta}(g_F)_{\alpha\gamma}\frac{\partial^2 f}{\partial x_k \partial x_\beta} \\
&= -\frac{1}{h}\delta_{\gamma\beta}\frac{\partial h}{\partial x_k}\frac{\partial f}{\partial x_\beta} + \delta_{\gamma\beta}\frac{\partial^2 f}{\partial x_k \partial x_\beta} \\
&= -\frac{1}{h}\frac{\partial h}{\partial x_k}\frac{\partial f}{\partial x_\gamma} + \frac{\partial^2 f}{\partial x_k \partial x_\gamma}.
\end{aligned}
$$

On the other hand, we have that $g(X_\alpha, X_\beta) = h^2 g_F(X_\alpha, X_\beta)$. Therefore

$$\mathrm{Hess}(f)(X_i, X_\alpha) = -\frac{1}{h}\frac{\partial h}{\partial x_i}\frac{\partial f}{\partial x_\alpha} + \frac{\partial^2 f}{\partial x_i \partial x_\alpha},$$

for every i and α, which implies 2.18.

Next, we are going to derivative the gradient of f in a vertical direction,

$$
\begin{aligned}
\nabla_{X_\gamma}\nabla f &= \nabla_{X_\gamma}\left(g_B^{ij}\frac{\partial f}{\partial x_j}X_i + \frac{1}{h^2}g_F^{\alpha\beta}\frac{\partial f}{\partial x_\beta}X_\alpha\right) \\
&= \nabla_{X_\gamma}\left(g_B^{ij}\frac{\partial f}{\partial x_j}X_i\right) + \nabla_{X_\gamma}\left(\frac{1}{h^2}g_F^{\alpha\beta}\frac{\partial f}{\partial x_\beta}X_\alpha\right) \\
&= g_B^{ij}\frac{\partial f}{\partial x_j}\nabla_{X_\gamma}X_i + g_B^{ij}\frac{\partial^2 f}{\partial x_\gamma \partial x_j}X_i + \frac{1}{h^2}g_F^{\alpha\beta}\frac{\partial f}{\partial x_\beta}\nabla_{X_\gamma}X_\alpha \qquad (2.23) \\
&\quad + \frac{1}{h^2}X_\gamma(g_F^{\alpha\beta})\frac{\partial f}{\partial x_\beta}X_\alpha + \frac{1}{h^2}g_F^{\alpha\beta}\frac{\partial^2 f}{\partial x_\gamma \partial x_\beta}X_\alpha.
\end{aligned}
$$

By replacing the third identity in (2.12) in (2.23) we obtain

$$\begin{aligned} \nabla_{X_\gamma}\nabla f &= \frac{1}{h}g_B^{ij}\frac{\partial f}{\partial x_j}\frac{\partial h}{\partial x_i}X_\gamma + g_B^{ij}\frac{\partial^2 f}{\partial x_\gamma \partial x_j}X_i + \frac{1}{h^2}g_F^{\alpha\beta}\frac{\partial f}{\partial x_\beta}\nabla_{X_\gamma}X_\alpha \\ &+\frac{1}{h^2}X_\gamma(g_F^{\alpha\beta})\frac{\partial f}{\partial x_\beta}X_\alpha + \frac{1}{h^2}g_F^{\alpha\beta}\frac{\partial^2 f}{\partial x_\gamma \partial x_\beta}X_\alpha. \end{aligned} \tag{2.24}$$

By replacing the third identity in (2.13) in (2.24) we obtain

$$\begin{aligned} \nabla_{X_\gamma}\nabla f &= \frac{1}{h}g_B^{ij}\frac{\partial f}{\partial x_j}\frac{\partial h}{\partial x_i}X_\gamma + g_B^{ij}\frac{\partial^2 f}{\partial x_\gamma \partial x_j}X_i - \frac{1}{h}g_F^{\alpha\beta}(g_F)_{\gamma\alpha}\frac{\partial f}{\partial x_\beta}\nabla_B h \\ &+\frac{1}{h^2}\Gamma^\lambda_{\gamma\alpha}g_F^{\alpha\beta}\frac{\partial f}{\partial x_\beta}X_\lambda + \frac{1}{h^2}X_\gamma(g_F^{\alpha\beta})\frac{\partial f}{\partial x_\beta}X_\alpha + \frac{1}{h^2}g_F^{\alpha\beta}\frac{\partial^2 f}{\partial x_\gamma \partial x_\beta}X_\alpha. \end{aligned}$$

From this, it follows that

$$\begin{aligned} \nabla_{X_\gamma}\nabla f &= \frac{1}{h}g_B^{ij}\frac{\partial f}{\partial x_j}\frac{\partial h}{\partial x_i}X_\gamma + g_B^{ij}\frac{\partial^2 f}{\partial x_\gamma \partial x_j}X_i - \frac{1}{h}\frac{\partial f}{\partial x_\gamma}\nabla_B h \\ &+\frac{1}{h^2}\Gamma^\lambda_{\gamma\alpha}g_F^{\alpha\beta}\frac{\partial f}{\partial x_\beta}X_\lambda + \frac{1}{h^2}X_\gamma(g_F^{\alpha\beta})\frac{\partial f}{\partial x_\beta}X_\alpha + \frac{1}{h^2}g_F^{\alpha\beta}\frac{\partial^2 f}{\partial x_\gamma \partial x_\beta}X_\alpha. \end{aligned}$$

Then

$$\begin{aligned} g(\nabla_{X_\gamma}\nabla f, X_\omega) &= hg_B^{ij}(g_F)_{\gamma\omega}\frac{\partial f}{\partial x_j}\frac{\partial h}{\partial x_i} + \Gamma^\lambda_{\gamma\alpha}g_F^{\alpha\beta}(g_F)_{\lambda\omega}\frac{\partial f}{\partial x_\beta} \\ &+X_\gamma(g_F^{\alpha\beta})(g_F)_{\alpha\omega}\frac{\partial f}{\partial x_\beta} + g_F^{\alpha\beta}(g_F)_{\alpha\omega}\frac{\partial^2 f}{\partial x_\gamma \partial x_\beta}. \end{aligned}$$

Therefore

$$\begin{aligned} g(\nabla_{X_\gamma}\nabla f, X_\omega) &= hg_B^{ij}(g_F)_{\gamma\omega}\frac{\partial f}{\partial x_j}\frac{\partial h}{\partial x_i} + \Gamma^\lambda_{\gamma\alpha}g_F^{\alpha\beta}(g_F)_{\lambda\omega}\frac{\partial f}{\partial x_\beta} \\ &+X_\gamma(g_F^{\alpha\beta})(g_F)_{\alpha\omega}\frac{\partial f}{\partial x_\beta} + \frac{\partial^2 f}{\partial x_\gamma \partial x_\omega}. \end{aligned} \tag{2.25}$$

By replacing the third identity in (2.9) in (2.25) we obtain

$$\begin{aligned} g(\nabla_{X_\gamma}\nabla f, X_\omega) &= hg_B^{ij}(g_F)_{\gamma\omega}\frac{\partial f}{\partial x_j}\frac{\partial h}{\partial x_i} + \Gamma^\lambda_{\gamma\alpha}g_F^{\alpha\beta}(g_F)_{\lambda\omega}\frac{\partial f}{\partial x_\beta} - \Gamma^\beta_{\gamma\omega}\frac{\partial f}{\partial x_\beta} - \\ &-\Gamma^\lambda_{\gamma\alpha}g_F^{\alpha\beta}(g_F)_{\lambda\omega}\frac{\partial f}{\partial x_\beta} + \frac{\partial^2 f}{\partial x_\gamma \partial x_\omega}. \end{aligned}$$

Then, one obtains that

$$g(\nabla_{X_\gamma}\nabla f, X_\omega) = hg_B^{ij}(g_F)_{\gamma\omega}\frac{\partial f}{\partial x_j}\frac{\partial h}{\partial x_i} - \Gamma^\beta_{\gamma\omega}\frac{\partial f}{\partial x_\beta} + \frac{\partial^2 f}{\partial x_\gamma \partial x_\omega},$$

for any $\gamma, \omega = 1, ..., m$. Therefore,

$$\text{Hess}(f)(X_\alpha, X_\beta) = h g_B^{ij} (g_F)_{\alpha\beta} \frac{\partial f}{\partial x_j} \frac{\partial h}{\partial x_i} - \Gamma_{\alpha\beta}^{\gamma} \frac{\partial f}{\partial x_\gamma} + \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta},$$

for all $\alpha, \beta = 1, ..., m$, which implies 2.19. ■

Theorem 2.2.5 (Corollary 43 in [11].). *Let (B^n, g_B) and (F^m, g_F) be Riemannian manifolds, where B is a Riemannian manifold with boundary and F is a Riemannian manifold without boundary. Let $(M = B \times_h F, g)$ be a warped product. Then The Ricci curvature of M satisfies the following identities:*

$$\begin{aligned} Ric(X, Y) &= Ric_B(X, Y) - \frac{m}{h} Hess_B(h)(X, Y), && (2.26) \\ Ric(X, V) &= 0, && (2.27) \\ Ric(V, W) &= Ric_F(V, W) - [-h\Delta_B h + (m-1)|\nabla_B h|^2] g_F(V, W), && (2.28) \end{aligned}$$

for any horizontal vector fields X, Y, and any vertical vector fields V, W. Here Ric_B and Ric_F are the Ricci curvature of B and F, respectively,

Proof. From (2.10) we have that $\Gamma_{ij}^{\alpha} = 0$, then $\nabla_{X_i} X_j = \sum_l \Gamma_{ij}^l X_l$. It follows that

$$\begin{aligned} \nabla_{X_s} \nabla_{X_i} X_j &= \sum_l \nabla_{X_s} (\Gamma_{ij}^l X_l) \\ &= \sum_l \left(\frac{\partial}{\partial x_s} (\Gamma_{ij}^l) X_l + \Gamma_{ij}^l \nabla_{X_s} X_l \right) \\ &= \nabla_{X_s}^B \nabla_{X_i}^B X_j. && (2.29) \end{aligned}$$

Since

$$\text{Ric}(X, Y) = \sum_i g(\text{R}(E_i, X)Y, E_i) + \sum_\alpha g(\text{R}(E_\alpha, X)Y, E_\alpha),$$

where $\{E_i\}$ and $\{E_\alpha\}$ are orthonormal frame to the B and F, respectively, and X and Y are horizontal vectors, one sees that

$$\text{R}(E_i, X)Y = \nabla_{E_i} \nabla_X Y - \nabla_X \nabla_{E_i} Y - \nabla_{[E_i, X]} Y.$$

By using the first identity into (2.12) with (2.29) we deduce

$$\text{R}(E_i, X)Y = \text{R}_B(E_i, X)Y,$$

where R_B denote the curvature tensor of B. Therefore,

$$\text{Ric}(X, Y) = \text{Ric}_B(X, Y) + \sum_\alpha g(\text{R}(E_\alpha, X)Y, E_\alpha).$$

By using the third identity in (2.12), a direct computation we obtain

$$\nabla_{X_i}\nabla_{X_\alpha}X_j = \frac{1}{h}\frac{\partial^2 h}{\partial x_i \partial x_j}X_\alpha, \tag{2.30}$$

$$\nabla_{X_\alpha}\nabla_{X_i}X_j = \sum_k \Gamma_{ij}^k \frac{1}{h}\frac{\partial h}{\partial x_k}X_\alpha, \tag{2.31}$$

for all i, j, and α. We have

$$\begin{aligned} g(\mathrm{R}(E_\alpha, X_i)X_j, E_\alpha) &= g(\nabla_{E_\alpha}\nabla_{X_i}X_j - \nabla_{X_i}\nabla_{E_\alpha}X_j - \nabla_{[E_\alpha, X_i]}X_j, E_\alpha) \\ &= g(\nabla_{E_\alpha}\nabla_{X_i}X_j - \nabla_{X_i}\nabla_{E_\alpha}X_j, E_\alpha), \end{aligned} \tag{2.32}$$

where $[E_\alpha, X_i] = 0$, because E_α does not depend of B, and X_i does not depend of F. Write $E_\alpha = \sum_\beta a_\beta X_\beta$, where each a_β does not depend of B. So, from a simple computation we obtain

$$\nabla_{E_\alpha}\nabla_{X_i}X_j = \sum_l \Gamma_{ij}^l \frac{1}{h}\frac{\partial h}{\partial x_l}E_\alpha, \tag{2.33}$$

$$\nabla_{X_i}\nabla_{E_\alpha}X_j = \frac{1}{h}\frac{\partial^2 h}{\partial x_i \partial x_j}E_\alpha, \tag{2.34}$$

for all i, j, and α. By replacing (2.33) and (2.34) into (2.32), one sees that

$$g(\mathrm{R}(E_\alpha, X_i)X_j, E_\alpha) = \sum_l \Gamma_{ij}^l \frac{1}{h}\frac{\partial h}{\partial x_l} - \frac{1}{h}\frac{\partial^2 h}{\partial x_i \partial x_j}.$$

Therefore,

$$\mathrm{Ric}(X_i, X_j) = \mathrm{Ric}_B(X_i, X_j) - \frac{m}{h}\mathrm{Hess}_B(h)(X_i, X_j).$$

By using the linearity in each entrance of Ric we obtain (2.26).

To prove (2.27) we proceed as follows. By using the third identity in (2.12) we obtain

$$\nabla_{X_i}\nabla_{X_j}X_\alpha = \frac{1}{h}\frac{\partial^2 h}{\partial x_i \partial x_j}X_\alpha, \tag{2.35}$$

for all i, j, and α. Now, let $\{E_i\}$ be a locally horizontal orthonormal frame, then it follows from (2.35) that

$$g(R(E_i, X_l)X_\beta, E_i) = g(\nabla_{E_i}\nabla_{X_l}X_\beta - \nabla_{X_l}\nabla_{E_i}X_\beta - \nabla_{[E_i, X_l]}X_\beta, E_i) = 0. \tag{2.36}$$

From a direct computation we get

$$\nabla_{X_\alpha}\nabla_{X_i}X_\beta = H_1 + \frac{1}{h}\frac{\partial h}{\partial x_i}\nabla^F_{X_\alpha}X_\beta, \tag{2.37}$$

where H_1 is a horizontal vector field. Since

$$\nabla_{X_\alpha} X_\beta = \sum_l \Gamma^l_{\alpha\beta} X_l + \sum_\gamma \Gamma^\gamma_{\alpha\beta} X_\gamma,$$

for all α and β, then

$$\nabla_{X_i} \nabla_{X_\alpha} X_\beta = H_2 + \frac{1}{h}\frac{\partial h}{\partial x_i} \nabla^F_{X_\alpha} X_\beta, \tag{2.38}$$

for all i, α, and β, where H_2 is a horizontal vector field. Let $\{E_\alpha\}$ be locally vertical orthonormal frame, from (2.37) and (2.38) we obtain

$$g(R(E_\alpha, X_i)X_\beta, E_\alpha) = 0, \tag{2.39}$$

for all i and β. We have

$$\mathrm{Ric}(X_i, X_\alpha) = \sum_j g(R(E_j, X_i)X_\alpha, E_j) + \sum_\beta g(R(E_\beta, X_i)X_\alpha, E_\beta),$$

then from (2.36) and (2.39) we obtain $\mathrm{Ric}(X_i, X_\alpha) = 0$. Therefore,

$$\mathrm{Ric}(X, V) = 0,$$

for all horizontal vector field X and vertical vector field V, which implies (2.27).

From (2.11) we obtain the following identity

$$\nabla_{X_\alpha} X_\beta = -h(g_F)_{\alpha\beta} \nabla_B h + \sum_\gamma \Gamma^\gamma_{\alpha\beta} X_\gamma, \tag{2.40}$$

for all α and β. The equation (2.40) gives the following expression

$$\nabla_{X_i} \nabla_{X_\alpha} X_\beta = -(g_F)_{\alpha\beta} \frac{\partial h}{\partial x_i} \nabla_B h - h(g_F)_{\alpha\beta} \nabla_{X_i} \nabla_B h + V_1, \tag{2.41}$$

for all i, α, and β, where V_1 is a vertical vector field.

From the third identity in (2.12) we get

$$\nabla_{X_\alpha} \nabla_{X_i} X_\beta = -(g_F)_{\alpha\beta} \frac{\partial h}{\partial x_i} \nabla_B h + V_2, \tag{2.42}$$

for all i, α, and β, where V_2 is a vertical vector field. From (2.41) and (2.42) we obtain

$$g(R(X_i, X_\alpha)X_\beta, X_i) = -h(g_F)_{\alpha\beta} g(\nabla_{X_i} \nabla_B h, X_i), \tag{2.43}$$

for all i, α, and β. The identity (2.43) gives the following expression

$$g(R(X_i, X_\alpha)X_\beta, X_i) = -\frac{1}{h} g_{\alpha\beta} g(\nabla_{X_i}\nabla_B h, X_i). \tag{2.44}$$

From (2.40) we obtain

$$\nabla_{X_\gamma}\nabla_{X_\alpha}X_\beta = H_3 - h(g_F)_{\alpha\beta}\nabla_{X_\gamma}\nabla_B h + \nabla^F_{X_\gamma}\nabla^F_{X_\alpha}X_\beta,$$

for all γ, α, and β. Since

$$\nabla_{X_\gamma}\nabla_B h = \frac{1}{h} g_B(\nabla_B f, \nabla_B f)X_\gamma,$$

for all γ, then

$$\nabla_{X_\gamma}\nabla_{X_\alpha}X_\beta = H_3 - (g_F)_{\alpha\beta} g_B(\nabla_B h, \nabla_B h)X_\gamma + \nabla^F_{X_\gamma}\nabla^F_{X_\alpha}X_\beta.$$

Which implies

$$\nabla_{X_\alpha}\nabla_{X_\gamma}X_\beta = H_4 - (g_F)_{\gamma\beta} g_B(\nabla_B h, \nabla_B h)X_\alpha + \nabla^F_{X_\alpha}\nabla^F_{X_\gamma}X_\beta,$$

where H_4 is a horizontal vector field. Therefore,

$$\sum_\gamma g(R(E_\gamma, X_\alpha)X_\beta, E_\gamma) = -\frac{m-1}{h^2}|\nabla_B h|^2 g_{\alpha\beta} + \sum_\gamma R_F(E_\gamma, X_\alpha, X_\beta, E_\gamma), \tag{2.45}$$

for all α and β, where R_F is the curvature tensor of F.

Therefore, from the definition of Ricci tensor, (2.43), and (2.45) we obtain (2.28). ■

Corollary 2.2.6. *Let (B^n, g_B) and (F^m, g_F) be Riemannian manifolds, where B is a Riemannian manifold with boundary and F is a Riemannian manifold without boundary. The warped product $(M = B \times_h F, g)$ is Einstein with $Ric = \lambda g$ if and only if*

$$\begin{aligned} Ric_B &= \lambda g_B + \frac{m}{h} Hess_B(h), && (2.46)\\ Ric_F &= \mu g_F, && (2.47)\\ \mu &= -h\Delta_B h + (m-1)|\nabla_B h|^2 + \lambda h^2. && (2.48) \end{aligned}$$

In what follows we have the following theorem whose proof uses the equations in Corollary 2.2.6.

Theorem 2.2.7. *Let (B^n, g_B) and (F^m, g_F) be compact Riemannian manifolds, where B is a connected manifold with boundary, F is without boundary and $m \geq 3$. Let $(M =$*

$B \times_h F, g)$ be an warped product such that M has non-positive scalar curvature and

$$\int_{\partial B} h \frac{\partial h}{\partial \mathcal{N}} d(\partial B) \geq 0. \tag{2.49}$$

So if M is Einstein and the maximum point of h is an inner point, then h is constant. Where $\mathcal{N}$ is the outward unit normal vector field along ∂B.

Proof. Since M is Einstein, then from (2.46) and (2.48) we have that

$$\begin{aligned} \mathrm{Ric}_B &= \lambda g_B + \frac{m}{h}\mathrm{Hess}_B(h), \\ \mathrm{Ric}_F &= [-h\Delta_B h + (m-1)|\nabla_B h|^2 + \lambda h^2] g_F, \end{aligned}$$

where $\mathrm{Ric} = \lambda g$. Since $m \geq 3$, then $\mu = -h\Delta_B h + (m-1)|\nabla_B h|^2 + \lambda h^2$ is constant on F. So, by proceeding in the same way that we did in Theorem 2.1.1, we can show that μ is constant on B. Therefore, μ is constant on M.

By using the properties of the divergence operator we obtain

$$\mu = \lambda h^2 + \mathrm{div}(h\nabla_B h) + (m-2)|\nabla_B h|^2.$$

It follows that

$$\mu = \frac{\lambda}{\mathrm{vol}(B)} \int_B h^2 + \frac{1}{\mathrm{vol}(B)} \int_B \mathrm{div}(h\nabla_B h) + \frac{m-2}{\mathrm{vol}(B)} \int_B |\nabla_B h|^2. \tag{2.50}$$

By applying the Divergence theorem on (2.50) we obtain

$$\mu = \frac{\lambda}{\mathrm{vol}(B)} \int_B h^2 + \frac{1}{\mathrm{vol}(B)} \int_{\partial B} h \frac{\partial h}{\partial \mathcal{N}} + \frac{m-2}{\mathrm{vol}(B)} \int_B |\nabla_B h|^2. \tag{2.51}$$

Let S be the scalar curvature of M. Since $\mathrm{Ric} = \lambda g$, then $S = \lambda(n + mh^2)$. Since $S \leq 0$, then $\lambda \leq 0$.

Let p_{max} be the maximum point of h, then $h(p_{max}) > 0$, $\nabla_B h(p_{max}) = 0$, and since

$\Delta_B h = -\text{tr Hess}_B(h)$, then $\Delta_B h(p_{max}) \geq 0$. It follows that

$$\begin{aligned}
0 &\leq h(p_{max})\Delta_B h(p_{max}) \\
&= \lambda h(p_{max})^2 - \mu \\
&= \frac{\lambda}{\text{vol}(B)} \int_B h(p_{max})^2 - \frac{\lambda}{\text{vol}(B)} \int_B h^2 - \frac{1}{\text{vol}(B)} \int_{\partial B} h \frac{\partial h}{\partial \mathcal{N}} + \frac{2-m}{\text{vol}(B)} \int_B |\nabla_B h|^2 \\
&= \frac{\lambda}{\text{vol}(B)} \int_B (h(p_{max})^2 - h^2) - \frac{1}{\text{vol}(B)} \int_{\partial B} h \frac{\partial h}{\partial \mathcal{N}} + \frac{2-m}{\text{vol}(B)} \int_B |\nabla_B h|^2 \\
&\leq 0.
\end{aligned}$$

Then $\nabla_B h = 0$, in B. Therefore, h is constant. ■

Chapter 3

Ricci Almost Solitons with Boundary

The definition of Ricci almost solitons was first introduced by Pigola, Rigoli, Rimoldi, and Setti in [15]. Many results have been obtained for gradient Ricci almost solitons without boundary. In this chapter we prove properties of gradient Ricci almost solitons with boundary and some characterizations. The first section we obtain a lower bound to a symmetric tensor. The second section we prove some properties for Ricci almost solitons and we finish this section with a characterization of gradient Ricci almost solitons with boundary where the gradient of the potential function is a conformal vector field. In the third section we obtain a characterization for totally geodesic boundaries by using an inequality. In the fourth section we characterize compact gradient Ricci almost solitons where the mean curvature of the boundary is positive. At the last section we show a rigidity theorem for the case where the boundary is immersed into the hyperbolic space.

3.1 A Lower Bound for the Ricci Curvature

A **Ricci almost soliton** $(M, g, \mathcal{X}, \lambda)$ is a Riemannian manifold (M, g) with (or without) boundary, a smooth vector field $\mathcal{X}$, and a smooth function $\lambda : M \longrightarrow \mathbb{R}$ satisfying the following equation

$$\mathrm{Ric} = \lambda g + \frac{1}{2}\mathcal{L}_{\mathcal{X}}(g). \tag{3.1}$$

If the vector field $\mathcal{X}$ vanishes, then the Ricci almost soliton is an Einstein manifold. If the vector field $\mathcal{X}$ is the gradient of some smooth function $f : M \longrightarrow \mathbb{R}$, then the Ricci almost soliton is called **gradient Ricci almost soliton**, and it is denoted by $(M, g, \nabla f, \lambda)$. By substituting Proposition 1.2.14 into (3.1) we obtain

$$\mathrm{Ric} = \lambda g + \mathrm{Hess}(f). \tag{3.2}$$

We shall see below always there exists a positive lower bound for any positive symmetric 2-tensor on a compact Riemannian manifold with or without boundary.

Theorem 3.1.1. *Let (M^n, g), $n \geq 2$, be a compact Riemannian manifold with (or without) boundary. Let $\mathcal{T} \in \mathscr{T}^2(M)$ be a symmetric 2-tensor on M. Set $\mathcal{T}(p) = \mathcal{T}_p$ for all $p \in M$. Then $\mathcal{T}_p$ is positive for all $p \in M$ if and only if there exists a positive constant $\alpha_0 \in \mathbb{R}$ such that*

$$\mathcal{T}_p(v, v) \geq \alpha_0 |v|^2, \tag{3.3}$$

for all $p \in M$, and $v \in T_pM$.

Proof. Since $\mathcal{T}$ is a symmetric 2-tensor on M, then $\mathcal{T}_p : T_pM \times T_pM \longrightarrow \mathbb{R}$ is a symmetric bilinear form for all $p \in M$. Thus, for all $p \in M$ there exists a self-adjoint operator $S_p : T_pM \longrightarrow T_pM$ such that

$$\mathcal{T}_p(u, v) = g_p(S_p(u), v),$$

for all $u, v \in T_pM$. From the Spectral Theorem of Linear Algebra we have that, for all $p \in M$, there exist a basis $\{(e_1)_p, ..., (e_n)_p\}$ for T_pM and constants $\lambda_1, (p), ..., \lambda_n(p) \in \mathbb{R}$ such that $S_p((e_i)_p) = \lambda_i(p)(e_i)_p$, for each $i = 1, ..., n$. For an arbitrary $v \in T_pM$ write

$$v = \sum_{i=1}^{n} a_i(p)(e_i)_p,$$

where $a_i(p) \in \mathbb{R}$ is the i-th component of v in the basis $\{(e_1)_p, ..., (e_n)_p\}$, for each $i = 1, ..., n$. It follows that

$$\mathcal{T}_p(v, v) = \sum_{i=1}^{n} \lambda_i(p) a_i(p)^2,$$

for all $p \in M$, and $v \in T_pM$.

Let $\mathbb{S}_1^{n-1}(p)$ be the subset of T_pM defined by

$$\mathbb{S}_1^{n-1}(p) = \{v \in T_pM : |v| = 1\}.$$

Since $\mathcal{T}_p$ is continuous and $\mathbb{S}_1^{n-1}(p)$ is compact, for all $p \in M$, then there exists $v_p \in \mathbb{S}_1^{n-1}(p)$ such that

$$\mathcal{T}_p(v_p, v_p) = \min_{v \in \mathbb{S}_1^{n-1}(p)} \mathcal{T}_p(v, v).$$

Since the tensor $\mathcal{T}$ is continuous and M is compact, then there exists $p_0 \in M$ such that

$$\mathcal{T}_{p_0}(v_{p_0}, v_{p_0}) = \min_{p \in M} \mathcal{T}_p(v_p, v_p).$$

Since T_{p_0} is positive and $v_0 \in \mathbb{S}_1^{n-1}(p_0)$, then $\mathcal{T}_{p_0}(v_{p_0}, v_{p_0}) > 0$. Define

$$\alpha_0 = \mathcal{T}_{p_0}(v_{p_0}, v_{p_0}).$$

Therefore, for all $p \in M$, and $v \in T_pM$, $v \neq 0$, we have

$$T_p(v,v) = T_p\left(\frac{v}{|v|}, \frac{v}{|v|}\right)|v|^2 \geq \alpha_0 |v|^2.$$

The converse is a straight computation. ■

3.2 The conformal case

In this section we are interested in gradient Ricci almost solitons with boundary $(M, g, \nabla f, \lambda)$ where the gradient ∇f of the potential function f is a conformal vector field, and f satisfies the following boundary conditions

$$\begin{aligned} f(p) > c_0, &\quad \text{for all} \quad p \in \text{int}(M), \\ f(p) = c_0, &\quad \text{for all} \quad p \in \partial M, \end{aligned} \tag{3.4}$$

where $c_0 \in \mathbb{R}$ is constant. In what follows we give a simple example of gradient Ricci almost soliton with boundary where the base is one dimensional.

Example 3.2.1. *Let $(M = [0, +\infty) \times_h \mathbb{H}^2(c), g = dt^2 + h(t)^2 g_{\mathbb{H}^2(c)})$ be the warped product, where $c < 0$ and $h(t) = \cosh(\sqrt{-c}\,t + b_0)$, $b_0 > 0$. Let f, λ be smooth functions on M given by*

$$\begin{aligned} f(t) &= \frac{1}{c}\sinh(\sqrt{-c}\,t + b_0), \\ \lambda(t) &= \sinh(\sqrt{-c}\,t + b_0) + 2c. \end{aligned}$$

By using Proposition 2.2.4 and Theorem 2.2.5 we can show that $(M, g, \nabla f, \lambda)$ is a gradient Ricci almost soliton with boundary.

Definition 3.2.2. *Let (M, g) be a Riemannian manifold with boundary. A smooth vector field $\mathcal{X} \in \mathscr{X}(M)$ is called* ***normally parallel to the boundary*** *if $\mathcal{X}$ is parallel along any geodesic $\gamma : [0, \epsilon) \longrightarrow M$ such that $\gamma(0) \in \partial M$, $\gamma((0, \epsilon)) \subset int(M)$, and $\gamma'(0) \perp T_{\gamma(0)}\partial M$.*

When the gradient of a smooth function which satisfies (3.4) is normally parallel to the boundary it has not to vanish on the boundary.

Lemma 3.2.3. *Let (M^n, g) be Riemannian manifold with boundary. If f is a smooth function which safisties (3.4) and ∇f is normally parallel to the boundary, then $\nabla f(p) \in (T_p\partial M)^\perp$ and $\nabla f(p) \neq 0$ for all $p \in \partial M$.*

Proof. For each $p \in \partial M$ and $v \in T_p\partial M$, there exist a real number $\delta > 0$ and a smooth curve $\beta : (-\delta, \delta) \longrightarrow \partial M$ such that $\beta(0) = p$ and $\beta'(0) = v$. Let $c : (-\delta, \delta) \longrightarrow \mathbb{R}$ be a smooth function fiven by $c(t) = f(\beta(t))$. Thus, $c(t) = c_0, \forall t \in (-\delta, \delta)$. It follows that

$$0 = c'(t) = df_{\beta(t)}(\beta'(t)) = g(\nabla f(\beta(t)), \beta'(t)),$$

$\forall t \in (-\delta, \delta)$. In particular, for $t = 0$ we have

$$g(\nabla f(p), v) = 0,$$

for each $p \in \partial M$ and $v \in T_p\partial M$. Therefore, $\nabla f(p) \in (T_p\partial M)^{\perp}$, for all $p \in \partial M$.

Let $p \in \partial M$ and $\epsilon > 0$ be such that $\gamma : [0, \epsilon) \longrightarrow M$ is an unit geodesic emanating of p, $\gamma((0, \epsilon)) \subset \text{int}(M)$ and $\gamma'(0)$ is orthogonal to ∂M at p. Set a function $s : [0, \epsilon) \longrightarrow \mathbb{R}$ by

$$s(t) = f(\gamma(t)), \tag{3.5}$$

which implies that $s(0) = 0$ and $s(t) \neq 0$ for all $t \in (0, \epsilon)$. So, it follows that

$$s'(t) = dh_{\gamma(t)}(\gamma'(t)) = g(\nabla f(\gamma(t)), \gamma'(t)), \tag{3.6}$$

and

$$\begin{aligned} s''(t) &= g\left(\frac{D}{dt}\nabla f(\gamma(t)), \gamma'(t)\right) \\ &= g(\nabla_{\gamma'(t)}\nabla f(\gamma(t)), \gamma'(t)) \\ &= \text{Hess}(f)(\gamma'(t), \gamma'(t)). \end{aligned} \tag{3.7}$$

Since ∇f is parallel along γ, we have

$$\text{Hess}(f)(\gamma'(t), \gamma'(t)) = g(\nabla_{\gamma'(t)}\nabla f(\gamma(t)), \gamma'(t)) = \frac{D}{dt}g(\nabla f(\gamma(t)), \gamma'(t)) = 0,$$

then $s''(t) = 0$, for all $t \in [0, \epsilon)$. From (3.5) and (3.6), we obtain

$$\begin{aligned} s''(t) &= 0, \\ s(0) &= c_0, \\ s'(0) &= g(\nabla f(p), \gamma'(0)). \end{aligned} \tag{3.8}$$

From the Existence and Uniqueness Theorem for Ordinary Differential Equations of second order we have that (3.8) admits only one solution. If we would have $g(\nabla f(p), \gamma'(0)) = 0$, then as we have $\nabla f(p) \in (T_p\partial M)^{\perp}$, we obtain $\nabla f(p) = 0$. Since ∇f is parallel along

γ, then $\nabla f \equiv 0$ along γ. Therefore, the unique solution for (3.8) would be $s \equiv c_0$ along all γ. But we know from (3.5) that $s(t) \neq 0$ for all $t \in (0, \epsilon)$, which is a contradiction! Therefore, $\nabla f(p) \neq 0$ for any $p \in \partial M$. ■

We can apply the Lemma 3.2.2 to conclude that every smooth function whose gradient is a Killing vector field shall have gradient non-zero on the boundary.

Theorem 3.2.4. *Let (M^n, g) be Riemannian manifold with boundary. If f is a smooth function which safisties* (3.4) *and ∇f is a Killing vector field, then $\nabla f(p) \in (T_p\partial M)^\perp$ and $\nabla f(p) \neq 0$ for all $p \in \partial M$.*

Proof. From Lemma 3.2.2 all what we have to do is to show that ∇f is normally parallel to the boundary. So, for any $p \in \partial M$, let $\gamma; [0, \epsilon) \longrightarrow M$ be a unit geodesic such that $\gamma(0) = p$, $\gamma((0, \epsilon)) \subset \text{int}(M)$, and $\gamma'(0) \perp T_p\partial M$. Let $\{e_1, ..., e_{n-1}, \gamma'(0)\}$ be an orthonormal basis for T_pM, then by using the parallel translation along γ we obtain an orthonormal basis $\{e_1(t), ..., e_{n-1}(t), \gamma'(t)\}$ for $T_{\gamma(t)}M$, for all $t \in [0, \epsilon)$, where $e_i(0) = e_i$ for every $i = 1, ..., n-1$. So, one obtains that

$$\nabla_{\gamma'(t)}\nabla f(\gamma(t)) = \sum_{i=1}^{n-1} g(\nabla_{e_i(t)}\nabla f(\gamma(t)), e_i(t))e_i(t) + g(\nabla_{\gamma'(t)}\nabla f(\gamma(t)), \gamma'(t))\gamma'(t),$$

for all $t \in [0, \epsilon)$. Therefore, since ∇f is a Killing vector field, then

$$\frac{D}{dt}\nabla f(\gamma(t)) = \nabla_{\gamma'(t)}\nabla f(\gamma(t)) = 0.$$

From Lemma 3.2.2 we conclude that $\nabla f(p) \in (T_p\partial M)^\perp$ and $\nabla f(p) \neq 0$, for all $p \in \partial M$.
■

In Lemma 3.2.2 we showed that if f is a smooth function on a Riemannian manifold with boundary (M^n, g), $n \geq 2$, such that f satisfies (3.4) and ∇f is normally parallel to the boundary, then $\nabla f(p) \in (T_p\partial M)^\perp$ and $\nabla f(p) \neq 0$, for all $p \in \partial M$. In what follows we show a characterization of a smooth function which satisfies (3.4) and do not vanish on the boundary.

Theorem 3.2.5. *Let (M^n, g) be a Riemannian manifold with boundary. Suppose f satisfies* (3.4)*. If for all $p \in \partial M$ there exists a function $\zeta_p : [0, 1] \longrightarrow \mathbb{R}$ such that $\zeta_p(0) = 0$, and for any geodesic $\gamma : [0, \epsilon) \longrightarrow M$, $\gamma(0) = p$, $\gamma((0, \epsilon)) \subset$ int(M), and $\gamma'(0) \perp T_p\partial M$, the following inequalities are satisfied*

$$f(\gamma(t)) - f(p) \geq \zeta_p(t), \tag{3.9}$$

for all $t \in [0, \epsilon)$, *and*

$$\limsup_{t \to 0} \frac{\zeta_p(t)}{t} > 0. \tag{3.10}$$

Then, $\nabla f(p) \neq 0$, *for all* $p \in \partial M$. *Conversely, if* $\nabla f(p) \neq 0$, *for all* $p \in \partial M$, *then for all* $p \in \partial M$ *there exists a function* $\zeta_p : [0, 1] \longrightarrow \mathbb{R}$ *such that* $\zeta_p(0) = 0$, *and for any geodesic* $\gamma : [0, \epsilon) \longrightarrow M$, $\gamma(0) = p$, $\gamma((0, \epsilon)) \subset int(M)$, *and* $\gamma'(0) \perp T_p\partial M$, (3.9) *and* (3.10) *are satisfied.*

Proof. The first implication is a straight computation. Conversely, suppose that $\nabla f(p) \neq 0$ for all $p \in \partial M$. Let $\gamma : [0, \epsilon) \longrightarrow M$ be an unit geodesic such that $\gamma(0) = p$, $\gamma((0, \epsilon)) \subset \text{int}(M)$, and $\gamma'(0) \perp T_p\partial M$. Define $s : [0, \epsilon) \longrightarrow \mathbb{R}$ by $s(t) = f(\gamma(t))$. It follows that

$$s'(0) = g_p(\nabla f(p), \gamma'(0)).$$

Since $\nabla f(p) \in (T_p\partial M)^\perp$ and $\nabla f(p) \neq 0$, then $s'(0) \neq 0$. From the Taylor's formula we have

$$s(t) = s(0) + s'(0)t + \frac{1}{2}s''(\theta(t))t^2,$$

where $0 < \theta(t) < t$, and $\lim_{t \to 0} s''(\theta(t))t = 0$. By choosing $\epsilon > 0$ small we obtain that γ is the unique geodesic which goes through p at $t = 0$ with speed $\gamma'(0)$. Define $\zeta_p : [0, \epsilon) \longrightarrow \mathbb{R}$ by

$$\zeta_p(t) = s'(0)t + \frac{1}{2}s''(\theta(t))t^2.$$

Since f satisfies (3.4), then $f(\gamma(t)) - f(p) > 0$ for all $t \in (0, \epsilon)$, which implies $s'(0) > 0$. We have

$$\frac{\zeta_p(t)}{t} = s'(0) + \frac{1}{2}s''(\theta(t))t,$$

for all $t \in (0, \epsilon)$. Therefore,

$$\lim_{t \to 0} \frac{\zeta_p(t)}{t} = s'(0) > 0.$$

■

Theorem 3.2.5 says that a smooth function $f : M \longrightarrow \mathbb{R}$ defined on a Riemannian manifold with boundary which satisfies (3.4), and does not go "too fast" to the boundary should have grandient non vanishing on ∂M. We shall see in the next example that the height function has gradient non vanishing on the boundary.

Example 3.2.6. *Let* f *be the function on* $\mathbb{R}^n_+$ *given by* $f(x_1, ..., x_n) = x_n$. *Then the gradient of* f *is* $\nabla f = e_n$, *where* $e_n = (0, ..., 1)$. *For each* $p \in \partial\mathbb{R}^n_+$, *set* $\gamma_p(t) = p + te_n$, $t \geq 0$. *We have that* γ_p *is the unique geodesic such that* $\gamma_p(0) = p$, $\gamma_p((0, +\infty)) \subset int(\mathbb{R}^n_+)$, *and* $\gamma'(0) \perp T_p\partial\mathbb{R}^n_+$. *Then, set* $\zeta_p(t) = t$. *Therefore,* $f(p + te_n) - f(p) = \zeta_p(t)$, *and* $\lim_{t \to 0} \frac{\zeta_p(t)}{t} = 1$.

The next example shows us a situation where the characterization of Theorem 3.2.5 is not satisfied.

Example 3.2.7. *Let* $f : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ *be the function given by*

$$f(p) = f(x_1, ..., x_n) = \begin{cases} e^{-\frac{1}{x_n^2}}, & \text{if } x_n > 0, \\ 0, & \text{if } x_n = 0. \end{cases}$$

So f *is smooth, and* $\nabla f(p) = 0$*, for all* $p \in \partial\mathbb{R}^n_+$*.*

Let (M, g) be an oriented Riemannian manifold with boundary. From Theorem 1.1.11 the boundary ∂M is a submanifold of B with codimension 1. For any $p \in \partial M$, let $\eta(p)$ be a normal vector to ∂M at p. The **second fundamental form of ∂M at p with respect to** $\eta(p) \in (T_p\partial M)^\perp$ is the map $II_{\eta(p)} : T_p\partial M \longrightarrow \mathbb{R}$ given by

$$II_{\eta(p)}(v) = -g_p(\nabla^T_v \mathcal{N}(p), v), \tag{3.11}$$

where ∇^T is the Levi-Civita connection of ∂M, and $\mathcal{N}$ is a locally extension of $\eta(p)$ which is normal along ∂M. The **shape operator of ∂M at $p \in \partial M$ with respect to** $\eta(p) \in (T_p\partial M)^\perp$ is the map $S_{\eta(p)} : T_p\partial M \longrightarrow T_p\partial M$ given by

$$S_{\eta(p)}(v) = -\nabla^T_v \mathcal{N}(p), \tag{3.12}$$

for all $v \in T_p\partial M$. It follows that $S_{\eta(p)}$ is a self-adjoint operator. From (3.11) and (3.12) we obtain

$$II_{\eta(p)}(v) = g_p(S_{\eta(p)}(v), v),$$

for all $p \in \partial M$, and $v \in T_p\partial M$.

Lemma 3.2.8. *Let* (M, g) *be a Riemannian manifold with boundary. If* $f : M \longrightarrow \mathbb{R}$ *is a smooth function which satisfies* (3.4) *and* ∇f *does not vanish on* ∂M*, then for all* $p \in \partial M$ *there exists a smooth function* ψ *locally defined in a neighborhood in* M *of* p *such that*

$$II_{\eta(p)}(v) = \psi(p) II_{\nabla f(p)}(v), \tag{3.13}$$

for all $\eta(p) \in (T_p\partial M)^\perp$*, and* $v \in T_p\partial M$*. Moreover, we obtain* $S_{\eta(p)} = \psi(p) S_{\nabla f(p)}$*.*

Proof. For each $\eta(p) \in (T_p\partial M)^\perp$, there exists a constant $\psi(p) \in \mathbb{R}$ such that

$$\eta(p) = \psi(p)\nabla f(p).$$

So, for any vector field $\mathcal{N} : U \longrightarrow TM$ locally defined in M which is a extension of $\eta(p)$ and $\mathcal{N}(\partial U) \subset (T\partial M)^{\perp}$, where $\partial U = U \cap \partial M$, there exists a smooth function $\psi : U \longrightarrow \mathbb{R}$ such that

$$\mathcal{N}(q) = \psi(q)\nabla f(q), \text{ and } \mathcal{N}(p) = \eta(p),$$

for all $q \in U$, and $p \in \partial U$. For every vector field $\mathcal{X}$, we have

$$\nabla_{\mathcal{X}}\mathcal{N} = \nabla_{\mathcal{X}}(\psi\nabla f) = \mathcal{X}(\psi)\nabla f + \psi\nabla_{\mathcal{X}}\nabla f.$$

For all $\eta(p) \in (T_p\partial M)^{\perp}$, we obtain

$$II_{\eta(p)}(v) = -g_p(\nabla_v\mathcal{N}(p), v) = -\psi(p)g_p(\nabla_v\nabla f(p), v) = \psi(p)II_{\nabla f(p)}(v).$$

Moreover, since $g_p(\nabla_v\mathcal{N}(p), v) = \psi(p)g_p(\nabla_v\nabla f, v)$, for all $v \in T_p\partial M$, then $S_{\eta(p)} = \psi(p)S_{\nabla f(p)}$, for all $p \in \partial M$. ■

For a more complete approach on second fundamental form, shape operator, or submanifolds in general we indicate the excellent reference [4].

Proposition 3.2.9. *Let (M^n, g), $n \geq 2$, be a Riemannian manifold with compact boundary. If $f : M \longrightarrow \mathbb{R}$ is a smooth function which satisfies (3.4) and ∇f does not vanish on ∂M, then ∂M is minimal if and only if the mean curvature of ∂M does not change sign.*

Proof. Define

$$\eta(p) = \frac{\nabla f(p)}{|\nabla f(p)|},$$

For all $p \in \partial M$. Since $\nabla f(p) \in (T_p\partial M)^{\perp}$, for all $p \in \partial M$, then $\eta(p)$ is an unit normal vector to the boundary ∂M at p. By definition, the mean curvature H of ∂M is given by

$$H(p) = \text{tr } S_{\eta(p)},$$

for all $p \in \partial M$. Let $\{e_1, ..., e_{n-1}, e_n = \eta(p)\}$ be an orthonormal basis for T_pM. Then

$$H(p) = -\sum_{i=1}^{n-1} g_p(\nabla_{e_i}\eta(p), e_i) = -\text{div}_{\partial M}(\eta(p)).$$

It follows that

$$H(p) = -\text{div}_{\partial M}\left(\frac{\nabla f(p)}{|\nabla f(p)|}\right),$$

for all $p \in \partial M$. From the Divergence Theorem, one sees that

$$\int_{\partial M} H(p)d(\partial M) = -\int_{\partial M} \text{div}_{\partial M}\left(\frac{\nabla f(p)}{|\nabla f(p)|}\right) d(\partial M).$$

Since $\partial(\partial M) = \varnothing$, then

$$\int_{\partial M} H(p) d(\partial M) = 0.$$

Thus, H does not change sign on ∂M if and only if $H \equiv 0$. So, H does not change sign on ∂M if and only if ∂M is minimal. ■

We shall see in the next lemma if (M, g) and f are as in Proposition 3.2.7 and the gradient of f is conformal, then we can relate the mean curvature the boundary of M and the gradient of f with the Hessian of f.

Lemma 3.2.10. *Let (M^n, g), $n \geq 2$, be a Riemannian manifold with boundary. Suppose f satisfies* (3.4) *and ∇f is a conformal vector field which does not vanish on ∂M. Then the mean curvature H of ∂M satisfies the following identity*

$$H(p)|\nabla f(p)| = -(n-1)\xi(p), \tag{3.14}$$

for all $p \in \partial M$, where $\xi : M \longrightarrow \mathbb{R}$ is the smooth function such that Hess$(f) = \xi g$.

Proof. For each $p \in \partial M$, let $\{e_1(p), ..., e_{n-1}(p), \eta(p)\}$ be an orthonormal basis for T_pM, where $\eta(p) = \dfrac{\nabla f(p)}{|\nabla f(p)|}$. Let $S_{\eta(p)}$ be the shape operator at p with respect to $\eta(p)$, then

$$\text{tr } S_{\eta(p)} = -\sum_{i=1}^{n-1} g(\nabla_{e_i(p)}\eta(p), e_i(p)).$$

Since $|\eta(p)| = 1$, then $2g(\nabla_{\eta(p)}\eta(p), \eta(p)) = 0$. Thus, it follows that

$$\text{tr } S_{\eta(p)} = -\sum_{i=1}^{n-1} g(\nabla_{e_i(p)}\eta(p), e_i(p)) - g(\nabla_{\eta(p)}\eta(p), \eta(p)).$$

On the other hand, we have

$$g(\nabla_{e_i(p)}\eta(p), e_i(p)) = \frac{1}{|\nabla f(p)|} g(\nabla_{e_i(p)}\nabla f(p), e_i(p)),$$

for all $i = 1, ..., n-1$, and

$$0 = g(\nabla_{\eta(p)}\eta(p), \eta(p)) = \eta\left(\frac{1}{|\nabla f(p)|}\right)|\nabla f(p)| + \frac{1}{|\nabla f(p)|} g(\nabla_{\eta(p)}\nabla f(p), \eta(p)).$$

Of which

$$\begin{aligned}\text{tr } S_{\eta(p)} &= -\frac{1}{|\nabla f(p)|}\sum_{i=1}^{n-1}\text{Hess}(f)(e_i(p), e_i(p)) - \frac{1}{|\nabla f(p)|}\text{Hess}(f)(\eta(p), \eta(p)) \\ &\quad -\eta\left(\frac{1}{|\nabla f(p)|}\right)|\nabla f(p)|.\end{aligned}$$

Thus

$$\text{tr } S_{\eta(p)} = -\frac{1}{|\nabla f(p)|}\text{tr Hess}(f) + \frac{1}{|\nabla f(p)|}\text{Hess}(f)(\eta(p), \eta(p)).$$

Then

$$\text{tr } S_{\eta(p)} = \frac{1}{|\nabla f(p)|}\Delta f(p) + \frac{1}{|\nabla f(p)|}\text{Hess}(f)(\eta(p), \eta(p)),$$

for all $p \in \partial M$. Since ∇f is a conformal vector field, then there exists a smooth function $\xi : M \longrightarrow \mathbb{R}$ such that $\text{Hess}(f) = \xi g$. Therefore, we obtain

$$-\Delta f(p) = n\xi(p) \quad \text{and} \quad \text{Hess}(f)(\eta(p), \eta(p)) = \xi(p),$$

for all $p \in \partial M$. We just obtain

$$\text{tr } S_{\eta(p)} = -\frac{1}{|\nabla f(p)|}(n-1)\xi(p),$$

for all $p \in \partial M$. Since $H(p) = \text{tr } S_{\eta(p)}$, then

$$H(p)|\nabla f(p)| = -(n-1)\xi(p),$$

for all $p \in \partial M$. ∎

Proposition 3.2.11. *Let (M^n, g), $n \geq 2$, be a Riemannian manifold with boundary. Suppose f satisfies* (3.4) *and ∇f is a conformal vector field which does not vanish on ∂M. Then, the boundary ∂M is minimal if and only if it is totally geodesic.*

Proof. Since $H \equiv 0$, then from Lemma 3.2.8 $\xi \equiv 0$ on ∂M, where ξ is a smooth function which satisfies $\text{Hess}(f) = \xi g$. We obtain

$$\text{Hess}(f) = 0,$$

on ∂M. Since, $II_{\nabla f} = \text{Hess}(f)$ on ∂M, then $II_{\nabla f} = 0$. Therefore, from Lemma 3.2.6 we conclude that ∂M is totally geodesic.

The converse follows from the definition of totally geodesic. ∎

Let (M, g) a Riemannian manifold with boundary. The boundary ∂M is called **umbilical** if for all $p \in \partial M$ the following identity is satisfies

$$S_{\eta(p)}(v) = g_p(\mathcal{H}(p), \eta(p))v, \tag{3.15}$$

for all $v \in T_p\partial M$, where $\eta(p)$ is the unit normal vector to the boundary at p. Here $\mathcal{H}(p)$ denotes the **mean curvature vector**.

Proposition 3.2.12. *Let (M^n, g) be a Riemannian manifold with boundary, $n \geq 2$. Suppose f satisfies* (3.4) *and ∇f is a conformal vector field which does not vanish on ∂M. Then ∂M is umbilical.*

Proof. For every $u, v \in T_p\partial M$ we have

$$g_p(S_{\nabla f(p)}(u), v) = -\text{Hess}(f)(u, v) = -\xi(p)g_p(u, v).$$

Thus $S_{\nabla f(p)} = -\xi(p)Id_{T_p\partial M}$. Let $\eta(p) \in (T_p\partial M)^{\perp}$ be given by

$$\eta(p) = \frac{\nabla f(p)}{|\nabla f(p)|}.$$

From Lemma 3.2.4 we have $S_{\eta(p)} = \dfrac{1}{|\nabla f(p)|}S_{\nabla f(p)}$. On the other hand, from Lemma 3.2.6 we deduce

$$-\xi(p) = \frac{|\nabla f(p)|}{n-1}H(p),$$

for all $p \in \partial M$. Thus, $S_{\eta(p)} = \dfrac{1}{n-1}H(p)Id_{T_p\partial M}$. Since $\mathcal{H}(p) = \dfrac{1}{n-1}H(p)\eta(p)$, then

$$S_{\eta(p)} = g_p(\mathcal{H}(p), \eta(p))Id_{T_p\partial M}.$$

■

The next theorem is going to be use to prove the main result of this section.

Theorem 3.2.13 (Theorem 4 in [18].). *Let (M^n, g) be a compact Riemannian manifold with boundary. Assume that there is a positive constant $\rho^2 > 0$ such that $Ric \geq (n-1)\rho^2 g$ and the mean curvature of ∂M is non negative. Then the first eigenvalue $\lambda_1(\Delta)$ of the Laplacian on M satisfies the inequality $\lambda_1(\Delta) \geq n\rho^2$. Moreover, $\lambda_1(\Delta) = n\rho^2$ if and only if M is isometric to a closed hemisphere of the Euclidean sphere $\mathbb{S}^n(\rho^2)$ of radius $\dfrac{1}{\rho}$.*

The next result characterize compact gradient Ricci almost solitons when the gradient of the potential function is conformal, the scalar curvature is positive, and the Ricci curvature satisfies an inequality.

Theorem 3.2.14. *Let $(M^n, g, \nabla f, \lambda)$, $n \geq 2$ be a compact gradient Ricci almost soliton with connected boundary, where f satisfies 3.4. Suppose the scalar curvature S of M is* *positive, and ∇f does not vanish on ∂M. Assume Hess(f) $= \xi\, g$, where $\xi \leq 0$ on ∂M. Then the mean curvature of ∂M is non negative, and there exists a positive constant $\rho \in \mathbb{R}$ such that the Ricci curvature of M satisfies*

$$Ric_p(v, v) \geq (n-1)\rho^2, \tag{3.16}$$

for all $p \in \partial M$, and $v \in T_p\partial M$, $|v| = 1$. Moreover, the first eigenvalue $\lambda_1(\Delta)$ of the Laplacian on M satisfies the inequality $\lambda_1(\Delta) \geq n\rho^2$. The equality holds if and only if M is isometric to a closed hemisphere of the Euclidean sphere $\mathbb{S}^n(\rho^2)$ of radius $\dfrac{1}{\rho}$.

Proof. From Lemma 3.2.10

$$H(p)|\nabla f(p)| = -(n-1)\xi(p),$$

for all $p \in \partial M$. Since $\xi \leq 0$ on ∂M, then $H \geq 0$.

Since

$$\mathrm{Ric}_p = (\lambda(p) + \xi(p))g_p,$$

for all $p \in M$, and the scalar curvature of M is positive, then from Theorem 3.1.1 we obtain

$$\min_{p\in M}(\lambda(p) + \xi(p)) > 0, \quad \text{and} \quad \mathrm{Ric}_p(v, v) \geq \min_{p\in M}(\lambda(p) + \xi(p))|v|^2,$$

for all $p \in M$, and $v \in T_pM$. Define

$$\rho^2 = \frac{\min_{p\in M}(\lambda(p) + \xi(p))}{n-1}.$$

So, $\mathrm{Ric}_p(v, v) \geq (n-1)\rho^2$, for all $p \in M$, and $v \in T_pM$, $|v| = 1$. The conclusion follows from Theorem 3.2.13. ■

We finish this section by giving an example of compact gradient Ricci almost soliton on the hemisphere.

Example 3.2.15. *Set*

$$\mathbb{S}^2_+ = \{p = (x, y, z) \in \mathbb{S}^2 : z \geq 0\}.$$

So $(\mathbb{S}^2_+, g_{\mathbb{S}^2_+})$ is a Riemannian manifold with boundary, where $g_{\mathbb{S}^2_+}$ is the metric induced from $\mathbb{S}^2$. Now, define the function $f : \mathbb{S}^2_+ \longrightarrow \mathbb{R}$ by

$$f(p) = \sin\, d_{\mathbb{S}^2}(p, \partial\mathbb{S}^2_+),$$

where the function $d_{\mathbb{S}^2}$ is given by

$$d_{\mathbb{S}^2}(p, \partial\mathbb{S}^2_+) = \inf_{q\in\partial\mathbb{S}^2_+} d_{\mathbb{S}^2}(p, q).$$

If we choose spherical coordinates $\psi(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$*, then*

$$d_{\mathbb{S}^2}(p, \partial\mathbb{S}^2_+) = \frac{\pi}{2} - \varphi,$$

where $p = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$*. It follows that the function* f *in these coordinates is given by*

$$f(p) = \cos\varphi.$$

We obtain $\dfrac{\partial f}{\partial\theta} = 0$ *and* $\dfrac{\partial f}{\partial\varphi} = -\sin\varphi$*. Since* $g_{\theta\theta} = g_{\mathbb{S}^2_+}\left(\dfrac{\partial}{\partial\theta}, \dfrac{\partial}{\partial\theta}\right)$, $g_{\theta\varphi} = g_{\mathbb{S}^2_+}\left(\dfrac{\partial}{\partial\theta}, \dfrac{\partial}{\partial\varphi}\right)$, *and* $g_{\varphi\varphi} = g_{\mathbb{S}^2_+}\left(\dfrac{\partial}{\partial\varphi}, \dfrac{\partial}{\partial\varphi}\right)$*, then* $g_{\theta\theta} = \sin^2\varphi$, $g_{\theta\varphi} = 0$*, and* $g_{\varphi\varphi} = 1$*. This implies*

$$\nabla f(p) = -\sin\varphi\frac{\partial}{\partial\varphi}.$$

Then

$$\nabla_{\frac{\partial}{\partial\theta}}\nabla f(p) = -\cos\varphi\frac{\partial}{\partial\theta} \quad \text{and} \quad \nabla_{\frac{\partial}{\partial\varphi}}\nabla f(p) = -\cos\varphi\frac{\partial}{\partial\varphi}.$$

Define $\overline{\dfrac{\partial}{\partial\theta}} = \dfrac{1}{\sin\varphi}\dfrac{\partial}{\partial\theta}$*. So the set* $\left\{\overline{\dfrac{\partial}{\partial\theta}}, \dfrac{\partial}{\partial\varphi}\right\}$ *is an orthonormal basis for* $T_p\mathbb{S}^2_+$*. Consequently,*

$$Hess_p(f)\left(\overline{\frac{\partial}{\partial\theta}}, \overline{\frac{\partial}{\partial\theta}}\right) = -\cos\varphi,$$

$$Hess_p(f)\left(\overline{\frac{\partial}{\partial\theta}}, \frac{\partial}{\partial\varphi}\right) = 0,$$

and

$$Hess_p(f)\left(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi}\right) = -\cos\varphi.$$

This implies that ∇f *is a conformal vector field, where* $Hess_p(f) = -f(p)g_{\mathbb{S}^2_+}(p)$*. Moreover,* ∇f *does not vanish on* $\partial\mathbb{S}^2_+$*.*

Let $\lambda : \mathbb{S}^2_+ \longrightarrow \mathbb{R}$ *be the function given by* $\lambda(p) = 1 + f(p)$*. Therefore, since the Ricci curvature of* $\mathbb{S}^2_+$ *is 1, then we obtain that* $(\mathbb{S}^2_+, g_{\mathbb{S}^2_+}, \nabla f, \lambda)$ *is a gradiente Ricci almost soliton with boundary, which is under the hypothesis of the Theorem 3.2.14.*

3.3 Totally Geodesic Boundaries

Let $(M, g.\nabla f, \lambda)$ be a gradient Ricci almost soliton with boundary. Now we are interested to know how the function λ affects the geometric structure of the boundary.

Definition 3.3.1. *Let* (M^n, g)*,* $n \geq 2$*, be a Riemannian manifold with boundary. Suppose that* f *is a smooth function which satisfies* (3.4) *and* ∇f *does not vanish on* ∂M*. For* *each $p \in \partial M$ define*

$$\eta(p) = \frac{\nabla f(p)}{|\nabla f(p)|}.$$

*Let $\kappa_1(p), ..., \kappa_{n-1}(p)$ be the eigenvalues of the shape operator $S_{\eta(p)}$ at p with respect $\eta(p)$. We call $\kappa_1(p), ..., \kappa_{n-1}(p)$ the **principal curvatures of** ∂M **at** p. Moreover, we call a tangent vetor $v \in T_p\partial M$ of **principal direction of** $S_{\eta(p)}$ **at** p if $S_{\eta(p)}(v) = \kappa_i(p)v$, for some $i = 1, ..., n-1$.*

We have the following proposition.

Proposition 3.3.2. *Let $(M^n, g, \nabla f, \lambda)$, $n \geq 2$, be a gradient Ricci almost soliton with boundary. Suppose f satisfies (3.4) and ∇f does not vanish on ∂M. Let $S_{\partial M}$ and Ric_M be the scalar curvature of ∂M and the Ricci tensor of M, respectively. Then ∂M is minimal and*

$$(n-1)\lambda(p) \leq S_{\partial M}(p) + Ric_M(\eta(p), \eta(p)), \tag{3.17}$$

for all $p \in \partial M$, where $\eta(p) = \dfrac{\nabla f(p)}{|\nabla f(p)|}$, if and only if ∂M is totally geodesic.

Proof. Let $\{e_1(p), ..., e_{n-1}(p)\}$ be an orthonormal basis for $T_p\partial M$ such that each $e_i(p)$ is a principal direction of $S_{\eta(p)}$ which satisfies $S_{\eta(p)}(e_i(p)) = \kappa_i(p)e_i(p)$, for all $i = 1, ..., n-1$. From definition of Ricci curvature and the Gauss equation we have, respectively,

$$\begin{aligned} \mathrm{Ric}_M(e_i(p), e_i(p)) &= \sum_{j \neq i} K_M(e_j(p), e_i(p)) + K_M(\eta(p), e_i(p)), \\ K_{\partial M}(e_j(p), e_i(p)) - K_M(e_j(p), e_i(p)) &= \kappa_j(p)\kappa_i(p), \forall j \neq i, \end{aligned}$$

where K_M and $K_{\partial M}$ are the sectional curvature of M and ∂M, respectively. From this, it follows that

$$\begin{aligned} \mathrm{Ric}_M(e_i(p), e_i(p)) &= \sum_{j \neq i} [K_{\partial M}(e_j(p), e_i(p)) - \kappa_j(p)\kappa_i(p)] + K_M(\eta(p), e_i(p)) \\ &= \mathrm{Ric}_{\partial M}(e_i(p), e_i(p)) - \left(\sum_{j \neq i} \kappa_j(p)\right)\kappa_i(p) + K_M(\eta(p), e_i(p)), \end{aligned}$$

for all $p \in \partial M$. Since the mean curvature is given by $H(p) = \kappa_1(p) + ... + \kappa_{n-1}(p)$, then

$$\mathrm{Ric}_M(e_i(p), e_i(p)) = \mathrm{Ric}_{\partial M}(e_i(p), e_i(p)) - H(p)\kappa_i(p) + \kappa_i(p)^2 + K_M(\eta(p), e_i(p)),$$

for all $p \in \partial M$. On the other hand, since $(M, g, \nabla f, \lambda)$ is a gradient Ricci almost soliton and $S_{\nabla f(p)} = |\nabla f(p)| S_{\eta(p)}$, for every $p \in \partial M$, then

$$\mathrm{Ric}_M(e_i(p), e_i(p)) = \lambda(p) - |\nabla f(p)|\kappa_i(p),$$

which implies

$$\lambda(p) - |\nabla f(p)|\kappa_i(p) = \mathrm{Ric}_{\partial M}(e_i(p), e_i(p)) - H(p)\kappa_i(p) + \kappa_i(p)^2 + K_M(\eta(p), e_i(p)).$$

Thus

$$\kappa_i(p)^2 + [|\nabla f(p)| - H(p)]\kappa_i(p) + \mathrm{Ric}_{\partial M}(e_i(p), e_i(p)) + K_M(\eta(p), e_i(p)) - \lambda(p) = 0,$$

for all $p \in \partial M$, and $i = 1, ..., n-1$. Set

$$a(p) = |\nabla f(p)| - H(p),$$

and

$$b_i(p) = \mathrm{Ric}_{\partial M}(e_i(p), e_i(p)) + K_M(\eta(p), e_i(p)) - \lambda(p).$$

Then for each i the principal curvature $\kappa_i(p)$ satisfies

$$\kappa_i(p)^2 + a(p)\kappa_i(p) + b_i(p) = 0.$$

From definition we have that

$$\begin{aligned}\sum_{i=1}^{n-1} b_i(p) &= \sum_{i=1}^{n-1}[\mathrm{Ric}_{\partial M}(e_i(p), e_i(p)) + K_M(\eta(p), e_i(p)) - \lambda(p)] \\ &= S_{\partial M}(p) + \mathrm{Ric}_M(\eta(p), \eta(p)) - (n-1)\lambda(p)).\end{aligned}$$

For each $p \in \partial M$, define $b(p) = \sum_{i=1}^{n-1} b_i(p)$. We obtain

$$\sum_{i=1}^{n-1} \kappa_i(p)^2 + a(p)H(p) + b(p) = 0.$$

If ∂M is minimal, then

$$\sum_{i=1}^{n-1} \kappa_i(p)^2 + b(p) = 0,$$

which implies $b(p) \leq 0$, for all $p \in \partial M$. Then

$$S_{\partial M}(p) + \mathrm{Ric}_M(\eta(p), \eta(p)) \leq (n-1)\lambda(p),$$

for all $p \in \partial M$. So if $(n-1)\lambda(p) \leq S_{\partial M}(p) + \mathrm{Ric}_M(\eta(p), \eta(p))$, then

$$(n-1)\lambda(p) = S_{\partial M}(p) + \mathrm{Ric}_M(\eta(p), \eta(p)), \forall p \in \partial M.$$

It follows that

$$\sum_{i=1}^{n-1} \kappa_i(p)^2 = 0.$$

Which implies that $\kappa_i(p) = 0$, for all $p \in \partial M$, and for all $i = 1, ..., n-1$. Therefore, ∂M is totally geodesic.

The converse follows from the definition of totally geodesic submanifold. ■

A consequence from the proof of the Proposition 3.4.2 is that if $(M^n, g, \nabla f, \lambda)$ is a gradient Ricci almost soliton with boundary, $n \geq 2$, such that f satisfies (3.4) and ∇f is non zero on whole ∂M, then the mean curvature H of ∂M satisfies the following equation

$$\sum_{i=1}^{n-1} \kappa_i(p)^2 + a(p)H(p) + b(p) = 0, \tag{3.18}$$

for all $p \in \partial M$, where $\kappa_i(p)$ is a principal curvature of ∂M at p, for all i, and $a(p)$ and $b(p)$ are given by

$$a(p) = |\nabla f(p)| - H(p), \quad \text{and} \quad b(p) = S_{\partial M}(p) + \mathrm{Ric}_M(\eta(p), \eta(p)) - (n-1)\lambda(p).$$

Thus we obtain that if ∂M is minimal but it is not totally geodesic, then

$$S_{\partial M}(p) + \mathrm{Ric}_M(\eta(p), \eta(p)) < (n-1)\lambda(p),$$

for all $p \in \partial M$.

3.4 Positive Mean Curvature

In the previous sections we study ridigity theorems when the boundary is minimal, but in this section we obtain a rigidity theorem for gradient Ricci almost solitons when the mean curvature of the boundary is positive.

Proposition 3.4.1. *Let $(M^n, g, \nabla f, \lambda)$, $n \geq 2$, be a gradient Ricci almost soliton manifold with boundary. If f is constant on ∂M, then every principal curvature $\kappa_i(p)$ of ∂M at p, $i = 1, ..., n-1$, satisfies*

$$\kappa_i(p)^2 + [g_p(\nabla f(p), \eta(p)) - H(p)]\kappa_i(p) + Ric_{\partial M}(e_i(p), e_i(p)) + \\ + K_M(e_i(p), \eta(p)) - \lambda(p) = 0, \tag{3.19}$$

where $Ric_{\partial M}$ and K_M is the Ricci curvature tensor of ∂M and the sectional curvature of M, respectively, $\eta(p)$ is an inward unit vector normal to the boundary at p, and $e_i(p)$ is the principal direction associated to $\kappa_i(p)$.

Proof. We have that $\nabla f(p) = g_p(\nabla f(p), \eta(p))\eta(p)$, for all $p \in \partial M$. Then

$$\mathrm{Hess}(f)(u, v) = g_p(\nabla f(p), \eta(p))g_p(\nabla_u \eta(p), v),$$

for all $p \in \partial M$, and $u, v \in T_p\partial M$. In particular, we obtain that

$$S_{\nabla f(p)}(v) = g_p(\nabla f(p), \eta(p))S_{\eta(p)}(v).$$

So, if $\kappa_i(p)$ is a principal curvature of ∂M at p, and $e_i(p)$ is an unit principal direction associated with $\kappa_i(p)$, then

$$\mathrm{Hess}(f)(e_i(p), e_i(p)) = -\kappa_i(p)g_p(\nabla f(p), \eta(p)),$$

for all $p \in \partial M$, and $i = 1, ..., n-1$. Therefore, by proceeding as in the proof of Proposition 3.4.2 we obtain (3.19). ∎

In what follows we give the statement of two theorems that we are going to use to prove the last theorem of this section.

Theorem 3.4.2 (Theorem 1 in [20].)**.** *Let M^n, $n \geq 1$, be a compact Riemannian manifold with boundary and non negative Ricci curvature. For each $p \in \partial M$, let $\eta(p)$ be the inward unit vector normal to the boundary at p. Assume that the principal curvatures of ∂M are bounded from below by a positive constant ρ. Then, the first eigenvalue $\lambda_1(\Delta_{\partial M})$ of the Laplacian $\Delta_{\partial M}$ on ∂M satisfies $\lambda_1(\Delta_{\partial M}) \geq n\rho^2$ with equality holding if and only if M is isometric to a closed Euclidean ball of radius $\frac{1}{\rho}$.*

Theorem 3.4.3 (Theorem 1 in [19].)**.** *Let M^n, $n \geq 1$, be a compact Riemannian manifold with boundary and non negative Ricci curvature. Let H be the mean curvature of ∂M. If H is positive everywhere, then*

$$\int_{\partial M} \frac{1}{H(p)} d(\partial M) \geq n vol(M).$$

The equality holds if and only if M is isometric to a Euclidean ball.

Next we have the main result of this section.

Theorem 3.4.4. *Let $(M^n, g, \nabla f, \lambda)$, $n \geq 2$, be a compact gradient Ricci almost soliton with boundary. Suppose f is constant on ∂M, and the Ricci curvature of M is non negative. If*

$$g_p(\nabla f(p), \eta(p)) < H(p), \text{ and } \lambda(p) < Ric_{\partial M}(v, v) + K_M(\eta(p), v), \tag{3.20}$$

for all $p \in \partial M$, where $\eta(p)$ is the inward unit vector normal to the boundary at p, $H(p)$

is the mean curvature of ∂M at p, and $v \in T_p\partial M$, $|v| = 1$, is a principal direction of the shape operator $S_{\eta(p)}$. Here, $Ric_{\partial M}$ and K_M denote the Ricci curvature of ∂M and the sectional curvature of M, respectively. Then, the following assertions are satisfies:

(i) The first eigenvalue $\lambda_1(\Delta_{\partial M})$ of the Laplacian on ∂M satisfies $\lambda_1(\Delta_{\partial M}) \geq (n-1)\rho^2$, for some positive constant $\rho \in \mathbb{R}$. Moreover, the equality holds if and only if M is isometric to an n-dimensional closed Euclidean ball of radius $\frac{1}{\rho}$.

(ii) The mean curvature satisfies

$$\int_{\partial M} \frac{1}{H(p)} d(\partial M) \geq n vol(M).$$

The equality holds if and only if M is isometric the a closed Euclidean ball.

Proof. (i) For all $p \in \partial M$, let $\{e_1(p), ..., e_{n-1}(p)\}$ be a basis for $T_p\partial M$, where each $e_i(p)$ is a principal direction of the shape operator $S_{\eta(p)}$ associated to the principal curvature $\kappa_i(p)$. From the hypothesis we have

$$\lambda(p) < \mathrm{Ric}_{\partial M}(e_i(p), e_i(p)) + K_M(\eta(p), e_i(p)),$$

for all $p \in \partial M$, and $i = 1, ..., n-1$. For each p and i, define

$$b_i(p) = \mathrm{Ric}_{\partial M}(e_i(p), e_i(p)) + K_M(\eta(p), e_i(p)) - \lambda(p),$$

then $b_i(p) > 0$. Set $a(p) = g_p(\nabla f(p), \eta(p)) - H(p)$, where $p \in \partial M$, and $H(p)$ is the mean curvature of ∂M at p. We have that

$$a(p)^2 - 4b_i(p) < a(p)^2.$$

From Proposition 3.5.1, every principal curvature $\kappa_i(p)$ satisfies (3.19), then

$$[g_p(\nabla f(p), \eta(p)) - H(p)]^2 - 4[\mathrm{Ric}_{\partial M}(e_i(p), e_i(p)) + K_M(\eta(p), e_i(p)) - \lambda(p)] \geq 0.$$

It follows that $a(p)^2 - 4b_i(p) \geq 0$, thus since $a(p) < 0$, we obtain

$$a(p) + \sqrt{a(p)^2 - 4b_i(p)} < 0,$$

for all $p \in \partial M$, and $i = 1, ..., n-1$. Since the principal curvature $\kappa_i(p)$ satisfies

$$\kappa_i(p) = -\frac{a(p)}{2} - \frac{\sqrt{a(p)^2 - 4b_i(p)}}{2}$$

or

$$\kappa_i(p) = -\frac{a(p)}{2} + \frac{\sqrt{a(p)^2 - 4b_i(p)}}{2}.$$

Thus $k_i(p)$ is positive for all $p \in \partial M$, and $i = 1, ..., n-1$. Then, $H > 0$ and the second fundamental form $II_{\eta(p)}$ of ∂M is positive defined. The converse is true, if $a(p) < 0$ and all principal curvatures are positives, then $b_i(p) > 0$, for all $p \in \partial M$, and $i = 1, ..., n-1$. Since M is compact, then there exists a positive constant κ_0 such that $\kappa_i(p) \geq \kappa_0$, for all $p \in \partial M$, and $i = 1, ..., n-1$. The conclusion follows from Theorem 3.5.2.

(ii) From the proof of item (i) the mean curvature $H(p) > 0$, for all $p \in \partial M$. Therefore, the conclusion follows from Theorem 3.5.3. ■

Example 3.4.5. *Let $\mathbb{B}_1^3$ be the Riemannian manifold with boundary defined by*

$$\mathbb{B}_1^3 = \{p \in \mathbb{R}^3 : |p| \leq 1\},$$

provided with the metric $\langle \cdot, \cdot \rangle$ induced from $\mathbb{R}^3$. It follows that $\partial \mathbb{B}_1^3 = \mathbb{S}^2$. Let $f : \mathbb{B}_1^3 \longrightarrow \mathbb{R}$ be the function given by

$$f(p) = \frac{1 - |p|^2}{4}.$$

It follows that

$$\nabla f(p) = -\frac{p}{2} \quad \text{and} \quad Hess(f) = -\frac{1}{2}\langle \cdot, \cdot \rangle_p,$$

for all $p \in \mathbb{B}_1^3$. Choose $\lambda = \frac{1}{2}$, then $(\mathbb{B}_1^3, \langle \cdot, \cdot \rangle, f, \lambda)$ is a gradient Ricci almost soliton with boundary. Moreover, we have that $(\mathbb{B}_1^3, \langle \cdot, \cdot \rangle, f, \lambda)$ is under the hypothesis of the Theorem 3.5.4.

3.5 Ricci Almost Soliton as a Hyperbolic Domain

So far we have been studying gradient Ricci almost solitons with boundary which are isometric to a closed hemisphere of a Euclidean sphere or a closed Euclidean ball. In this section we obtain inequalities which implies that a gradient Ricci almost soliton with boundary is isometric to a domain in some hyperbolic space.

First, we start recalling some basic definitions.

Definition 3.5.1. *We denote by $\mathbb{L}^{n+1}$ the set $\mathbb{R}^{n+1}$ provided the scalar product*

$$\langle u, v \rangle_{\mathbb{L}} = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i, \tag{3.21}$$

where $(x_0, x_1, ..., x_n), (y_0, y_1, ..., y_n) \in \mathbb{R}^{n+1}$. The space $\mathbb{L}^{n+1}$ is called the ***Lorentzian space****. For some $r > 0$, we define the* ***hyperbolic space of dimension*** *n,* ***and cur-***

vature $c = -\dfrac{1}{r^2}$, *as the subset of* $\mathbb{L}^{n+1}$ *defined by*

$$\mathbb{H}^n(c) = \{v \in \mathbb{L}^{n+1} : \langle v, v\rangle_{\mathbb{L}} = \frac{1}{c},\ \text{and } x_0 > 0\}. \tag{3.22}$$

We denote $\mathbb{H}^n(-1)$ *by* $\mathbb{H}^n$ *and we just call it* ***hyperbolic space of dimension*** n.

Definition 3.5.2. *The* ***Lorentzian norm on*** $\mathbb{L}^{n+1}$ *is the function* $|\cdot|_{\mathbb{L}} : \mathbb{L}^{n+1} \longrightarrow \mathbb{C}$ *given by*

$$|v|_{\mathbb{L}} = \sqrt{\langle v, v\rangle_{\mathbb{L}}} \tag{3.23}$$

For a treatment much more systematic about the hyperbolic space see [17].

Example 3.5.3. *Set* $P = \{(x, y, z) \in \mathbb{L}^3 : x = \sqrt{2}\}$. *Let* D^2 *be the domain of* $\mathbb{H}^2$ *defined by the subset of* $\mathbb{H}^2$ *which is "under"* P. *Let* f *be the function on* D^2 *given by*

$$f(p) = \cosh d_{\mathbb{H}^2}(p, \partial D^2),$$

where $d_{\mathbb{H}^2}(p, \partial D^2) = \inf_{q \in \partial D^2} d_{\mathbb{H}^2}(p, q)$. *If we choose the coordinates*

$$\psi(\theta, \varphi) = (\cosh\varphi, \sinh\varphi\ \cos\theta, \sinh\varphi \sin\theta),$$

then

$$d_{\mathbb{H}^2}(p, \partial D^2) = \varphi,$$

where $p = (\cosh\varphi, \sinh\varphi\cos\theta, \sinh\varphi\sin\theta)$. *It follows that the function* f *in this coordinates is given by*

$$f(p) = \cosh\varphi.$$

Thus $\dfrac{\partial f}{\partial\theta} = 0$ *and* $\dfrac{\partial f}{\partial\varphi} = \sinh\varphi$. *Since*

$$\begin{aligned}\frac{\partial}{\partial\theta} &= (0, -\sinh\varphi\sin\theta, \sinh\varphi\cos\theta),\\ \frac{\partial}{\partial\varphi} &= (\sinh\varphi, \cosh\varphi\cos\theta, \cosh\varphi\sin\theta),\end{aligned}$$

then

$$\left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right\rangle_{\mathbb{L}} = \sinh^2\varphi,\ \left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}\right\rangle_{\mathbb{L}} = 0,\ \text{and}\ \left\langle \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi},\right\rangle_{\mathbb{L}} = 1.$$

We have

$$\nabla f = \sinh\varphi\frac{\partial}{\partial\varphi}.$$

We have

$$\nabla_{\frac{\partial}{\partial\theta}}\nabla f = \sinh\varphi\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\varphi} = \sinh\varphi\left(\Gamma^{\theta}_{\theta\varphi}\frac{\partial}{\partial\theta} + \Gamma^{\varphi}_{\theta\varphi}\frac{\partial}{\partial\varphi}\right) = \cosh\varphi\frac{\partial}{\partial\theta},$$

and since $\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi} = 0$*, then*

$$\nabla_{\frac{\partial}{\partial\varphi}}\nabla f = \cosh\varphi\frac{\partial}{\partial\varphi}.$$

Thus Hess$(f) = \cosh\varphi\langle\cdot,\cdot\rangle_{\mathbb{L}}$*. If we choose* $\lambda(p) = -1 - f(p)$*, then* $(D^2, \langle\cdot,\cdot\rangle_{\mathbb{L}}, \nabla f, \lambda)$ *is a gradient Ricci almost soliton with boundary.*

The next theorem is going to be applied in the prove of the main result of this section.

Theorem 3.5.4 (Theorem 1.6 in [10].)**.** *Let* (M^n, g)*,* $n \geq 1$*, be a Riemannian manifold with boundary. Suppose*

- $Ric \geq -(n-1)g$*;*
- *there is an isometric immersion* $\psi : \partial M \longrightarrow \mathbb{H}^m$*, where* $\mathbb{H}^m$ *is the hyperbolic space of dimension* $m \geq n$*;*
- *for each* $p \in \partial M$*,* $II_p(v,v) \geq |II_p^{\mathbb{H}}(v,v)|_{\mathbb{L}}$*, for all* $v \in T_p\partial M$*. Here* II *is the second fundamental form of* ∂M *in* M *and* $II^{\mathbb{H}}$ *is the vector-valued second fundamental form of the immersion* ψ*.*

If ∂M *is simply connected, then* M *is isometric to a domain in* $\mathbb{H}^n$*.*

The following theorem allows us to describe a gradient Ricci almost soliton with boundary as a hyperbolic domain.

Theorem 3.5.5. *Let* $(M^n, g, \nabla f, \lambda)$*,* $n \geq 3$*, be a compact gradient Ricci almost soliton with simply connected boundary. Suppose* f *satisfies* (3.4) *and* ∇f *does not vanish on* ∂M*. If there exists an isometric immersion of* ∂M *into the* $\mathbb{H}^{n+n_0}$*,* $n_0 \geq 0$*, and*

$$1 - n - \lambda(p) \leq Hess(f)(v,v) \leq -|\nabla f(p)||II_p^{\mathbb{H}}(v,v)|_{\mathbb{L}}, \tag{3.24}$$

for all $p \in \partial M$*,* $v \in T_p\partial M$*,* $|v|_{\mathbb{L}} = 1$*. Here,* $II^{\mathbb{H}}$ *denotes the vector-valued second fundamental form of* ∂M *in* $\mathbb{H}^{n+n_0}$*, then* M *is isometric to a domain in* $\mathbb{H}^n$*.*

Proof. Since $g_p(S_{\nabla f(p)}(u), v) = -\text{Hess}(f)(u,v)$, for any $u, v \in T_p\partial M$, then

$$g_p(S_{\nabla f(p)}(v), v) \geq |\nabla f(p)|_{\mathbb{L}}|II_p^{\mathbb{H}}(v,v)|_{\mathbb{L}},$$

for all $v \in T_p\partial M$, $|v|_{\mathbb{L}} = 1$, which implies that $II_{\nabla f(p)}(v,v) \geq |\nabla f(p)|_{\mathbb{L}}|II_p^{\mathbb{H}}(v,v)|_{\mathbb{L}}$, for all $p \in \partial M$, and $v \in T_p\partial M$, $|v|_{\mathbb{L}} = 1$. Where $II_{\nabla f(p)}$ is the second fundamental form of ∂M

at p with respect $\nabla f(p)$ in M. On the other hand, we have that

$$\lambda(p) + \mathrm{Hess}(f)(v,v) \geq -(n-1),$$

for all $p \in \partial M$, and $v \in T_p\partial M$, $|v|_{\mathbb{L}} = 1$. Since $(M, g, \nabla f, \lambda)$ is a gradient Ricci almost soliton, then $\mathrm{Ric} \geq -(n-1)g$. The conclusion follows from Theorem 3.5.4. ■

Example 3.5.6. *Set* $D^3 = \{(x,y,z,w) \in \mathbb{H}^3 : x \leq \sqrt{2}\}$. *If we choose* $f(p) = -\cosh d_{\mathbb{H}}(p, \partial D^3)$ *and* $\lambda(p) = -1 + f(p)$, *then* $(D^3, \langle\cdot,\cdot\rangle_{\mathbb{L}}, \nabla f, \lambda)$ *is a gradient Ricci almost soliton.*

Chapter 4

Gradient Ricci Almost Solitons on Warped Products with Boundary

Our goal in this chapter is to describe gradient Ricci almost solitons on warped products $(M = B \times_h F, g, \nabla f, \lambda)$, where B is a Riemannian manifold with boundary, and F has no boundary. In order to do so, we shall use the rigidity theorems that we obtained in Chapter 3. Moreover, we apply some identities obtained by Borges and Tenenblat in [2], namely, Theorem 2.1 and Theorem 2.3. We divide this chapter in two sections. In the first section we obtain basic properties for gradient Ricci almost solitons on warped product with boundary, and then we give the statement of Theorem 2.1 in [2], thus we obtain properties of the topology and geometry of the basis of $M = B \times_h F$, and its boundary. By using these properties we conclude that the gradient vector field of the warping function h is a Killing vector field. In the second section, we use Theorem 2.3 in [1], Theorem 3.4.2, Theorem 3.8.2, and Theorem 3.9.3 to characterize the basis of the gradient Ricci almost solitons on warped product, where the it is compact. At the last section we give two examples of non trivial gradient Ricci almost solitons on warped product.

4.1 Some properties of gradient Ricci almost solitons

The first Proposition of this section is a simple and very useful result which show us how decompose a smooth function on a warped product as the sum of a function constant on the fiber and a function constant on the basis.

Proposition 4.1.1 (Proposition 4.1 in [2].). *Let (B, g_B) and (F, g_F) be Riemannian manifolds, where B is a smooth manifold with boundary and F is a smooth manifold without boundary. Let $(M = B \times_h F, g)$ be an warped product. If $(M, g, \nabla f, \lambda)$ is a gradient Ricci* *almost soliton, then*

$$f = \Lambda + h\Phi, \tag{4.1}$$

where $\Lambda : B \longrightarrow \mathbb{R}$ *and* $\Phi : F \longrightarrow \mathbb{R}$ *are smooth functions.*

Proof. From (2.18) and (2.27) we obtain

$$-\frac{1}{h}X(h)U(f) + X(U(f)) = 0,$$

for all horizontal vector field X and vertical vector field U. Since

$$X(U(fh^{-1})) = X(h^{-1}U(f)) = -\frac{1}{h^2}X(h)U(f) + \frac{1}{h}X(U(f)),$$

then $X(U(fh^{-1})) = 0$, for all horizontal vector field X and vertical vector field U. which implies there exist smooth functions $\overline{\Lambda} : B \longrightarrow \mathbb{R}$ and $\Phi : F \longrightarrow \mathbb{R}$ such that $fh^{-1} = \overline{\Lambda} + \Phi$. Therefore,

$$f = \Lambda + h\Phi,$$

where $\Lambda = h\overline{\Lambda}$. ■

From (2.17), (2.19) and (4.1) we obtain the following lemma.

Lemma 4.1.2. *Let* (B^m, g_B) *and* (F^k, g_F) *be Riemannian manifolds, where* B *is a smooth manifold with boundary and* F *is a smooth manifold without boundary. Let* $(M = B \times_h F, g)$ *be an warped product. If* $(M, g, \nabla f, \lambda)$ *is a gradient Ricci almost soliton, then*

$$\begin{aligned} Ric_B &= \lambda g_B + \frac{k}{h}Hess_B(h) + Hess_B(\Lambda) + \Phi Hess_B(h), && (4.2)\\ Ric_F &= [\lambda h^2 - h\Delta_B h + (k-1)|\nabla_B h|^2 + hg_B(\nabla_B\Lambda, \nabla_B h)]g_F + h|\nabla_B h|^2\Phi g_F + \\ &\quad + hHess_F(\Phi), && (4.3) \end{aligned}$$

where $\Lambda : B \longrightarrow \mathbb{R}$ *and* $\Phi : F \longrightarrow \mathbb{R}$ *are smooth functions which satifies* (4.1).

The following corollary is a more general version of Theorem 2.2.7.

Corollary 4.1.3. *Let* (B^m, g_B) *and* (F^k, g_F), $k \geq 3$ *be Riemannian manifolds such that* B *is a manifold with boundary,* F *with no boundary. Let* $(M = B \times_h F, g, \nabla f, \lambda)$ *be a gradient Ricci soliton. Assume the maximum point of* h *is an interior point. If* f *is not constant on* F*, then the warped product* $M = B \times_h F$ *is a Riemannian product, i.e.,* h *is constant.*

Proof. Let p_0 be any point in B. Consider an unit horizontal vector field X locally defined in a neighborhood of p_0, namely, $D \subset B$. By using (4.2) we have

$$\mathrm{Ric}_B(X,X) = \lambda + \frac{k}{h}\mathrm{Hess}_B(h)(X,X) + \Phi\mathrm{Hess}_B(h)(X,X),$$

then for any point $(p,q) \in D \times F$, we have

$$\Phi(q)\mathrm{Hess}_B(h)(X,X)(p) = \mathrm{Ric}_B(X,X)(p) - \lambda - \frac{k}{h(p)}\mathrm{Hess}_B(h)(X,X)(p). \tag{4.4}$$

Since Φ is not constant, then there exists a vertical vector field U defined in an open subset $G \subset F$ such that $U(\Phi) \neq 0$ in G. By applying U in (4.4) we get to

$$U(\Phi)\mathrm{Hess}_B(h)(X,X) = 0,$$

in $D \times G$. Thus,

$$\mathrm{Hess}_B(h)(X,X)(p_0) = 0,$$

for any $p_0 \in B$ and any unit horizontal vector field X defined in a neighborhood of p_0. It follows that

$$\Delta_B h = 0, \quad \text{in} \quad B,$$

From the Maximum Principle we conclude that h is constant. ∎

For more applications of the Laplacian on the Riemannian manifolds with boundary we indicate [21].

Proposition 4.1.4. *Let (B^m, g_B) and (F^k, g_F) be Riemannian manifolds, where B is a smooth manifold with boundary and F with no boundary. Let $(M = B \times_h F, g)$ be an warped product. If $(M, g, \nabla f, \lambda)$ is a gradient Ricci almost soliton and the warped product $M = B \times_h F$ is a Riemannian product, then λ is constant.*

Proof. Let c_0 be the positive constant such that $h \equiv c_0$. From (4.2) and (4.3) we have that

$$\begin{aligned} \mathrm{Ric}_B &= \lambda g_B + \mathrm{Hess}_B(\Lambda), \\ \mathrm{Ric}_F &= c_0^2\lambda g_F + c_0\mathrm{Hess}_F(\Phi). \end{aligned}$$

Let $(p_0, q_0) \in B \times F$ be any point. Take X any unit horizontal vector field defined on an open set $D_0 \subset B$ and U an unit vertical vector field defined on an open set $G_0 \subset F$ such that $(p_0, q_0) \in D_0 \times G_0$. For each $(p, q) \in D_0 \times G_0$ it follows that

$$\begin{aligned}\lambda(p, q) &= \mathrm{Ric}_B(X, X)(p) - \mathrm{Hess}_B(\Lambda)(X, X)(p),\\ \lambda(p, q) &= \frac{1}{c_0^2}\mathrm{Ric}_F(U, U)(q) - \frac{1}{c_0}\mathrm{Hess}_F(\Phi)(U, U)(q).\end{aligned}$$

By setting $\lambda(p_0, q_0) = \lambda_0$, it follows that $\lambda \equiv \lambda_0$ on $D_0 \times G_0$. So we have proved that for each $(p_0, q_0) \in B \times F$ there exists a neighborhood $D_0 \times G_0 \in B \times F$ of (p_0, q_0) such that $\lambda(p, q) = \lambda(p_0, q_0)$. Let $x_0 \in \mathbb{R}$ be such that $\lambda^{-1}(x_0) \neq \varnothing$. Set

$$(B \times F)(x_0) = \{(p, q) \in B \times F : \lambda(p, q) = x_0\}.$$

We have that $(B \times F)(x_0)$ is closed because $(B \times F)(x_0) = \lambda^{-1}(x_0)$. And $(B \times F)(x_0)$ is open because for each point $(p, q) \in (B \times F)(x_0)$ there exists a neighborhood $D_p \times G_P \subset B \times F$ of (p, q) such that $\lambda \equiv x_0$ in $D_p \times G_P$. As $B \times F$ is connected, then $\lambda \equiv x_0$. ■

The following theorem provide us with important identities that we shall use in the next section.

Theorem 4.1.5 (Theorem 2.1 in [2].). *Let (B^m, g_B) and (F^k, g_F) be Riemannian manifolds, where B is with boundary and F with no boundary. Let $(M = B \times_h F, g)$ be a non trivial warped product, i.e., h is not constant. Then $(M, g, \nabla f, \lambda)$ is a gradient Ricci almost soliton, with f non constant on F if and only if $f = \Lambda + h\Phi$, where $\Lambda : B \longrightarrow \mathbb{R}$ and $\Phi : F \longrightarrow \mathbb{R}$ are smooth functions such that*

$$\begin{aligned}Hess_B(h) &= a_0 h g_B, & (4.5)\\ Ric_B &= -\left[\frac{1}{h}g_B(\nabla_B h, \nabla_B \Lambda) - \frac{a}{h} + (m-1)a_0\right] g_B + Hess_B(\Lambda), & (4.6)\\ Hess_F(\Phi) &= (-c\Phi - a)g_F, & (4.7)\\ Ric_F &= c(k-1)g_F, & (4.8)\end{aligned}$$

for some constants $a_0, a, c \in \mathbb{R}$. Moreover, the function λ is given by

$$\lambda = \frac{1}{h}g_B(\nabla_B h, \nabla_B \Lambda) - \frac{a}{h} + (m + k - 1)a_0 - a_0 h\Phi, \tag{4.9}$$

and the constants a_0 and c are related to h by the equation

$$|\nabla_B h|^2 - a_0 h^2 = c. \tag{4.10}$$

Proof. See [2]. ■

By using the Theorem 4.1.5 and the results that we obtained in Chapter 3, we have the following corollary.

Corollary 4.1.6. *Let (B^m, g_B) and (F^k, g_F) be Riemannian manifolds, where B is oriented, connected with boundary and F with no boundary. Let $(M = B \times_h F, g)$ be an warped product, where h is a positive function which satisfies* (3.4)*. Suppose $(M, g, \nabla f, \lambda)$ is a gradient Ricci almost soliton such that f is non constant on F. Then, the following assertions are satisfied:*

(i) If B is compact, then $\nabla_B h \neq 0$ and $H_{\partial B} > 0$. In particular, $II_{\nabla_B h}$ is positive.

(ii) Assume $\nabla_B h \neq 0$ on ∂B. If $II_{\nabla_B h} \leq 0$, then $|\nabla_B h|$ is constant and B is non compact. In particular, ∂B is totally geodesic.

(iii) Let Λ be the funcion which satisfies (4.1)*. Suppose Λ is constant on ∂B, and $m \geq 3$. Then, $g_B(\nabla_B h, \nabla_B \Lambda)$ is constant in B if and only if B is Ricci flat and $\nabla_B \Lambda$ is a Killing vector field.*

Proof. (i) Suppose that B is compact. So, take the trace in (4.5). We obtain

$$\Delta_B h = -a_0 m h.$$

Since B is compact, then $0 < \lambda_1(\Delta_B) = -a_0 m$, where $\lambda_1(\Delta_B)$ is the first eigenvalue of Δ_B. Let $\eta(p)$ be an unit normal vector to the boundary ∂B at p. So

$$\nabla_B h(p) = g_p(\nabla_B h(p), \eta(p))\eta(p),$$

which implies that

$$\mathrm{Hess}_B(h)(u, v) = g_p(\nabla_B h(p), \eta(p))g_p(\nabla_u \eta(p), v),$$

for all $u, v \in T_p\partial B$. From (4.5) we obtain

$$a_0 h(p) g_p(u, v) = g_p(\nabla_B h(p), \eta(p))g_p(\nabla_u \eta(p), v), \forall u, v \in T_p\partial B,$$

for all $p \in \partial B$. Since $a_0 h(p) < 0$ for every $p \in B$, then

$$g_p(\nabla_B h(p), \eta(p))g_p(\nabla_v \eta(p), v) < 0,$$

for all $v \in T_p\partial B$, $|v| = 1$. Therefore, $\nabla_B h \neq 0$ on ∂B.

On the other hand, from Lemma 3.2.6 we have

$$H_{\partial B}(p)|\nabla_B h(p)| = -(m-1)a_0 h(p),$$

for all $p \in \partial B$, where $H_{\partial B}$ is the mean curvature of ∂B. Since $|\nabla_B h| \neq 0$ on ∂B, then

$H_{\partial B} > 0$. Moreover, since $II^B_{\nabla_B h(p)}(u,v) = -\text{Hess}(h)(u,v)$, for all $u, v \in T_p\partial B$, then

$$II^B_{\nabla_B h(p)}(v,v) = -a_0 h(p),$$

for all $p \in \partial B$, and $v \in T_p\partial B$, $|v| = 1$.

(ii) Since h satisfies (3.4), and from (4.5) $\nabla_B h$ is a conformal vector field with $\nabla_B h \neq 0$ on ∂B, then from Proposition 3.2.12 ∂B is umbilical. The second fundamental form with respect to $\nabla_B h$ is given by $II^B_{\nabla_B h} = -\text{Hess}_B(h)$. From item (i) we have that B is non compact. Since

$$\int_B \Delta_B h dB = -\int_B \text{div}(\nabla_B h) dB = -\int_{\partial B} g_B(\nabla_B h, \eta) d(\partial B) \geq 0,$$

where η is a outward normal vector to the boundary, and $\text{Hess}_B(h) = a_0 h g_B$, then $a_0 h \leq 0$, which implies $II_{\nabla_B h} \geq 0$. On the other hand, $II_{\nabla_B h} \leq 0$ from the hypothesis. Therefore, $II_{\nabla_B h} \equiv 0$ on ∂B, which implies that ∂B is totally geodesic and $\text{Hess}_B(h) \equiv 0$ on B. Since

$$\text{Hess}_B(h)(X, \nabla_B h) = \frac{1}{2} X(|\nabla_B h|^2),$$

for every horizontal vector field X, then $|\nabla_B h|$ is constant.

(iii) For any $p \in \partial B$, we have $\nabla_B \Lambda(p) \in (T_p \partial B)^\perp$. Thus

$$\nabla_B \Lambda(p) = g_B(\nabla_B \Lambda(p), \nabla_B h(p)) \nabla_B h(p),$$

on ∂B.

If $g_B(\nabla_B \Lambda, \nabla_B h)$ is constant, then

$$\nabla_B \Lambda(p) = g_B(\nabla_B \Lambda(p), \nabla_B h(p)) \nabla_B h(p),$$

for all $p \in B$. Thus

$$\nabla_{X_p} \nabla_B \Lambda(p) = [X_p g_B(\nabla_B \Lambda(p), \nabla_B h(p))] \nabla_B h(p) +$$

$$+ g_B(\nabla_B \Lambda(p), \nabla_B h(p)) \nabla_B h(p) \nabla_{X_p} \nabla_B h(p).$$

Then

$$\nabla_{X_p} \nabla_B \Lambda(p) = g_B(\nabla_B \Lambda(p), \nabla_B h(p)) \nabla_B h(p) \nabla_{X_p} \nabla_B h(p),$$

which implies

$$\text{Hess}_B(\Lambda)(u,v) = g_B(\nabla_B \Lambda(p), \nabla_B h(p)) \text{Hess}_B(h)(u,v), u, v \in T_pB.$$

Since $\text{Hess}_B(h) \equiv 0$ in B, then $\text{Hess}_B(\Lambda) \equiv 0$ in B. So $\nabla_B \Lambda$ is a Killing vector field. From

(4.6) one shows that

$$\mathrm{Ric}_B = -\left[\frac{1}{h(p)} g_B(\nabla_B \Lambda(p), \nabla_B h(p)) - \frac{a}{h(p)}\right] g_B.$$

Thus, there exists a constant $\alpha_0 \in \mathbb{R}$ such that

$$g_B(\nabla_B \Lambda(p), \nabla_B h(p)) - a = \alpha_0 h(p)$$

Since $g_B(\nabla_B \Lambda, \nabla_B h)$ and a are constant, and h is not constant, then $\alpha_0 = 0$. Therefore, B is Ricci flat.

Conversely, if B is Ricci flat and $\nabla_B \Lambda$ is a Killing vector field, then from (4.6) we obtain that $g_B(\nabla_B \Lambda, \nabla_B h)$ is constant in B. ∎

4.2 Compact Ricci Almost Solitons on Warped Product

In this section we study gradient Ricci almost solitons $(M = B \times_h F, g, \nabla f, \lambda)$, where the function f is not constant on the fiber F.

The following theorem follows from Theorem 2.3 in [2], which was proved in local coordinates.

Theorem 4.2.1. *Let (B^m, g_B) and (F^k, g_F) be Riemannian manifolds, where B is a smooth manifold with boundary and F is a smooth manifold without boundary. Let $(M = B \times_h F, g)$ be a non trivial warped product, i.e., h is not constant. Then $(M, g, \nabla f, \lambda)$ is a gradient Ricci almost soliton, with f constant on F if and only if*

$$\begin{aligned} Ric_B &= \lambda g_B + \frac{k}{h} Hess_B(h) + Hess_B(f), && (4.11)\\ \lambda h^2 &= h g_B(\nabla_B f, \nabla_B h) - (k-1)|\nabla_B h|^2 + h\Delta_B h + c(k-1), && (4.12)\\ Ric_F &= c(k-1) g_F, && (4.13) \end{aligned}$$

for some constant $c \in \mathbb{R}$.

Proof. See [2]. ∎

Next, we have the main theorem of this chapter. It provide us characterizations for a gradient Ricci almost soliton $(M = B \times_h F, g, \nabla f, \lambda)$, where the basis B is a compact Riemannian manifold with boundary.

Theorem 4.2.2. *Let (B^m, g_B) and (F^k, g_F), $m \geq 2$ be Riemannian manifolds, where B is oriented, connected, compact, and with connected boundary, and F with no boundary.*

Let $(M = B \times_h F, g)$ be an warped product, where h is a positive function which satisfies (3.4). *Suppose $(M, g, \nabla f, \lambda)$ is a gradient Ricci almost soliton such that f is constant on F. Moreover, suppose that $\nabla_B h$ is a conformal vector field. Then, the following assertions are satisfies:*

(i) If f satisfies (3.4)*, $\nabla f \neq 0$ on ∂M, the scalar curvature of B is positive, and Hess$(f) = \xi\, g$, where $\xi \leq 0$ on ∂B, then there exists a positive constant $\rho \in \mathbb{R}$ such that the first eigenvalue $\lambda_1(\Delta_B)$ of the Laplacian on B satisfies the inequality $\lambda_1(\Delta_B) \geq m\rho^2$. Moreover, the equality holds if and only if B is isometric to a closed hemisphere of the Euclidean sphere $\mathbb{S}^m(\rho^2)$ of radius $\dfrac{1}{\rho}$.*

(ii) Let $H_{\partial B}$, $Ric_{\partial B}$, and K_B be the mean curvature, the Ricci curvature and the sectional curvature of B, respectively. Suppose f constant on ∂M, and the Ricci curvature of B is non negative. Assume $H_{\partial B}$, $Ric_{\partial B}$, and K_B are such that the following system

$$\begin{aligned} k g_B(\nabla_B h(p), \eta(p)) + h(p)|\nabla f(p)| &< h(p) H_{\partial B}(p), \\ \lambda(p) &< Ric_{\partial B}(v, v) + K_B(\eta(p), v), \end{aligned} \tag{4.14}$$

is satisfied, for all $p \in \partial B$, where $\eta(p) = \frac{\nabla f(p)}{|\nabla f(p)|}$. Then the first eigenvalue $\lambda_1(\Delta_{\partial B})$ of the Laplacian on ∂B satisfies $\lambda_1(\Delta_{\partial B}) \geq (m-1)\rho^2$, for some positive constant $\rho \in \mathbb{R}$. Moreover, the equality holds if and only if B is isometric to an m-dimensional closed Euclidean ball of radius $\dfrac{1}{\rho}$.

(iii) Let f, $H_{\partial B}$, $Ric_{\partial B}$, and K_B be as in item (ii). If $H_{\partial B}$, $Ric_{\partial B}$, and K_B are such that (4.14) *is satisfied, then the mean curvature $H_{\partial B}$ satisfies*

$$\int_{\partial B} \frac{1}{H_{\partial B}(p)} d(\partial B) \geq m vol(B).$$

Moreover, the equality holds if and only if B is isometric to a closed Euclidean ball.

(iv) Suppose that ∂B is simply connected, $m \geq 3$, $\nabla f \neq 0$ on ∂M, and f satisfies (3.4)*. If there exists an isometric immersion of ∂B into the $\mathbb{H}^{m+m_0}$, $m_0 \geq 0$, and the Hessian of f is such that*

$$\begin{aligned} \frac{(1 - n - \lambda(p))h(p)|\nabla f(p)|}{k\, g_B(\nabla_B h(p), \eta(p)) + h(p)|\nabla f(p)|} &\leq Hess(f)(v, v) \leq \\ &\leq -|\nabla f(p)|_{\mathbb{L}} |II_p^{\mathbb{H}}(v, v)|_{\mathbb{L}}, \end{aligned} \tag{4.15}$$

is satisfied, then B is isometric to a m-dimensional hyperbolic domain. Here, $II^{\mathbb{H}}$ denotes the vector-valued second fundamental form of ∂B in $\mathbb{H}^{m+m_0}$.

Proof. To prove item (i) we shall act as in the prove of Theorem 3.2.14. Since f is constant on F, $\nabla f \neq 0$, and $\text{Hess}(f) = \xi\, g$, then from Lemma 3.2.10 we obtain that

$$H_{\partial B}(p)|\nabla f(p)| = -(n-1)\xi(p),$$

for all $p \in \partial B$. It follows that $H_{\partial B} \geq 0$. Since $(M, g, \nabla f, \lambda)$ is a gradient Ricci almost soliton, then from (4.11) there exists a positive constant ρ such that

$$\text{Ric}_B \geq (m-1)\rho^2.$$

The conclusion follows from Theorem 3.2.13.

To prove item (ii), we proceed as in the proof of Proposition 3.3.2, consequently we obtain

$$\text{Ric}_B(e_i(p), e_i(p)) = \lambda(p) - \frac{k\, g_B(\nabla_B h(p), \eta(p)) + h(p)|\nabla f(p)|}{h(p)}\kappa_i(p),$$

for all $p \in \partial B$, where $e_i(p)$ is the i-th principal direction associated to the i-th principal curtavure $\kappa_i(p)$ at p.

By acting as in Proposition 3.4.1 we obtain

$$\kappa_i(p)^2 + \frac{k\, g_B(\nabla_B h(p), \eta(p)) + h(p)|\nabla f(p)| - h(p)H(p)}{h(p)}\kappa_i(p) + \text{Ric}_{\partial B}(e_i(p), e_i(p)) + \\ + K_B(e_i(p), \eta(p)) - \lambda(p) = 0,$$

for all $p \in \partial B$. The conclusion follows by following the same steps in the proof of the Theorem 3.4.4.

Since the statement of item (iii) is basically the same as item (ii) of Theorem 3.4.4, then the conclusion follows.

From (4.11) and(4.15) we have that

$$\begin{aligned} II_{\nabla f(p)}(v, v) &\geq |\nabla f(p)||II_p^{\mathbb{H}}(v, v)|_{\mathbb{L}}, \\ \text{Ric}_B(v, v) &\geq -(m-1), \end{aligned}$$

for all $p \in \partial B$, $v \in T_p\partial B$, $|v| = 1$, where $II_{\nabla f(p)}$ denotes the second fundamental form at p with respect $\nabla f(p)$. The conclusion of item (iv) follows from Theorem 3.5.4. ■

4.3 Some examples

In this section we compute two non trivial examples of gradient Ricci almost solitons with boundary on warped product.

Example 4.3.1. *Let c_1, c_2 be positive constants. For $i = 1, 2$, let $\mathbb{S}^2(c_i)$ be the Euclidean* *sphere of radius* $\frac{1}{\sqrt{c_i}}$*. Set*

$$\mathbb{S}^2_+(c_i) = \{(x, y, z) \in \mathbb{S}^2(c_i) : z \geq 0\},$$

for each $i = 1, 2$*. For any* $r \in \left(0, \frac{1}{\sqrt{c_1}}\right)$*, define the set*

$$D^2_r = \{(x, y, z) \in \mathbb{S}^2_+(c_1) : z \geq r\}.$$

We shall consider the warped product

$$(M = D^2_r \times_h \mathbb{S}^2(c_2), g = g_{\mathbb{S}^2_+(c_1)} + h(p)^2 g_{\mathbb{S}^2(c_2)}),$$

where

$$h(p) = \sqrt{\frac{c_2}{c_1}} \sin[\sqrt{c_1} d_{\mathbb{S}^2_+(c_1)}(p, \partial\mathbb{S}^2_+(c_1))].$$

Let TD^2_r *and* $T\mathbb{S}^2(c_2)$ *be the tangent bundle of* D^2_r *and* $\mathbb{S}^2(c_2)$*, respectively. For all* $X, Y \in TD^2_r$ *and* $V, W \in T\mathbb{S}^2(c_2)$*, we have*

$$\begin{aligned} Ric_M(X, Y) &= Ric_{D^2_r(c_1)} - \tfrac{2}{h} Hess(h)(X, Y), \\ Ric_M(X, V) &= 0, \\ Ric_M(V, W) &= Ric_{\mathbb{S}^2(c_2)} - [-h\Delta h + |\nabla h|^2] g_{\mathbb{S}^2(c_2)}. \end{aligned}$$

If we choose the coordinates $\psi(\theta, \varphi) = \frac{1}{\sqrt{c_1}}(\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi)$*, then*

$$\begin{aligned} \frac{\partial}{\partial\theta} &= \frac{1}{\sqrt{c_1}}(-\sin\varphi \sin\theta, \sin\varphi \cos\theta, 0), \\ \frac{\partial}{\partial\varphi} &= \frac{1}{\sqrt{c_1}}(\cos\varphi \cos\theta, \cos\varphi\ \sin\theta, -\sin\varphi). \end{aligned}$$

Since $(g_{\mathbb{S}^2_+(c_1)})_{\theta\theta} = \frac{1}{c_1} \sin^2\varphi$*,* $(g_{\mathbb{S}^2_+(c_1)})_{\theta\varphi} = 0$*, and* $(g_{\mathbb{S}^2_+(c_1)})_{\varphi\varphi} = \dfrac{1}{c_1}$*, then*

$$d_{\mathbb{S}^2_+(c_1)}(p, \partial\mathbb{S}^2_+(c_1)) = \frac{1}{\sqrt{c_1}} \left(\frac{\pi}{2} - \varphi\right).$$

Thus, $h(\psi(\theta, \varphi)) = \sqrt{\frac{c_2}{c_1}} \cos\varphi$*. From now on, we shall consider* h *on the coordinates* ψ*. From the definition, we have*

$$\nabla h = (g_{\mathbb{S}^2_+(c_1)})^{\theta\theta} \frac{\partial h}{\partial\theta} \frac{\partial}{\partial\theta} + (g_{\mathbb{S}^2_+(c_1)})^{\theta\varphi} \frac{\partial h}{\partial\varphi} \frac{\partial}{\partial\theta} + (g_{\mathbb{S}^2_+(c_1)})^{\varphi\theta} \frac{\partial h}{\partial\theta} \frac{\partial}{\partial\varphi} + (g_{\mathbb{S}^2_+(c_1)})^{\varphi\varphi} \frac{\partial h}{\partial\varphi} \frac{\partial}{\partial\varphi},$$

which implies $\nabla h = -\sqrt{c_1 c_2}\,\sin\varphi \frac{\partial}{\partial\varphi}$. *It follows that*

$$\nabla_{\frac{\partial}{\partial\theta}} \nabla h = -\sqrt{c_1 c_2}\,\sin\varphi \nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\varphi}.$$

Since $\nabla_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\varphi} = \Gamma^{\theta}_{\theta\varphi}\frac{\partial}{\partial\theta} + \Gamma^{\varphi}_{\theta\varphi}\frac{\partial}{\partial\varphi}$, *where* $\Gamma^{\theta}_{\theta\varphi} = cotg\,\varphi$, *and* $\Gamma^{\varphi}_{\theta\varphi} = 0$, *then*

$$\nabla_{\frac{\partial}{\partial\theta}} \nabla h = -\sqrt{c_1 c_2}\,\cos\varphi \frac{\partial}{\partial\theta}.$$

Since $\nabla_{\frac{\partial}{\partial\varphi}} \frac{\partial}{\partial\varphi} = 0$, *then* $\nabla_{\frac{\partial}{\partial\varphi}} \nabla h = -\sqrt{c_1 c_2}\,\cos\varphi \frac{\partial}{\partial\varphi}$. *Therefore,* $Hess(h) = -c_1 h g_{\mathbb{S}^2_+(c_1)}$, *which implies* $\Delta h = 2c_1 h$. *Then*

$$-h\Delta h + |\nabla h|^2 = -2c_2\,\cos^2\varphi + c_2\sin^2\varphi = -3c_2\cos^2\varphi + c_2.$$

We obtain

$$\begin{aligned} Ric_M(X,Y) &= 3c_1 g_{\mathbb{S}^2_+(c_1)}(X,Y), \\ Ric_M(V,W) &= 3c_2\cos^2\varphi\, g_{\mathbb{S}^2(c_2)}(V,W). \end{aligned}$$

Now, we shall consider functions $f, \lambda : M \longrightarrow \mathbb{R}$ *given by*

$$\begin{aligned} f(p,q) &= \frac{1}{c_1}\sin[\sqrt{c_1} d_{\mathbb{S}^2_+(c_1)}(p, \partial\mathbb{S}^2_+(c_1))]\sin[\sqrt{c_2} d_{\mathbb{S}^2(c_2)}(q, \partial\mathbb{S}^2_+(c_2))], \\ \lambda(p,q) &= \sin[\sqrt{c_1} d_{\mathbb{S}^2_+(c_1)}(p, \partial\mathbb{S}^2_+(c_1))]\sin[\sqrt{c_2} d_{\mathbb{S}^2(c_2)}(q, \partial\mathbb{S}^2_+(c_2))] + 3c_1. \end{aligned}$$

We have

$$\begin{aligned} Hess(f)(X,Y) &= -(\nabla^D_X Y)f + (XY)f, \\ Hess(f)(X,V) &= -\frac{1}{h}X(h)V(f) + (XV)f, \\ Hess(f)(V,W) &= hg_{\mathbb{S}^2_+(c_1)}(\nabla_D f, \nabla h) g_{\mathbb{S}^2(c_2)}(V,W) - (\nabla^{\mathbb{S}^2(c_2)}_V W)f + (VW)f, \end{aligned}$$

where ∇^D *and* $\nabla^{\mathbb{S}^2(c_2)}$ *denote the Levi-Civita conections on* D^2_r *and* $\mathbb{S}^2(c_2)$, *respectively. Moreover, set*

$$\nabla_D f = \sum_{i,j\in\{\theta,\varphi\}} (g_{\mathbb{S}^2_+(c_1)})^{ij}\frac{\partial f}{\partial j}\frac{\partial}{\partial i}.$$

On $\mathbb{S}^2(c_2)$, *we shall consider the coordinates*

$$\Theta(\xi,\zeta) = \frac{1}{\sqrt{c_2}}(\sin\xi\,\cos\zeta, \sin\xi\sin\zeta, \cos\xi).$$

Thus

$$\begin{aligned} \frac{\partial}{\partial\zeta} &= \frac{1}{\sqrt{c_2}}(-\sin\xi\,\sin\zeta, \sin\xi\cos\zeta, 0), \\ \frac{\partial}{\partial\xi} &= \frac{1}{\sqrt{c_2}}(\cos\xi\,\cos\zeta, \cos\xi\sin\zeta, -\sin\xi). \end{aligned}$$

Define $(g_{\mathbb{S}^2(c_2)})_{\zeta\zeta} = g_{\mathbb{S}^2(c_2)}\left(\frac{\partial}{\partial\zeta},\frac{\partial}{\partial\zeta}\right)$, $(g_{\mathbb{S}^2(c_2)})_{\zeta\xi} = g_{\mathbb{S}^2(c_2)}\left(\frac{\partial}{\partial\zeta},\frac{\partial}{\partial\xi}\right)$, *and* $(g_{\mathbb{S}^2(c_2)})_{\xi\xi} = g_{\mathbb{S}^2(c_2)}\left(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\xi}\right)$. *Then* $(g_{\mathbb{S}^2(c_2)})_{\zeta\zeta} = \frac{1}{c_2}\sin^2\xi$, $(g_{\mathbb{S}^2(c_2)})_{\zeta\xi} = 0$, *and* $(g_{\mathbb{S}^2(c_2)})_{\xi\xi} = \frac{1}{c_2}$. *On the coordinates* $\Theta(\xi,\zeta)$ *we have*

$$d_{\mathbb{S}^2(c_2)}(q,\partial\mathbb{S}^2_+(c_2)) = \frac{1}{\sqrt{c_2}}\left(\frac{\pi}{2}-\xi\right).$$

Therefore, $f(p,q) = \frac{1}{c_1}\cos\varphi\,\cos\xi$ *and* $\lambda(p,q) = \cos\varphi\,\cos\xi + 3c_1$. *Since* $f(p,q) = \Lambda(p) + h(p)\Phi(q)$, *then* $Hess(f)(X,V) = 0$, *for all* $X \in TD^2_r$ *and* $V \in T\mathbb{S}^2(c_2)$. *Moreover,*

$$\nabla_D f = -\sin\varphi\cos\xi\frac{\partial}{\partial\varphi}.$$

Since $\nabla^{\mathbb{S}^2(c_2)}_{\frac{\partial}{\partial\zeta}}\frac{\partial}{\partial\xi} = cotg\,\xi\frac{\partial}{\partial\zeta}$, *and* $\nabla^{\mathbb{S}^2(c_2)}_{\frac{\partial}{\partial\xi}}\frac{\partial}{\partial\xi} = 0$, *then*

$$-(\nabla^{\mathbb{S}^2(c_2)}_V W)f + (VW)f = -c_2 f(p,q)(g_{\mathbb{S}^2(c_2)})(V,W),$$

for all $V,W \in T\mathbb{S}^2(c_2)$. *Moreover,*

$$-(\nabla^D_X)f + (XY)f = -c_1 f(p,q) g_{\mathbb{S}^2_+(c_1)}(X,Y),$$

for each $X,Y \in TD^2_r$. *We have*

$$\begin{aligned} h g_{\mathbb{S}^2_+(c_1)}(\nabla_D f,\nabla h) g_{\mathbb{S}^2(c_2)} &= \sqrt{\frac{c_2}{c_1}}\cos\varphi\, g_{\mathbb{S}^2_+(c_1)}\left(-\sin\varphi\,\cos\xi\frac{\partial}{\partial\varphi}, -\sqrt{c_1c_2}\,\sin\varphi\frac{\partial}{\partial\varphi}\right) g_{\mathbb{S}^2(c_2)} \\ &= \frac{c_2}{c_1}\sin^2\varphi\,\cos\varphi\,\cos\xi\, g_{\mathbb{S}^2(c_2)}. \end{aligned}$$

Now, by using the identities that we have just obtained, we get to

$$\begin{aligned} \lambda\, g(X,Y) + Hess(f)(X,Y) &= (\cos\varphi\,\cos\xi + 3c_1) g_{\mathbb{S}^2_+(c_1)}(X,Y) - \cos\varphi\,\cos\xi\, g_{\mathbb{S}^2_+(c_1)}(X,Y), \\ \lambda\, g(X,V) + Hess(f)(X,V) &= 0, \\ \lambda\, g(V,W) + Hess(f)(V,W) &= (\cos\varphi\,\cos\xi + 3c_1)\frac{c_2}{c_1}\cos^2\varphi\, g_{\mathbb{S}^2(c_2)}(V,W) + \\ &\quad + \frac{c_2}{c_1}\sin^2\varphi\,\cos\varphi\,\cos\xi\, g_{\mathbb{S}^2(c_2)}(V,W) - \frac{c_2}{c_1}\cos\varphi\,\cos\xi\, g_{\mathbb{S}^2(c_2)}(V,W), \end{aligned}$$

for all $X,Y \in TD^2_r$ *and* $V,W \in T\mathbb{S}^2(c_2)$. *Therefore,*

$$\begin{aligned} \lambda\, g(X,Y) + Hess(f)(X,Y) &= 3c_1 g_{\mathbb{S}^2_+(c_1)}(X,Y) \\ \lambda\, g(X,V) + Hess(f)(X,V) &= 0 \\ \lambda\, g(V,W) + Hess(f)(V,W) &= 3c_2\,\cos^2\varphi\, g_{\mathbb{S}^2(c_2)}(V,W). \end{aligned}$$

Thus $Ric_M = \lambda\, g + Hess(f)$.

The next example is very analogous as Example 4.3.1. We just replace spherical to hyperbolic domains.

Example 4.3.2. *Let c_1, c_2 be negative constants. Set*

$$D_r^2 = \{(x, y, z) \in \mathbb{H}^2(c_1) : x \le r\},$$

where $\frac{1}{\sqrt{-c_1}} < r$. *We shall consider the warped product*

$$(M = D_r^2 \times_h \mathbb{H}^2(c_2), g = \langle \cdot, \cdot \rangle_{\mathbb{L}} + h(p)^2 \langle \cdot, \cdot \rangle_{\mathbb{L}}),$$

where the warping function is given by

$$h(p) = \sqrt{\frac{c_2}{c_1}} \cosh[\sqrt{-c_1} d_{\mathbb{H}^2(c_1)}(p, \partial D_{r+\epsilon}^2)],$$

where $\epsilon > 0$. *Let* f, λ *be functions on* M *given by*

$$\begin{aligned} f(p,q) &= \frac{1}{c_1} \cosh[\sqrt{-c_1} d_{\mathbb{H}^2(c_1)}(p, \partial D_{r+\epsilon}^2)] \cosh[\sqrt{-c_2} d_{\mathbb{H}^2(c_2)}(q, e_1)], \\ \lambda(p,q) &= \cosh[\sqrt{-c_1} d_{\mathbb{H}^2(c_1)}(p, \partial D_{r+\epsilon}^2)] \cosh[\sqrt{-c_2} d_{\mathbb{H}^2(c_2)}(q, e_1)] + 3c_1, \end{aligned}$$

where $e_1 = (1, 0, 0)$. *We shall consider the following coordinates on* $\mathbb{H}^2(c_1)$ *and* $\mathbb{H}^2(c_2)$, *respectively,*

$$\begin{aligned} \psi(\theta, \varphi) &= \frac{1}{\sqrt{-c_1}} (\cosh \varphi, \sinh \varphi \cos \theta, \sinh \varphi \sin \theta), \\ \Theta(\xi, \zeta) &= \frac{1}{\sqrt{-c_2}} (\cosh \xi, \sinh \xi\, \cos \zeta, \sinh \xi\, \sin \zeta). \end{aligned}$$

On these coordinates, the functions h, f, λ *are written by* $h(p) = \sqrt{\frac{c_2}{c_1}} \cosh \varphi$, $f(p,q) = \frac{1}{c_1} \cosh \varphi\, \cosh \xi$, *and* $\lambda(p,q) = \cosh \varphi\, \cosh \xi + 3c_1$. *We have that*

$$\begin{aligned} \frac{\partial}{\partial \theta} &= \frac{1}{\sqrt{-c_1}} (0, -\sinh \varphi\, \sin \theta, \sinh \varphi\, \cos \theta), \\ \frac{\partial}{\partial \varphi} &= \frac{1}{\sqrt{-c_1}} (\sinh \varphi, \cosh \varphi\, \cos \theta, \cosh \varphi\, \sin \theta). \end{aligned}$$

Thus $\left\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right\rangle_{\mathbb{L}} = -\frac{1}{c_1} \sinh^2 \varphi$, $\left\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi} \right\rangle_{\mathbb{L}} = 0$, *and* $\left\langle \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi} \right\rangle_{\mathbb{L}} = -\frac{1}{c_1}$. *It follows that*

$$\nabla h = -c_1 \sqrt{\frac{c_2}{c_1}} \sinh \varphi \frac{\partial}{\partial \varphi}.$$

Then

$$\nabla_{\frac{\partial}{\partial \theta}} \nabla h = -c_1 \sqrt{\frac{c_2}{c_1}} \sinh \varphi \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial \varphi} = -c_1 \sqrt{\frac{c_2}{c_1}} \cosh \varphi \frac{\partial}{\partial \theta},$$

and

$$\nabla_{\frac{\partial}{\partial\varphi}}\nabla h = -c_1\sqrt{\frac{c_2}{c_1}}\cosh\varphi\frac{\partial}{\partial\varphi}.$$

Therefore, $Hess(h) = -c_1 h(p)\langle,\rangle_{\mathbb{L}}$. *Set*

$$\nabla_{\mathbb{H}^2(c_1)} f = \sum_{i,f\in\{\theta,\varphi\}} g^{ij}\frac{\partial f}{\partial j}\frac{\partial}{\partial i},$$

which implies $\nabla_{\mathbb{H}^2(c_1)} f = -\sinh\varphi\,\cosh\xi\frac{\partial}{\partial\varphi}$. *Since* $Hess(f)(X,Y) = -(\nabla_X^{\mathbb{H}^2(c_1)}Y)f + (XY)f$, *then* $Hess(f)(X,Y) = -c_1 f(p,q)\langle X,Y\rangle_{\mathbb{L}}$, *for every* $X,Y\in TD_r^2$. *We have*

$$\begin{aligned} h(p)\langle\nabla_{\mathbb{H}^2(c_1)} f,\nabla h\rangle_{\mathbb{L}} &= \sqrt{\frac{c_2}{c_1}}\cosh\varphi\left\langle -\sinh\varphi\,\cosh\xi\frac{\partial}{\partial\varphi}, -c_1\sqrt{\frac{c_2}{c_1}}\sinh\varphi\frac{\partial}{\partial\varphi}\right\rangle_{\mathbb{L}} \\ &= -\frac{c_2}{c_1}\cosh\varphi\,\cosh\xi\,\sinh^2\varphi. \end{aligned}$$

Since $\nabla^{\mathbb{H}^2(c_2)}_{\frac{\partial}{\partial\zeta}}\frac{\partial}{\partial\xi} = cotgh\,\xi\frac{\partial}{\partial\zeta}$, $\nabla^{\mathbb{H}^2(c_2)}_{\frac{\partial}{\partial\zeta}}\frac{\partial}{\partial\zeta} = -\sinh\xi\,\cosh\xi\frac{\partial}{\partial\xi}$ *e* $\nabla^{\mathbb{H}^2(c_2)}_{\frac{\partial}{\partial\xi}}\frac{\partial}{\partial\xi} = 0$, *then*

$$-(\nabla_V^{\mathbb{H}^2(c_2)}W)f + (VW)f = -c_2 f(p,q)\langle,\rangle_{\mathbb{L}}.$$

We have that

$$\begin{aligned} Ric_M(X,Y) &= 3c_1\langle,\rangle_{\mathbb{L}}, \\ Ric_M(X,V) &= 0, \\ Ric_M(V,W) &= 3c_2\cosh^2\varphi\langle,\rangle_L, \end{aligned}$$

for all $X,Y\in TD_r^2$ *e* $V,W\in T\mathbb{H}^2(c_2)$. *On the other hand, we have*

$$\begin{aligned} \lambda\,g(X,Y) + Hess(f)(X,Y) &= 3c_1\langle,\rangle_{\mathbb{L}}, \\ \lambda\,g(X,V) + Hess(f)(X,V) &= 0, \\ \lambda\,g(V,W) + Hess(f)(V,W) &= 3c_2\cosh^2\varphi\langle,\rangle_{\mathbb{L}} \end{aligned}$$

Therefore, $(M = D_r^2\times_h\mathbb{H}^2(c_2), g, \nabla f, \lambda)$ *is a gradient Ricci almost soliton with boundary.*

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